The Problem Solving Approach in the Teaching of Number Theory

TOH, Pee Choon*
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: peechoon.toh@nie.edu.sg

LEONG, Yew Hoong
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: yewhoong.leong@nie.edu.sg

TOH, Tin Lam
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: tinlam.toh@nie.edu.sg

DINDYAL, Jaguthsing
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: jaguthsing.dindyal@nie.edu.sg

QUEK, Khiok Seng
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: khiokseng.quek@nie.edu.sg

TAY, Eng Guan
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: engguan.tay@nie.edu.sg

HO, Foo Him
National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,
Singapore 637616; Email: foohim.ho@nie.edu.sg

Keywords: Problem Solving, Mathematics Teacher Education, Number Theory

* Corresponding author

Abstract

Mathematical problem solving is the mainstay of the mathematics curriculum for Singapore schools. In the preparation of prospective mathematics teachers, the authors, who are mathematics teacher educators, deem it important that pre-service mathematics teachers experience non-routine problem solving and acquire an attitude that predisposes them to adopt a Pólya-style approach in learning mathematics. The Practical Worksheet is an instructional scaffold we adopted to help our pre-service mathematics teachers develop problem solving dispositions alongside the learning of the subject matter. The Worksheet was initially used in a design experiment aimed at teaching problem solving in a secondary school. In this paper, we describe an application and adaption of the MProSE (Mathematical Problem Solving for Everyone) design experiment to a university level Number Theory course for pre-service mathematics teachers. The goal of the enterprise was to help the pre-service mathematics teachers develop problem solving dispositions alongside the learning of the subject matter. Our analysis of the pre-service mathematics teachers' work shows that the MProSE design holds promise for mathematics courses at the tertiary level.

The study reported in this paper arose from an effort to improve the classroom practice of the first author (henceforth referred to as the first person singular pronoun) in the context of teaching undergraduate level mathematics to pre-service teachers (henceforth referred to as student teachers). I was assigned to teach two first year courses that were commonly taken by student teachers in the BA (Education) and BSc (Education) programmes. Student teachers participating in these programmes learn the subject matter of the disciplines in Arts or Sciences (hence BA or BSc) and their respective pedagogies for teaching them (hence the Education qualification). I was assigned to teach essentially the same group of student teachers over two consecutive semesters within one academic year for two mathematics content courses. This relatively long period of interaction with them provided an opportunity for me to learn about their learning difficulties and to address them.

My reflections from the first course

The first course I taught to a group of student teachers (which was also their first course in undergraduate mathematics) was entitled *Calculus I*. There were 58 students enrolled on this 36-hour course. The content of the course included limits, differentiation and integration of single-variable functions. Throughout the course, one observation stood out: when the student teachers were confronted with problems that appeared novel to them, that is, problems with no clear procedures to apply, many of them would do nothing and would wait for me to provide the solution steps. However, when it came to following a set of procedures I had taught them in familiar exercises, I noticed they were proficient in completing the exercises. But when it came to unfamiliar problems, many students were unable to even start or put pen to paper to explore the problem, and simply waited for me to demonstrate the procedure of solution. This was not how I envisioned mathematics learning at the undergraduate level.

The above clearly pointed towards their lack of a mathematical problem solving disposition that would have enabled them to "take a stab" at a non-routine mathematics problem. It was also clear then to me what I needed to address in the second course with them the next semester: teach problem solving! That the second course was on number theory strengthened this resolve, since many theorems in number theory lend themselves well to problem solving approaches. Also, the nature of number theoretic problems is such that if students do not attempt to use suitable problem solving heuristics at all, they would not be able to make progress in proving the theorems, which means they are unlikely to do well in the course. At that point in time, I was not plugged into the formal genre of mathematical problem solving, which I came to learn of from my colleagues in the department. These colleagues are active researchers in the teaching of problem solving in secondary school mathematics. They were involved in a funded research project on secondary school students known as Mathematical Problem Solving for Everyone (MProSE). In talking to these colleagues, I found the theoretical basis of the project to be sound and thus helpful to my enterprise of helping my student teachers with their learning of problem solving. The research reported here draws heavily from the work done in the MProSE project.

MProSE

There has been much interest in mathematical problem solving since Pólya published his first book on mathematical problem solving in 1945. From the 1980s onwards—and with the added contributions of significant theory builders like Schoenfeld (1985)—there has also been a world-wide push for problem solving to be the central focus of the school mathematics curriculum. For example, in the United States, the National Council of Teachers of Mathematics (NCTM) in their *Agenda for Action* recommended that "[p]roblem solving should be the central focus of the mathematics curriculum" (NCTM, 1980). In the UK, the Cockcroft Report (1982) advocated that mathematics teaching at all levels should include opportunities for problem solving.

Although there is general agreement among mathematicians and mathematics educators that problem solving is one of the fundamental goals of teaching mathematics, that goal remains one of the most elusive ones (Lester, 1994; Stacey, 2005). There remains a gap to fill with regards to translating the theories on problem solving to sustainable classroom practice. The MProSE project aims to do the translational research. It is not the aim of the MProSE project to re-invent the wheel of decades of theory building about problem solving. The MProSE research project adopts as its theoretical underpinnings the ideas of George Pólya (1945) and Alan Schoenfeld (1985). The well-known cornerstones of [1]Pólya's stages and heuristics as well as Schoenfeld's framework of problem solving are readily recognised by the mathematics education community. As such we seek to build on their contributions, focusing especially on the work of translating these theories into workable practices, as proposed by Schoenfeld (2007, p. 539):

> That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985, 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. … The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms.

[1] As Pólya's (1945) four-stage model and Schoenfeld's (1985) framework of four components are now well-known, we assume the readers are familiar with them and thus are not included here.

The MProSE team (henceforth referred to as "we") comprises mathematics educators, research mathematicians, and classroom practitioners. In our experience, mathematics students are generally resistant to applying the stages of Pólya's model. These students also do not consciously use and manage heuristics productively. Even the higher-achieving students who could solve the given problems do not generally make the extra effort to undergo the fourth stage of Pólya's model, namely, *Looking Back* to check their solutions and learn from their experience of solving the problem.

In an attempt to help mathematics students develop good problem solving habits (namely, those explicated by Pólya and Schoenfeld) which are especially useful to them when they are confronted with a novel mathematics problem, we used a worksheet like that used in science practical lessons. The students would work through the worksheet as they would in a science laboratory class. We call the worksheet *The Practical Worksheet.* (A condensed version of the Practical Worksheet, as well as samples with data of students' work, are shown in the later sections of this paper.) The students would treat problem-solving lessons as mathematics "practical" classes. In this way, we hoped to achieve a paradigm shift in the way learners look at these "difficult, unrelated" problems which had to be done in this "special" classroom setting – a form of mathematics "practical".

The use of practical work to achieve the learning of the scientific processes has a long history of at least a hundred years and can be traced to Henry Edward Armstrong (Woolnough & Allsop, 1985). Woolnough and Allsop (1985) stated clearly what is to be achieved in science education:

> As we look at the nature of science we see two quite distinct strands. The knowledge, the important content and concepts of science and their interrelationships, and also the processes which a scientist uses in his working life. In teaching science we should be concerned both with introducing students to the important body of scientific knowledge, that they might understand and enjoy it, and also with familiarizing students with the way a problem-solving scientist works, that they too might develop such habits and use them in their own lives. (p.32)

It is instructive to note that we could just replace 'science' with 'mathematics' and the preceding passage reads just as true, which any mathematics educator would agree. As such, it is certainly conceivable that similar specialised lessons and materials for mathematics may be necessary to teach the mathematical processes, including and via problem solving (Toh, Quek & Tay, 2008). The envisaged mathematics practical approach could elicit the learning of the processes of problem solving, analogous to the processes of science practical skills of scientists in their working life.

To realise this paradigmatic change in the teaching of mathematical problem solving we must address the issue of student assessment. What we see as the root of the lack of success for previous attempts to implement problem solving in classrooms is that mathematical problem solving within the school curriculum is not assessed, especially in a way that matters to the students. Because it is not assessed, students and teachers do not place much emphasis on the processes of problem solving; students are more interested to learn the other components of the curriculum which would be assessed. Thus, MProSE has designed a formal assessment system complementing the innovative curriculum that assesses not just the product but also the processes of problem solving. If the curriculum were a dog and assessment its tail, it would be a painful truth that "the tail wags the dog". Most students would study what counts in an exam. Leong (2012, p. 133) interviewed five students after they had attended a MProSE problem solving course. Leong reported that "[t]he responses from the five students seem to indicate that making the results of the module count as part of school assessment would very likely make the students put in greater effort for the course."

The Practical Worksheet that we designed contains sections explicitly guiding the students to go through the Pólya's stages when solving non-routine problem solving, and the problem-solving heuristics to use. We used the Practical Worksheet together with the practical paradigmatic thinking in a number of Secondary schools in Singapore. The results so far have been encouraging. Teachers in the MProSE's research project schools demonstrated strong fidelity to the MProSE design in their classroom teaching of problem solving (Leong, Dindyal, Toh, Quek, Tay, & Lou, 2011; Leong, Toh, Dindyal, Ho, & Tay, 2012); students were attempting the practical worksheets, with a significant number even making good progress to Pólya's fourth stage of extending and generalizing the problems (Leong, Tay, Toh, Quek, & Dindyal, 2011; Dindyal, Tay, Toh, Leong, & Quek, 2012). This gives us an impetus to trial the worksheet and paradigm also with undergraduate level students. In particular, I seek to examine whether student teachers, after substantial opportunities to use the Practical Worksheet to learn about problem solving in the context of the second course on number theory, would be able to (1) solve unfamiliar problems which I consider challenging for typical first year undergraduate students and (2) apply Pólya's stages (including Stage Four) and heuristics appropriately.

Aligning the second course to MProSE design

Like the first course, *Number Theory* was taught over a period of 36 hours. Fifty-nine student teachers enrolled in the course, of whom 56 were from my *Calculus I* course. The content covered in *Number Theory* was typical of similar courses taught elsewhere and includes for example divisibility, congruences, Diophantine equations, and Euler's generalization of Fermat's little theorem. I was aware that Pólya's model would be formally introduced to these student teachers in mathematics methods courses in the third year of their respective BA/BSc (Education) programmes. As such, my focus in *Number Theory* was in exemplifying the use of the model rather than to explicitly discuss its theoretical basis and pedagogical implications.

My priority in structuring *Number Theory* was to keep to the mathematical content of what is usually included in such a course at the undergraduate level; in other words, while I intend to make problem solving a prominent feature in my course, I did not wish to compromise the rigor of the mathematical content. As such, my primary challenge was to weave in problem solving components in such a way that it is integrated with the content rather than present it as a separate strand. The overall strategy was this: I taught the course in the usual way; but whenever there were results in the course that I reckoned were problems amenable to a demonstration of the problem solving process, I seized upon the opportunity to model Pólya's stages and heuristics. To elaborate on this point about what I considered amenable for explicating Pólya's processes, this is one example: *Prove that if the product of two integers is odd, then both the integers are odd*. I considered this problem as suitable for a number of reasons. An obvious one is that student teachers may confuse this with the converse, and thus provides a natural platform to introduce "Understand the Problem". In contrast, other typical questions in the course, such as "*prove, using mathematical induction, that* $1 + 3 + 5 + \ldots + (2n - 1) = n^2$" is straightforward in the sense that a particular method is prescribed and thus not suitable to be 'forced' into the problem solving mould.

I realised that my way of teaching would have to align to these structural changes in the course. Instead of directly presenting results as theorems followed by the demonstrated proofs, I used a style that gave student teachers the impression that I was genuinely solving the problems to prove; in other words, when I presented a theorem to prove, I talked aloud my problem solving thoughts and invited student teachers into this group talk. In the process, I consciously brought in the language of Pólya's stages and heuristics such as "substitute numbers", "use a suitable notation", and "restate the problem". As an example of how this was actually done in my class, I continue to elaborate on the example mentioned in the previous paragraph. Under "Understand the problem", I offered the heuristic of substituting numbers to help them make sense of the theorem; in "Devise a plan" I anticipated that student teachers would use a strategy similar to how one would prove the converse statement (such as writing the product in the form $2n + 1$ for some integer n) but might not be productive in this case. This served as a good point for me to show the need to loop back to the first stage before devising another plan. In this case, it also provided the motivation to use the method of indirect proof as another plan. I then proceeded with this plan and completed the proof. As this problem was presented in the early part of the course, I did not progress to the fourth stage—which was introduced in the later part of the course.

The place of problem solving in the second course

The 36-hour course was divided into 24 hours of lectures (one hour for each lecture slot and twice a week)—taught solely by me—and 12 hours of tutorials. All the student teachers sat for the same lectures; for tutorials, they were divided into four groups of roughly equal numbers. I taught two of these groups and another colleague in the department—not a member of the MProSE project team—taught the other two. Since only the lecture slots were common for all student teachers, the problem solving elements were made explicit solely during the lectures. Table 1 presents an overview of the content of the course and the place of problem solving at various junctures of the course.

Table 1:
Overview of How Problem Solving Elements are Weaved into the Course

Weeks	Content Taught	Significant Problem Solving Milestones
1 – 2	Number systems and methods of proof.	Introduce and model Pólya's first three stages.
3 – 4	Prime numbers, divisibility, the Euclidean algorithm and gcd.	Actively involve students in the problem solving process.
5 – 6	Diophantine equations.	Formally introduce Pólya's model (including the fourth stage) and the Practical Worksheet.
7 – 8	The fundamental theorem of arithmetic, congruences.	Students submit the first problem solving assignment.
9 – 10	Modular number systems and cryptography.	Return the graded assignment with feedback.
11– 12	Exponents and Euler's generalization of Fermat's little theorem.	Students submit the second problem solving assignment.

Since most students did not have prior exposure to the language of Pólya, I introduced the first three stages by explicitly modelling them whenever there were suitable opportunities. The heuristics were explicated whenever they were used. At this early period, the predominant mode was that of teacher demonstration of the stages. In the next two weeks, I retained the use of this structure in the presentation of the problems in the lecture notes but left blanks instead under each of the three stages to allow student teachers to participate more actively in their problem solving attempts instead of relying solely on me for the answers.

In the first four weeks, although I used the language of Pólya, I only presented the first three stages of Pólya's model. The fourth stage was introduced in Week 5. In fact, I devoted one full lecture for this formal introduction of Pólya's model. Within the same lecture, the student teachers were presented with the following problem, which was chosen from Toh, Quek, Leong, Dindyal & Tay (2011):

You are given two jugs, one holds 5 litres of water when full and the other holds 3 litres of water when full. There are no markings on either jug and the cross-section of each jug is not uniform. Show how to measure out exactly 4 litres of water from a fountain.

This problem was chosen because it lends itself well to a demonstration of Pólya's fourth stage of extension; and leads naturally to the Number Theory topic of Diophantine equations. The student teachers were asked to attempt the problem using the Practical Worksheet. After that lecture, I provided a sample of how I solved the problem (including the generalisation to linear Diophantine equations in Stage Four) on the Practical Worksheet for their reference. For the rest of the course, I continued to structure the solution strategies according to Pólya's stages (including the fourth stage) and used the language of heuristics whenever an appropriate problem was introduced.

In line with the paradigm of placing formal assessment as an important motivation for the student teachers, they were required to work on two problem solving assignments using the Practical Worksheet. These, together with another ten regular homework assignments, contributed to ten per cent of their final grades for the course. The student teachers submitted their first assignment in Week 7. This assignment was graded, with corrections on their mathematical errors highlighted, and returned to the student teachers for their feedback. A sample of how I solved this first assignment problem on the Practical Worksheet was also made available to them after that. The second assignment was collected at the end of the Week 12. This was graded but not returned to the student teachers as all formal classes had ended. The objective of the assignment problem and the student teachers' attempts of the problem are discussed in the following sections.

Method

The student teachers' second assignment submission in the form of the Practical Worksheet formed the data source for this study, as we wanted to know what student teachers learnt at the end of the course. (An exploratory analysis was carried out for student teachers' submissions on the first assignment and the outcomes are reported in Toh, Toh, Ho, and Quek (2012). The methods used in that study was modified and refined for the research reported here.) Figure 1 shows a condensed format of the [2]Practical Worksheet, with all the guiding instructions. As mentioned in an earlier section, the worksheet was organised in such a way as to guide student teachers to use Pólya's stages and problem solving heuristics to solve the mathematics problems. Structured in this way, the worksheet allows the instructor and the MProSE researchers to study the student teachers' attempts under each of the stages to examine if they apply them appropriately to solving the problem at hand.

The problem to be solved in the students' second assignment was:

Let a, b, c be natural numbers satisfying $a + b + c = 2012$. *If we have*

$$a!b!c! = m\,10^n$$

for some integers m and n, where 10 *does not divide m, find the smallest possible value of n.*

This was a challenging and novel problem for the student teachers because there were three important key steps for a complete solution of the problem. The first step was to find a formula to compute the number of "trailing zeroes" in $a!$ for a general positive integer a, namely

$$\left\lfloor\frac{a}{5}\right\rfloor + \left\lfloor\frac{a}{25}\right\rfloor + \left\lfloor\frac{a}{125}\right\rfloor + \left\lfloor\frac{a}{625}\right\rfloor,$$

for $a < 3125$. This result was not taught in class and was not recorded in the prescribed textbook. However, I believe that this was something that student teachers with a problem solving disposition could figure out. The second step was to decide which combination of a, b and c would lead to the smallest value of n. Finally, once a certain conclusion was reached, the student teachers would still have to check that their choice of a, b and c indeed yielded the minimum n.

[2] We have renamed the fourth stage of Looking Back as Check and Expand.

I	**Understand the problem (UP)** (You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.) (a) Write down your feelings about the problem. Does it bore you? scare you? challenge you? (b) Write down the parts you do not understand or that you misunderstood. (c) Write down the heuristics you used to understand the problem.
II	**Devise a plan (DP)** (You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.) (a) Write down the key concepts that might be involved in solving the question. (b) Do you think you have the required resources to implement the plan? (c) Write out each plan concisely and clearly.
III	**Carry out the plan (CP)** (You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.) (i) Write down in the *Control* column, the key points where you make a decision or observation, for eg., go back to check, try something else, look for resources, or totally abandon the plan. (ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.
IV	**Check and Expand (C/E)** (a) Write down how you checked your solution. (b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of. (c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Figure 1. Instructions on a Practical Worksheet

The student teachers were given three weeks to work on the assignment with the understanding that they could discuss with their classmates but they had to submit their own solutions on the Practical Worksheet. They were also informed that they would be graded on both their problem solving processes, based on their presentation using the Practical Worksheet and the correctness of the final answer in their worksheet. We do not rule out the possibility of student teachers making reference to solutions that already exist in the open domain. We were, however, not able to verify this directly.

We examined the student teachers' worksheets along these categories: (1) student teachers' solution approaches; (2) whether they apply the first three stages of Pólya appropriately; (3) whether they move on to the fourth stage of "Check and Expand"; and (4) the use of heuristics.

For (1), we first coded the worksheets by whether the solutions were correct. We further classified their solutions by the methods used. For (2), we first looked at whether the student teachers' written work on the worksheets reflected elements of each of the stages; we then examined evidence for links in between adjacent stages, that is, whether student teachers built on "Understand the problem" to "Devise a Plan", and also whether the "Carry out the Plan" matched the plan they devised. For (3), we were interested to know if student teachers would indeed go beyond the solution of the given problem to "Check and Expand". We further coded their responses in this section according to (i) whether they checked the solution correctly and (ii) whether they attempted to generalise the problem. As to (4), we listed the heuristics used by the student teachers and how they had been employed in the problem solving process.

In the following sections, for the ease of readers in following the analysis of our data, we have chosen to integrate the various data streams into the chronology of the solution process as it is presented in the student teachers' Practical Worksheet, that is, along the sequence of Pólya's Stages.

Problem solving processes used by the student teachers: Stage 1

Out of a total of 59 student teachers enrolled in the course, four did not submit their assignments. Of the 55 who submitted, 46 displayed some evidence of attempting to understand the problem in their Practical Worksheet. These student teachers demonstrated ability to use heuristics to make progress in their solution. The three most commonly used heuristics: "look for patterns", "use smaller numbers" and "guess and check" are tabulated in Table 2. Moreover, the student teachers typically used more than one heuristic to help them to better understand the problem.

Table 2:
Heuristics Used by Students

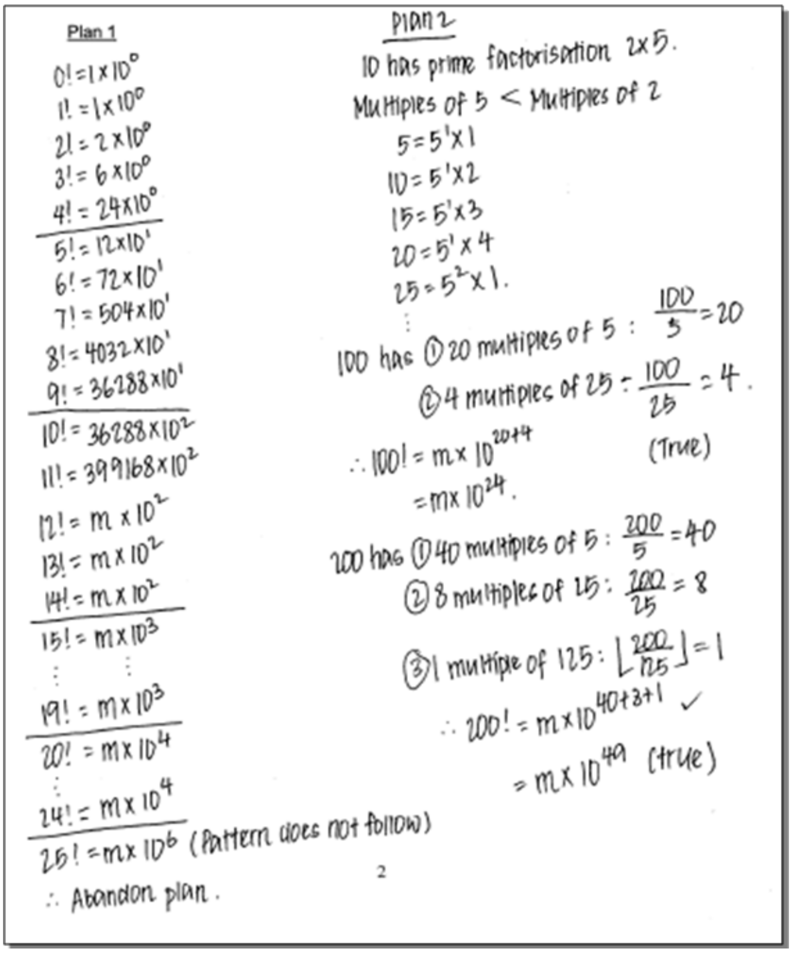
Heuristics Used	Number of Student Teachers
Look for patterns	25
Use smaller numbers	28
Guess and check	37

One heuristic adopted by many student teachers was "use smaller numbers". For example, they would try to study the easier problem with $a + b + c = 10$ instead of 2012. A number such as "10" was perhaps chosen as it is small enough to be computed with a calculator. This contributes to helping them better understand the problem through concretising it. Moreover, it was easy to work out all the possibilities in this special case and see that the minimum value of n was 0 as long as none of the unknowns exceeded 5. A number of student teachers also managed to arrive at the conjecture that choices of a, b and c, where the three unknowns are approximately the same size seemed to yield a smaller value of n.

A typical example of the use of the heuristic "look for patterns" is shown in Figure 2. The student teacher [3]Alan, a middle ability student, began by making a list of values from 0! to 25! and attempted to find a pattern. Alan noted that the number of trailing zeroes seemed to increase by 1 for every 5 natural numbers, but the pattern terminated at 25!. He decided at this point to abandon his plan, and moved on to his second plan which was to consider prime factorizations. Alan noticed that there were many more multiples of 2 compared to the multiples of 5 and further noted that $25 = 5^2$. Hence he focused on the powers of 5 and eventually figured out the correct formula for computing the trailing zeroes of $a!$, although this was not very clearly reflected in his Practical Worksheet. Figure 3 shows how another middle ability student teacher, Beatrice, provided more details on

[3] All names of student teachers used in this article are pseudonyms.

how she arrived at the formula $\left\lfloor\frac{a}{5}\right\rfloor + \left\lfloor\frac{a}{25}\right\rfloor + \left\lfloor\frac{a}{125}\right\rfloor + \left\lfloor\frac{a}{625}\right\rfloor$ for $a < 3125$, using similar reasoning.[4]


Plan 1

$0! = 1 \times 10^0$
$1! = 1 \times 10^0$
$2! = 2 \times 10^0$
$3! = 6 \times 10^0$
$4! = 24 \times 10^0$
$5! = 12 \times 10^1$
$6! = 72 \times 10^1$
$7! = 504 \times 10^1$
$8! = 4032 \times 10^1$
$9! = 36288 \times 10^1$
$10! = 36288 \times 10^2$
$11! = 399168 \times 10^2$
$12! = m \times 10^2$
$13! = m \times 10^2$
$14! = m \times 10^2$
$15! = m \times 10^3$
$\vdots$
$19! = m \times 10^3$
$20! = m \times 10^4$
$\vdots$
$24! = m \times 10^4$
$25! = m \times 10^6$ (Pattern does not follow)
$\therefore$ Abandon plan.

Plan 2

10 has prime factorisation 2×5.
Multiples of 5 < Multiples of 2
$5 = 5^1 \times 1$
$10 = 5^1 \times 2$
$15 = 5^1 \times 3$
$20 = 5^1 \times 4$
$25 = 5^2 \times 1$.
$\vdots$
100 has (1) 20 multiples of 5 : $\frac{100}{5} = 20$
(2) 4 multiples of 25 : $\frac{100}{25} = 4$.
$\therefore 100! = m \times 10^{20+4}$ (True)
$= m \times 10^{24}$.
200 has (1) 40 multiples of 5 : $\frac{200}{5} = 40$
(2) 8 multiples of 25 : $\frac{200}{25} = 8$
(3) 1 multiple of 125 : $\left\lfloor\frac{200}{125}\right\rfloor = 1$
$\therefore 200! = m \times 10^{40+8+1}$ ✓
$= m \times 10^{49}$ (true)

2

Figure 2. Use of the heuristic "look for pattern" by Alan.

A total of ten student teachers began with the same approach of listing values from 0!, but because they did not get as far as 25!, they arrived at the false conclusion that the number of trailing zeroes increase by 1 for every 5 natural numbers. These student teachers subsequently built upon this conjecture which was false and produced incorrect solutions.

[4] The work of these student teachers as reported in this paragraph and the one before may have been considered by them to be stages 2 and 3 of Polya's model. Indeed some of them place these workings under sections 2 and 3 of the worksheet (see Figure 2). However, we consider these as preliminary attempts prior to the main attack of the stated problem, so we classified them under stage 1.

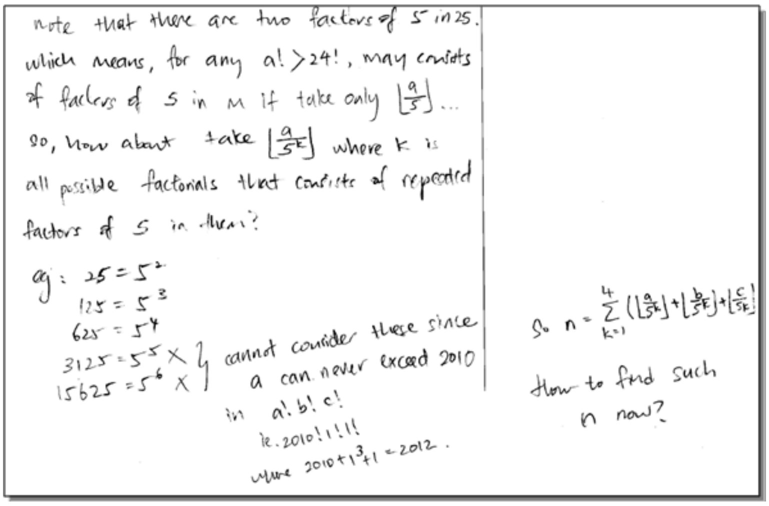


Figure 3. How Beatrice arrived at the correct formula.

Problem solving processes used by the student teachers: Stages 2 and 3

A total of 50 out of 55 student teachers proceeded to the next stage of "Devise a plan" after the first stage. In the second stage, most student teachers continued with the heuristic of "guess and check" as their main line of attack. They demonstrated it in Stage 3 of the Practical Worksheet in their attempts to decide the allocation of the values of *a*, *b* and *c*. A total of 34 student teachers chose to look at the two extreme cases of either one unknown is very large or all the three unknowns are of the same size. In other words, either $a = b = 1$ and $c = 2010$ or $a = 670$, and $b = c = 671$. Those who have already obtained the correct formula for trailing zeroes would realize that the latter case gave smaller values for *n.* Some of them then proceeded to search for other values of *a*, *b* and *c*, where the three unknowns are of similar size.

A total of 14 student teachers arrived at the correct answer by using "guess and check" for several combinations of *a*, *b* and *c.* A few presented their working in ways that show greater deliberateness in the guessing and the checking. They tend to provide more cases of substitutions to show an orientation towards the final answer. But most of the student teachers' worksheets presented a 'jump' from remote guesses into the final correct answer. An example of which is given in Figure 4, where the student teacher, Charles, did not provide any rationale to why the correct answer of $a = b = 624$ and $c = 764$, resulting in the minimum value of $n = 493$, was chosen. We grant that these student teachers could have done the substantial trialling elsewhere and only chose to present the correct answer. Nevertheless, it is not a practice we advocate as clear communication of the reasoning is a valued attribute within the mathematics community.

Another nine student teachers also arrived at the correct solution using "guess and check". What distinguished this group from the previous one was the fact that these student teachers noted in their Practical Worksheets that the values 5, 25, 125 and 625 played an important part in the problem. For example, the number 624! has 152 trailing zeroes but the number 625! has 156 trailing zeroes. They reasoned that because of this jump, in order to have the minimum number of trailing zeroes, two of the unknowns should be fixed at 624. One example (Figure 5) is the reasoning given by Desmond, a high ability student teacher. Although such reasoning was sound, this solution was not completely rigorous as the student teachers did not go on to show that this solution was indeed the minimum value. This was exemplified by the following comment extracted from another high ability student teacher, Eric's Practical Worksheet with regards to his satisfaction level of his solution. Eric wrote, "[I am] not too satisfied as I feel that I have to show n < 493 is not possible. But I am totally lost."

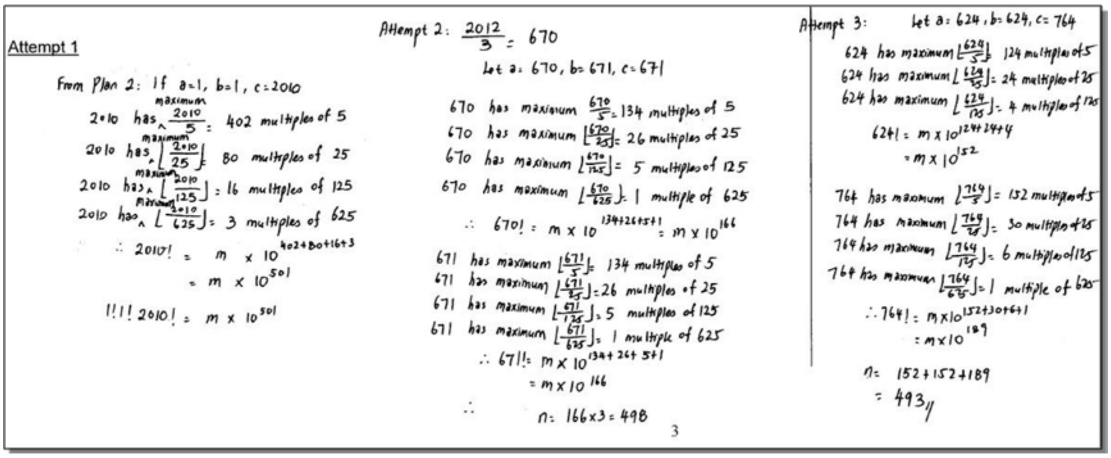


Figure 4. Use of the heuristic "guess and check" by Charles.

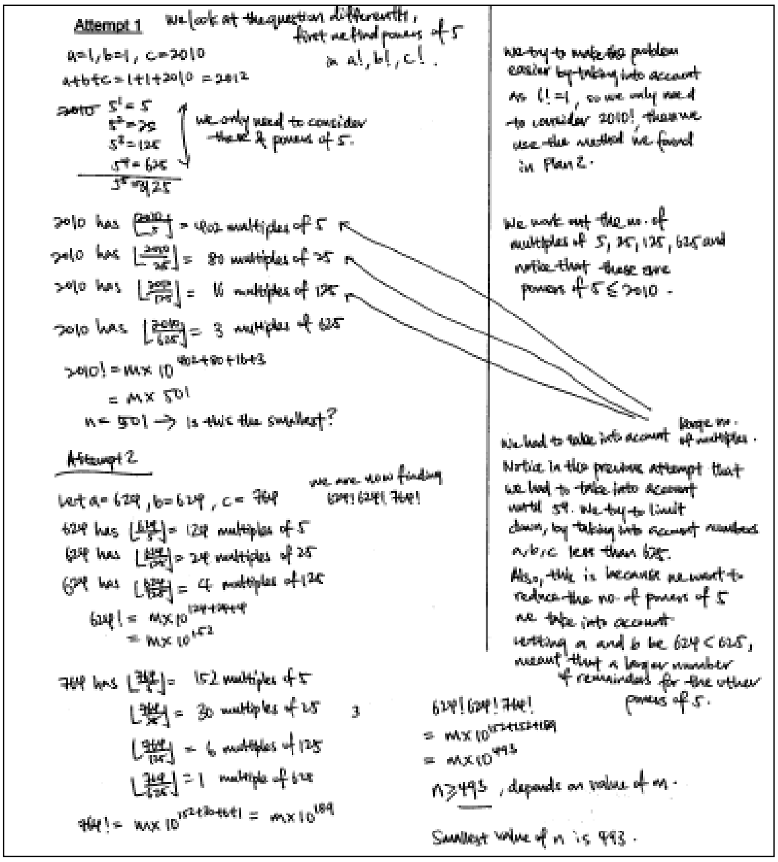
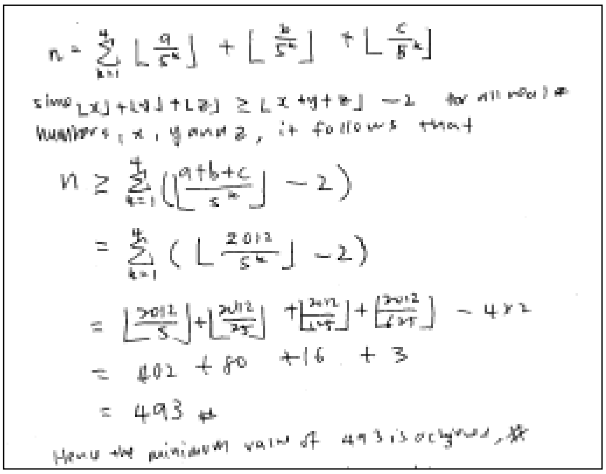


Figure 5. Desmond's reasoning that picking $a = b = 624$ would reduce the number of trailing zeroes.

A third group of seven student teachers presented the complete solution by proving that the value of n = 493 was the minimum value achievable. They managed to calculate a lower bound for n using the argument that

$$\lfloor a \rfloor + \lfloor b \rfloor + \lfloor c \rfloor \geq \lfloor a + b + c \rfloor - 2.$$

Figure 6 is an extract of Fiona's solution. We honestly did not expect the student teachers' solutions to go this far and as such we were impressed by their efforts.



$$n = \sum_{k=1}^{4} \left\lfloor \frac{a}{5^k} \right\rfloor + \left\lfloor \frac{b}{5^k} \right\rfloor + \left\lfloor \frac{c}{5^k} \right\rfloor$$

since $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor z \rfloor \geq \lfloor x + y + z \rfloor - 2$ for all real numbers, x, y and z, it follows that

$$n \geq \sum_{k=1}^{4} \left(\left\lfloor \frac{a+b+c}{5^k} \right\rfloor - 2 \right)$$

$$= \sum_{k=1}^{4} \left(\left\lfloor \frac{2012}{5^k} \right\rfloor - 2 \right)$$

$$= \left\lfloor \frac{2012}{5} \right\rfloor + \left\lfloor \frac{2012}{25} \right\rfloor + \left\lfloor \frac{2012}{125} \right\rfloor + \left\lfloor \frac{2012}{625} \right\rfloor - 4 \times 2$$

$$= 402 + 80 + 16 + 3$$

$$= 493$$

Hence the minimum value of 493 is achieved.

Figure 6. Fiona's work showing how the lower bound was calculated.

Problem solving processes used by the student teachers: Stage 4

In Stage Four of the Practical Worksheet, 48 out of 55 student teachers attempted to check their solutions. This, in particular, helped those who had quickly jumped to the conclusion that a = 670, and b = c = 671 would give the minimum to revise their answers. Geraldine wrote the following in her worksheet: "I am quite happy that I checked my solution as I discovered careless mistakes in my discovery of powers of 10." Taking into account the first author's instructional history with these students – that checking was not a common practice among them, we think that the explicit stipulation in the last stage of the Practical Worksheet provided the impetus for them to perform the required checks for their solutions.

The last stage of Pólya for this problem also includes the search for alternative solutions. In this regard, only one student teacher made attempts at finding alternative solutions and, in fact, obtained more than one correct solution, for example: a = 1249, b = 624 and c = 139, which gave the same value of n = 493.

Also, what we would have liked to see more of was student teachers attempting to go beyond the solution of the given problem, that is, to extend the problem. A total of 33 student teachers tried to extend the problem by either changing the values of 2012, or increasing the number of variables beyond a, b, and c. However, none of them attempted to discuss how their solutions could be adapted to solve the more general problem.

Looking back and looking ahead

In summary, I have described my experience in attempting to introduce problem solving into the teaching of number theory. Together with the MProSE team, I tried to examine—based on the student teachers' assignments—the ability of the student teachers to solve a challenging problem and their use of Pólya's model at the end of the course. From their solutions, almost 70% of the student teachers managed to arrive at the correct answer of what I consider a difficult problem at the first year of the undergraduate level. More than half of these student teachers also demonstrated clear reasoning in their solutions of the problem. Their responses far exceeded my prior conceptions of how the problem would be tackled.

Returning to a reflective stance as a teacher of the second course and in light of my efforts to help the students develop a problem solving disposition, I am encouraged that these student teachers were not merely performing calculations and following procedures; they were largely applying suitable heuristics along the lines of Pólya's stages in making productive advances towards solutions to the problem. In contrast to the image that I described at the beginning of this paper—that of student teachers waiting for my answer when confronted with an unfamiliar problem—, the solutions provided by the student teachers in their Practical Worksheets portrayed a different picture—they made serious attempts at attacking the problem without my guidance.

I surmise that one main reason for this difference was assessment. The fact that this assessment contributed to their final grade, albeit a very small percentage, provided impetus for most of them to continue to engage the problem, in spite of its difficulty. This reaffirms one of the key parameters of the MProSE design—that the problem solving process must be assessed in order for the student teachers to treat it seriously. Another factor is the use of the Practical Worksheet—that it is organised along the lines of Pólya's stages and heuristics—provided a scaffold for the student teachers to follow this overarching model and process. Moreover, that the student teachers had the opportunity to observe my attempts at modelling the problem solving processes during my lectures could have helped in setting expectations for what they can and should do for their assignments.

There is, however, still room for improvement when it comes to Stage 4. Pólya's vision of the fourth stage goes beyond student teachers merely checking their solution steps or changing the number of variables in the problem. In extending the problem, student teachers should also consider how their original strategy can be modified to address the new problem. A thorough consideration should lead to a fuller understanding of the original problem. Upon further reflection, this vision of the fourth stage might not have been sufficiently communicated to the student teachers in my lectures. Due to the competing demands of delivering the course content while incorporating elements of problem solving, I had perhaps placed insufficient emphasis on extensions of problems. This will be an area for improvement in future runs of *Number Theory*.

On a more personal note, it was a worthwhile enterprise, with a substantial investment of time notwithstanding. It is satisfying—as any instructor of an undergraduate mathematics course would—to see the student teachers show signs of developing problem solving dispositions. This, however, should not be a one-off experience for them; rather, it would be a worthwhile goal to engage student teachers in problem solving throughout the course of their entire undergraduate experience. The next step in this direction is to work closely with my colleagues within the same department to achieve this end.

Acknowledgement

This research was partially supported by the National Institute of Education research grant OER 22/12 TTL.

References

Cockcroft, W. (1982). *Mathematics counts*. London: HMSO.

Dindyal, J., Tay, E.G., Toh, T.L., Leong, Y.H. & Quek, K.S. (2012). Mathematical problem solving for everyone: A new beginning. *The Mathematics Educator*, *13*(2), 1-20.

Leong, Y.H., Dindyal, J., Toh, T.L., Quek, K.S., Tay, E.G. & Lou, S.T. (2011). Teacher education for a problem-solving curriculum in Singapore. *ZDM: The International Journal on Mathematics Education, 43*(6-7), 819-831.

Leong, Y.H., Tay, E.G., Toh, T.L., Quek, K.S., & Dindyal, J. (2011). Reviving Pólya's "Look Back" in a Singapore school. *Journal of Mathematical Behavior*, *30*(3), 181-193.

Leong, Y.H., Toh, P.C., Dindyal, J., Ho, F.H. & Tay, E.G. (2012). Diffusion of the Mathematics Practical to mainstream secondary schools in Singapore, paper presented at the 35th annual conference of the Mathematics Education Research Group of Australasia.

Leong, Y.W. (2012). The practical worksheet: Scaffolding and assessing the problem solving process. *Unpublished Masters dissertation,* National Institute of Education, Singapore.

Lester, F.K. (1994). Musing about mathematical problem-solving research: 1970–1994. *Journal of Research in Mathematics Education*, 25, 660–676.

National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics for the 1980s*. Reston, VA: Author.

Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.

Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: Research and theory, practice and politics. *ZDM: The International Journal on Mathematics Education*, 39(5–6), 537–551

Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior*, *24*, 341-350.

Toh, P.C., Toh, T.L., Ho, F.H. & Quek, K.S. (2012). Use of Practical Worksheet in Teacher Education at the Undergraduate and Postgraduate Levels. In J. Dindyal, L.P. Cheng & S.F. Ng (Eds.), Mathematics education: *Expanding horizons,* (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia, eBook, pp. 736-743). Singapore:MERGA, Inc.

Toh, T.L., Quek, K.S., & Tay, E.G. (2008). Mathematical problem solving – a new paradigm. In J. Vincent, R. Pierce, & J. Dowsey (Eds.), *Connected Maths* (pp. 356- 365). Melbourne: Mathematical Association of Victoria.

Toh, T.L., Quek, K.S., Leong, Y.H., Dindyal, J. & Tay, E.G. (2011). *Making Mathematics Practical: An Approach to Problem Solving*. Singapore: World Scientific.

Woolnough, B., & Allsop, T. (1985). *Practical work in science*. Cambridge: Cambridge University Press.