ON CERTAIN PAIRS OF q-SERIES IDENTITIES

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Abstract. Hirschhorn recently proved a pair of q-series identities that interlinked the coefficients of two infinite products. We use the theory of modular forms to extend his results from $p = 5$ to other primes and provide other examples of infinite products sharing similar properties.

1. Introduction

In [2], Hirschhorn proved a pair of interesting q-series identities that interlinked the coefficients of these two infinite products

$$\sum_{n\geq 0} \tilde{a}(n)q^n = \frac{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^3}{(q;q)_\infty} \quad \text{and} \quad \sum_{n\geq 0} \tilde{b}(n)q^n = \frac{(q;q)_\infty^3 (q^2;q^2)_\infty^2}{(q^4;q^4)_\infty}$$

where we adopted the standard notation

$$(q;q)_\infty = \prod_{j=1}^{\infty}(1-q^j).$$

The pair of identities are:

$$\sum_{n\geq 0} \tilde{a}(5n)q^n + 5\sum_{n\geq 0} \tilde{a}(n)q^{5n+3} = \sum_{n\geq 0} \tilde{b}(n)q^n, \tag{1.1a}$$

$$\sum_{n\geq 0} \tilde{b}(5n+3)q^n + 5\sum_{n\geq 0} \tilde{b}(n)q^{5n} = 16\sum_{n\geq 0} \tilde{a}(n)q^n. \tag{1.1b}$$

As a corollary, the following arithmetic relations hold.

$$\tilde{a}(625n+390) - 6\tilde{a}(25n+15) + 25\tilde{a}(n) = 0, \tag{1.2a}$$

$$\tilde{b}(625n+78) - 6\tilde{b}(25n+3) + 25\tilde{b}(n) = 0. \tag{1.2b}$$

Hirschhorn asked whether this is an isolated phenomenon or whether there are more of such instances. We will show that (1.1) is a special case of a more general phenomenon by establishing analogous relations between the coefficients $\tilde{a}(n)$ and $\tilde{b}(n)$ for all primes $p \equiv 5 \pmod 8$. We will also introduce another two pairs of infinite products that satisfy relations analogous to (1.1) for infinitely many primes.

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2. Modular Forms

If we set $q = e^{2\pi i\tau}$, for some τ in the complex upper half plane, we can define the Dedekind eta-function as

$$\eta(\tau) = q^{1/24}(q;q)_\infty.$$

The pair of infinite products considered by Hirschhorn are essentially the following modular forms of weight 2 and level 256,

$$A(\tau) = \frac{\eta(16\tau)^2\eta(32\tau)^3}{\eta(8\tau)} = q^5 + q^{13} + q^{29} + \cdots,$$
$$B(\tau) = \frac{\eta(8\tau)^3\eta(16\tau)^2}{\eta(32\tau)} = q - 3q^9 - 2q^{17} + \cdots,$$

and Hirschhorn's identities (1.1) can be interpreted as actions of the Hecke operator [3, p. 161]. Recall that the action of the Hecke operator T_p on a level N modular form of weight k, say $\sum \alpha(n)q^n$, is defined by

$$T_p \sum_{n\geq 0} \alpha(n)q^n = \sum_{n\geq 0} \left(\alpha(pn) + p^{k-1}\alpha(n/p)\right) q^n,$$

for $p \nmid N$ and we set $\alpha(n/p) = 0$ if $p \nmid n$.

So (1.1) means that the operator T_5 maps $A(\tau)$ to a multiple of $B(\tau)$ and vice versa. Specifically

$$T_5 A(\tau) = B(\tau),$$
$$T_5 B(\tau) = 16A(\tau).$$

From this perspective, it is natural to consider the actions of T_p for other primes and the following generalization of (1.1) holds.

Theorem 2.1. *If*

$$A(\tau) = \sum_{n\geq 0} a(n)q^n = \frac{\eta(16\tau)^2\eta(32\tau)^3}{\eta(8\tau)} = q^5 + q^{13} + q^{29} + \cdots,$$
$$B(\tau) = \sum_{n\geq 0} b(n)q^n = \frac{\eta(8\tau)^3\eta(16\tau)^2}{\eta(32\tau)} = q - 3q^9 - 2q^{17} + \cdots,$$

then for odd primes p,

(2.1a) $$T_p A(\tau) = \begin{cases} b(p)A(\tau) & \text{if } p \equiv 1 \pmod 8 \\ a(p)B(\tau) & \text{if } p \equiv 5 \pmod 8 \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

(2.1b) $$T_p B(\tau) = \begin{cases} b(p)B(\tau) & \text{if } p \equiv 1 \pmod 8 \\ 16a(p)A(\tau) & \text{if } p \equiv 5 \pmod 8 \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

The following is a generalization of (1.2). For ease of comparison, we state the results in terms of $\tilde{a}(n)$ and $\tilde{b}(n)$.

Corollary 2.2. *Let $p \equiv 5 \pmod 8$ be a prime, then*

$$\tilde{a}\left(p^4 n + \frac{5}{8}(p^4-1)\right) + \left(2p - 16\tilde{a}\left(\frac{p-5}{8}\right)^2\right)\tilde{a}\left(p^2 n + \frac{5}{8}(p^2-1)\right) + p^2\tilde{a}(n) = 0,$$

$$\tilde{b}\left(p^4 n + \frac{1}{8}(p^4-1)\right) + \left(2p - 16\tilde{a}\left(\frac{p-5}{8}\right)^2\right)\tilde{b}\left(p^2 n + \frac{1}{8}(p^2-1)\right) + p^2\tilde{b}(n) = 0.$$

Besides the pair of infinite products considered by Hirschhorn, one can also investigate the space of modular forms of a different level to find other pairs that behave analogously. We illustrate with the following theorem.

Theorem 2.3. *If*

$$C(\tau) = \sum_{n\geq 0} c(n)q^n = \frac{\eta(8\tau)^3\eta(64\tau)^2}{\eta(32\tau)} = q^5 - 3q^{13} + 5q^{29} + \cdots,$$

$$D(\tau) = \sum_{n\geq 0} d(n)q^n = \frac{\eta(8\tau)^3\eta(32\tau)^5}{\eta(16\tau)^2\eta(64\tau)^2} = q - 3q^9 + 2q^{17} + \cdots,$$

then for odd primes p,

$$T_pC(\tau) = \begin{cases} d(p)C(\tau) & \text{if } p \equiv 1 \pmod 8 \\ c(p)D(\tau) & \text{if } p \equiv 5 \pmod 8 \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{2.3a}$$

$$T_pD(\tau) = \begin{cases} d(p)D(\tau) & \text{if } p \equiv 1 \pmod 8 \\ 4c(p)C(\tau) & \text{if } p \equiv 5 \pmod 8 \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.3b}$$

Since the proofs of Theorems 2.1 and 2.3 are nearly identical, we shall only prove the latter, which is slightly simpler. We require the following lemma.

Lemma 2.4. *If*

$$\sum_{n\geq 0}\alpha(n)q^n = \sum_{x\equiv 1 \pmod 4}\sum_{y} xq^{(mx)^2+y^2},$$

then for any prime $p \equiv 3 \pmod 4$ and $p \nmid m$, we have

$$\alpha(pn) = -p\alpha(n/p).$$

Proof. We note that the coefficient $\alpha(pn)$ is non-zero only when $(mx)^2 + y^2 = pn$ or

$$(mx)^2 + y^2 \equiv 0 \pmod p.$$

As $p \equiv 3 \pmod 4$ and $p \nmid m$, the only solutions for the congruence are $x \equiv y \equiv 0 \pmod p$. In other words,

$$x = x_1(-p),\ y = y_1 p \ \text{ and } n = n_1 p, \text{ for some integers } x_1, y_1, n_1.$$

In particular $x_1 \equiv x \equiv 1 \pmod 4$. Thus we have

$$(mx_1)^2 + y_1^2 = n_1$$

and

$$\alpha(pn) = x = -px_1 = -p\alpha(n_1) = -p\alpha(n/p). \qquad \square$$

Since we can use the Jacobi triple product identity to expand all four infinite products $A(\tau)$, $B(\tau)$, $C(\tau)$ and $D(\tau)$ into the form described in the previous lemma, the following is an immediate consequence.

Corollary 2.5. *If* $p \equiv 3 \pmod 4$*, then*

$$T_pA(\tau) = T_pB(\tau) = T_pC(\tau) = T_pD(\tau) = 0.$$

We are now ready to prove Theorem 2.3.

Proof. We begin with the following identity

$$\frac{\eta(8\tau)^3\eta(32\tau)^5}{\eta(16\tau)^2\eta(64\tau)^2} + 2\frac{\eta(8\tau)^3\eta(64\tau)^2}{\eta(32\tau)} = \frac{\eta(8\tau)^8}{\eta(4\tau)^2\eta(16\tau)^2}. \tag{2.4}$$

To prove the identity, replace q^4 with q, and simplify to

$$\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2} + 2q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}.$$

The Jacobi triple product identity can then be used to reduce this product identity to the following summation.

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{(2n)^2} + \sum_{n=-\infty}^{\infty} q^{(2n+1)^2}.$$

We next appeal to the fact the function on the right side of (2.4) is the unique newform of weight 2 and level 64 [1, p. 147]. (See also [4, p. 4853].) Thus if we define

$$E(\tau) = \sum_{n\geq 0} e(n)q^n = \frac{\eta(8\tau)^8}{\eta(4\tau)^2\eta(16\tau)^2},$$

we have

$$T_pE(\tau) = e(p)E(\tau). \tag{2.5}$$

From (2.4), we know that $e(n) = d(n) + 2c(n)$. Furthermore, from their definitions, $d(n) = 0$ unless $n \equiv 1 \pmod 8$ and likewise, $c(n) = 0$ unless $n \equiv 5 \pmod 8$. These combined with (2.5) allows us to deduce (2.3).

For primes $p \equiv 1 \pmod 8$, we have $e(p) = d(p)$. For $n \equiv 1 \pmod 8$, the coefficients of n in (2.5) satisfy

$$d(pn) + pd(n/p) + 2c(pn) + 2pc(n/p) = d(p)\big(d(n) + 2c(n)\big).$$

Since $np \equiv 1 \pmod 8$ and $n/p \equiv 1 \pmod 8$ whenever $p \mid n$, we have

$$c(n) = c(pn) = c(n/p) = 0.$$

The coefficients of $n \equiv 1 \pmod 8$ in (2.5) reduces to

$$d(pn) + pd(n/p) = d(p)d(n),$$

which is equivalent to

$$T_pD(\tau) = d(p)D(\tau).$$

On the other hand for primes $p \equiv 5 \pmod 8$, we have $e(p) = 2c(p)$. For $n \equiv 5 \pmod 8$, we have $np \equiv 1 \pmod 8$ and $n/p \equiv 1 \pmod 8$ whenever $p \mid n$. In this case,

$$d(n) = c(np) = c(n/p) = 0,$$

and the coefficients of $n \equiv 5 \pmod 8$ in (2.5) satisfy

$$d(pn) + pd(n/p) = 2c(p)\,(2c(n)) = 4c(p)c(n).$$

So

$$T_p D(\tau) = 4c(p)C(\tau).$$

The computations for $T_p C(\tau)$ follow in the same manner. □

The proof of Theorem 2.1 is virtually identical and is based on the fact that $4A(\tau) + B(\tau)$ is a newform of weight 2 and level 256. Although, in this case, $4A(\tau) + B(\tau)$ does not appear to be expressible as an infinite product.

In both cases, the pair of eta-quotients generate a two dimensional subspace invariant under the action of Hecke operators. Using this fact and a table of modular forms (for example [5] or the software SAGE [6]), one can construct possibly many other similar examples. The eta-quotients $A(\tau)$ and $B(\tau)$ discussed by Hirschhorn, and our examples $C(\tau)$ and $D(\tau)$ can all be represented as theta series associated with binary quadratic forms. One can also use the same approach to construct examples of weight 2 modular forms that can be represented by theta series associated with quaternary quadratic forms. We illustrate with an example from level 48.

Theorem 2.6. *If*

$$F(\tau) = \sum_{n\geq 0} f(n)q^n = \frac{\eta(4\tau)^2\eta(16\tau)^2\eta(24\tau)^4}{\eta(8\tau)^2\eta(48\tau)^2} = q - 2q^5 + q^9 + \cdots,$$

$$G(\tau) = \sum_{n\geq 0} g(n)q^n = \frac{\eta(8\tau)^4\eta(12\tau)^2\eta(48\tau)^2}{\eta(16\tau)^2\eta(24\tau)^2} = q^3 - 4q^{11} - 2q^{15} + \cdots,$$

then we have

$$F(\tau) + G(\tau) = \frac{\eta(4\tau)^4\eta(12\tau)^4}{\eta(2\tau)\eta(6\tau)\eta(8\tau)\eta(24\tau)}.$$

Moreover, for primes $p \nmid 6$,

$$T_pF(\tau) = \begin{cases} f(p)F(\tau) & \text{if } p \equiv 1 \pmod 4 \\ g(p)G(\tau) & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{2.6a}$$

$$T_pG(\tau) = \begin{cases} f(p)G(\tau) & \text{if } p \equiv 1 \pmod 4 \\ g(p)F(\tau) & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.6b}$$

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