A PROBLEM ON EGYPTIAN FRACTIONS

TAY ENG GUAN AND TOH PEE CHOON

Egyptian fractions.

The ancient Egyptians used only fractions with unit numerator. Thus, instead of writing $\frac{3}{4}$, they would use $\frac{1}{2}+\frac{1}{4}$ instead. Similarly, $\frac{7}{11}$ would be expressed as $\frac{1}{2}+\frac{1}{8}+\frac{1}{88}$ but in their own hieroglyphs, of course.

This expression of fractions as sums of Egyptian fractions can be quite practical even today. Suppose 11 friends bought 7 loaves of bread and want to share them equally. That would mean $\frac{7}{11}$ each. One could cut every loaf into 11 equal slices and give each friend 7 slices. If each loaf were a cuboid and each slice has equal cross-section, this would take 70 cuts. Can we do it with fewer cuts? If we express $\frac{7}{11}=\frac{1}{2}+\frac{1}{8}+\frac{1}{88}$, then this can translate into cutting the 7 loaves into halves first, followed by cutting 3 halves into eighths (of a whole), and ending with cutting one eighth into eighty-eighths. Each person now gets a half, an eighth and an eighty-eighth of a loaf. The number of cuts is $7+9+10=26$.

Quite some work has been done on Egyptian fractions (you may refer to the excellent website [2]). For example, it has been proved that each fraction can always be written as sums of Egyptian fractions and that there are an infinite number of ways of doing so. New results concerning Egyptian fractions continue to appear [1]. In this paper, we are interested in the number of ways an Egyptian fraction can itself be represented as the sum of two other Egyptian fractions.

Classifying solutions to $\frac{1}{n}$ as a sum of two Egyptian fractions.

Let $x, y, n \in \mathbb{Z}^+, x \neq y$ and assume that $\gcd(x,y,n)=1$. We wish to classify all solutions to the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}.$$

Since $\frac{1}{y}<\frac{1}{n}$, we have $y>n$ and we can write $y=n+k$ for some $k>0$. So

$$\frac{1}{x}=\frac{1}{n}-\frac{1}{n+k} \implies n(n+k)=xk.$$

Let $\gcd(n,k)=d$, and write $n=dr$ and $k=ds$ where $\gcd(r,s)=1$. We further observe that $d \mid y$ and so $\gcd(d,x)=1$.

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Our equation now simplifies to

$$dr(r+s) = xs.$$

By Euclid's Lemma, since $\gcd(d, x) = 1$, we have $d \mid s$. Similarly, $r \mid x$ and $(r+s) \mid x$. This means that we have the following equations

$$x = r(r+s) \text{ and } s = d.$$

In other words, the original equation can be written as

$$\frac{1}{r(r+d)} + \frac{1}{d(r+d)} = \frac{1}{rd}, \quad \text{where } \gcd(r, d) = 1.$$

Number of representations of $\frac{1}{n}$ as a sum of two Egyptian fractions.

We shall prove our main result with the help of the following two lemmas.

Lemma 1. *Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorisation of the positive integer n. Let $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$, where $x, y \in \mathbb{Z}^+, x \neq y$. If $\gcd(x, y, n) = 1$, then the number of such representations is 2^{k-1}.*

Proof. If $\gcd(x, y, n) = 1$, then by the classification above, the number of such representations is the number of ways of expressing n as the product of two coprime factors r and d. This is the number of ways of dividing k distinct objects (the k distinct primes) into 2 identical boxes, where the boxes may be allowed to be empty, i.e. 2^{k-1} ways. □

Lemma 2. *Let $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$, where $x, y \in \mathbb{Z}^+, x \neq y$. If $\gcd(x, y, n) = d$, then the number of such representations is the number of representations of*

$$\frac{1}{n'} = \frac{1}{x'} + \frac{1}{y'},$$

where $n' = \frac{n}{d}$, $x' = \frac{x}{d}$, $y' = \frac{y}{d}$ and $\gcd(x', y', n') = 1$.

Theorem 3. *Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorisation of the positive integer n. Let $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$, where $x, y \in \mathbb{Z}^+, x \neq y$. The total number of such representations is*

$$\sum_{i=1}^{k} 2^{i-1} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant k} a_{j_1} a_{j_2} \dots a_{j_i}.$$

Proof. By Lemma 2, we can divide into cases by considering the number of representations of $\frac{1}{n'} = \frac{1}{x'} + \frac{1}{y'}$, where $n' = \frac{n}{d}$ and $\gcd(x', y', n') = 1$.

Let i be the number of distinct primes dividing n'. Suppose

$$n' = p_{j_1}^{b_{j_1}} p_{j_2}^{b_{j_2}} \dots p_{j_i}^{b_{j_i}}, \quad \text{where } 1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant k$$

and for each r from 1 to i, $1 \leqslant b_{j_r} \leqslant a_{j_r}$. By Lemma 1, the number of representations is 2^{i-1}. The total number of possible choices for n' is $a_{j_1} a_{j_2} \dots a_{j_i}$ for this set of i primes. Thus the number of representations is

$2^{i-1}a_{j_1}a_{j_2}\dots a_{j_i}$. Summing across all possible combinations of i primes, we have that the total number of representations when there are i primes is

$$2^{i-1}\sum_{1\leqslant j_1<j_2<\cdots<j_i\leqslant k} a_{j_1}a_{j_2}\dots a_{j_i}.$$

Finally, we sum across all i to obtain the total number of representations as

$$\sum_{i=1}^{k} 2^{i-1}\sum_{1\leqslant j_1<j_2<\cdots<j_i\leqslant k} a_{j_1}a_{j_2}\dots a_{j_i}.$$

□

We conclude with two examples to illustrate the main result.

Example 4. *If $n = p_1^{a_1}p_2^{a_2}p_3^{a_3}$ where p_i are distinct primes, then the total number of representations of $\frac{1}{n}$ as a sum of two Egyptian fractions is*

$$4a_1a_2a_3 + 2(a_1a_2 + a_1a_3 + a_2a_3) + a_1 + a_2 + a_3.$$

Example 5. *If $n = p^2q$, where p and q are distinct primes, then there are 7 distinct representations of $\frac{1}{n}$ as a sum of two Egyptian fractions. These are listed in the table below, arranged by possible divisors of n.*

d	2^{i-1}	$\frac{1}{x}+\frac{1}{y}$
1	2	$\frac{1}{1+p^2q}+\frac{1}{p^2q(1+p^2q)}$ $\frac{1}{p^2(p^2+q)}+\frac{1}{q(p^2+q)}$
p	2	$\frac{1}{p(1+pq)}+\frac{1}{p^2q(1+pq)}$ $\frac{1}{p^2(p+q)}+\frac{1}{pq(p+q)}$
p^2	1	$\frac{1}{p^2(1+q)}+\frac{1}{p^2q(1+q)}$
q	1	$\frac{1}{q(1+p^2)}+\frac{1}{p^2q(1+p^2)}$
pq	1	$\frac{1}{pq(1+p)}+\frac{1}{p^2q(1+p)}$
p^2q	0	

References

[1] S. Butler, P. Erdös, R. Graham, *Egyptian fractions with each denominator having three distinct prime divisors,* Integers 15 (2015) #A51. Published Dec 11, 2015.

[2] R. Knott, *Egyptian Fractions*, Retrieved Dec 8, 2015 from `http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/egyptian.html`.

Mathematics & Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616