ON REPRESENTATIONS BY FIGURATE NUMBERS: A UNIFORM APPROACH TO THE CONJECTURES OF MELHAM.

PEE CHOON TOH

Abstract. Using known formulas for $R_{(a,b)}(n)$, the number of representations of n as a times a square plus b times a square, we prove 21 conjectured formulas of Melham on the number of representations of n as sums of triangular, pentagonal and heptagonal numbers. We also demonstrate how our technique can be used to prove the other 277 conjectured formulas of Melham concerning representations by other figurate numbers.

1. Introduction and Statement of Results

Jacobi's two square theorem [12] states that the number of representations of a positive integer n as a sum of two squares, counting order and sign, is four times the difference of the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4. In terms of Lambert series and the Legendre-Jacobi symbol [11], this result can be represented as

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+y^2} = 1 + 4\sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) \frac{q^n}{1-q^n}. \tag{1.1}$$

The representations of a number n by a quadratic form is a subject that has intrigued number theorists throughout history. A discussion of (1.1) and its various extensions, including extensive notes on recent progress in the subject, can be found in Berndt's book [5, Chpt. 3]. Some recent work on representations by binary quadratic forms include [3, 4, 6, 8, 9, 13, 16, 18, 19, 20, 21].

In [15], Melham published 21 conjectured analogues of (1.1) where the squares are replaced by triangular, pentagonal or heptagonal numbers. These 21 formulas were extracted from an unpublished manuscript [14] which contained a total of 298 conjectured formulas. An example of his conjectures involving triangular numbers [15, eq.(9)] is the following.

Date: May 19, 2017.

2010 *Mathematics Subject Classification.* Primary—11E16 Secondary—11E25.

Key words and phrases. binary quadratic forms, figurate numbers, sum of squares, Lambert series.

Conjecture 1.1.

$$\sum_{x,y=0}^{\infty} q^{\left(\frac{x^2+x}{2}\right)+10\left(\frac{y^2+y}{2}\right)} = \sum_{n=0}^{\infty}\left(\frac{q^{11n}+q^{19n+1}}{1-q^{40n+5}} - \frac{q^{17n+5}+q^{33n+11}}{1-q^{40n+15}} + \frac{q^{7n+3}+q^{23n+13}}{1-q^{40n+25}} - \frac{q^{21n+17}+q^{29n+24}}{1-q^{40n+35}}\right).$$

Another two conjectures [14, eq.(213) and eq.(226)] involving pentagonal and nonagonal numbers are given below.

Conjecture 1.2.

$$\begin{aligned}\sum_{x,y=-\infty}^{\infty} & q^{\left(\frac{3x^2-x}{2}\right)+\left(\frac{7y^2-5y}{2}\right)} \\ = \sum_{n=0}^{\infty} & \left(\frac{q^{11n+2}}{1-q^{84n+19}} + \frac{q^{5n+1}}{1-q^{84n+25}} - \frac{q^{13n+4}}{1-q^{84n+29}} + \frac{q^{23n+8}}{1-q^{84n+31}}\right. \\ & + \frac{q^{17n+7}}{1-q^{84n+37}} + \frac{q^{n}}{1-q^{84n+41}} - \frac{q^{53n+38}}{1-q^{84n+61}} - \frac{q^{47n+37}}{1-q^{84n+67}} \\ & \left. + \frac{q^{55n+46}}{1-q^{84n+71}} - \frac{q^{65n+56}}{1-q^{84n+73}} - \frac{q^{59n+55}}{1-q^{84n+79}} - \frac{q^{43n+42}}{1-q^{84n+83}}\right).\end{aligned}$$

Conjecture 1.3.

$$\begin{aligned}\sum_{x,y=-\infty}^{\infty} & q^{\left(\frac{3x^2-x}{2}\right)+22\left(\frac{7y^2-5y}{2}\right)} \\ = \sum_{n=0}^{\infty} & \left(\frac{q^{227n+5}+q^{395n+16}+q^{563n+27}+q^{899n+49}+q^{1403n+82}}{1-q^{1848n+121}}\right. \\ & \left. - \frac{q^{29n-9}+q^{197n-4}+q^{365n+1}+q^{701n+11}+q^{1205n+26}}{1-q^{1848n+55}} + \cdots\right).\end{aligned}$$

Conjecture 1.3, when written out in full, contains another 46 terms of the form

$$\frac{q^{f_1(n)}+q^{f_2(n)}+q^{f_3(n)}+q^{f_4(n)}+q^{f_5(n)}}{1-q^{1848n+M}},$$

and takes up two whole pages in Melham's manuscript.

It is our aim in this article to present a uniform approach in proving the conjectures in [14]. The explicit Lambert series representations used in the above three examples can be cumbersome and so we require more succinct representations. For positive integers a, b and nonnegative n, define

$$R_{(a,b)}(n) = \left|\{(x,y) \in \mathbb{Z}\times\mathbb{Z} : n = ax^2+by^2\}\right|, \tag{1.2}$$

$$T_{(a,b)}(n) = \left|\left\{(x,y) \in \mathbb{N}\times\mathbb{N} : n = a\left(\frac{x^2+x}{2}\right)+b\left(\frac{y^2+y}{2}\right)\right\}\right|, \tag{1.3}$$

$$PN_{(a,b)}(n) = \left|\left\{(x,y) \in \mathbb{Z}\times\mathbb{Z} : n = a\left(\frac{3x^2-x}{2}\right)+b\left(\frac{7y^2-5y}{2}\right)\right\}\right|. \tag{1.4}$$

In other words,

$$\sum_{x,y=-\infty}^{\infty} q^{ax^2+by^2} = 1+\sum_{n=1}^{\infty} R_{(a,b)}(n)q^n$$

and so (1.1) can be restated as

$$R_{(1,1)}(n) = 4\sum_{d|n}\left(\frac{-4}{d}\right), \text{ for } n \geq 1.$$

Similarly, Conjectures 1.1 to 1.3 are equivalent to the following theorem.

Theorem 1.4.

$$T_{(1,10)}(n) = \frac{1}{2}\sum_{d|8n+11}\left(\frac{-40}{d}\right)\left(1-\left(\frac{10}{d}\right)\right), \tag{1.5}$$

$$PN_{(1,1)}(n) = \frac{1}{2}\sum_{d|168n+82}\left(\frac{-84}{d}\right)\left(1-\left(\frac{-7}{d}\right)\right), \tag{1.6}$$

$$PN_{(1,22)}(n) = \frac{1}{2}\sum_{d|168n+1657}\left(\frac{-1848}{d}\right)\left(1-\left(\frac{-11}{d}\right)\right). \tag{1.7}$$

We will prove (1.5) and the other 20 conjectures from [15] in Section 2. Section 3 will be devoted to our general approach to the remaining 277 conjectured formulas in [14]. The proofs of (1.6) and (1.7) will be presented as prototypical examples to illustrate our method.

We end this introduction by illustrating the equivalence of (1.5) and Conjecture 1.1. Now

$$\sum_{n\geq 0} T_{(1,10)}(n)q^{8n+11} = \frac{1}{2}\sum_{n\geq 0}\sum_{d|8n+11}\left(\frac{-40}{d}\right)\left(1-\left(\frac{10}{d}\right)\right)q^{8n+11}.$$

Since

$$\left(\frac{-40}{d}\right)\left(1-\left(\frac{10}{d}\right)\right) = \begin{cases} 2, & \text{if } d \equiv 7,11,19,23 \pmod{40}; \\ -2, & \text{if } d \equiv 17,21,29,33 \pmod{40}; \\ 0, & \text{otherwise,} \end{cases}$$

we can write this sum as

$$\sum_{k,m\geq 0}\Big(q^{(40k+11)(8m+1)}+q^{(40k+19)(8m+1)}-q^{(40k+17)(8m+3)}-q^{(40k+33)(8m+3)}$$

$$+q^{(40k+7)(8m+5)}+q^{(40k+23)(8m+5)}-q^{(40k+21)(8m+7)}-q^{(40k+29)(8m+7)}\Big)$$

$$=q^{11}\sum_{m\geq 0}\left(\frac{q^{8(11m)}+q^{8(19m+1)}}{1-q^{8(40m+5)}}-\frac{q^{8(17m+5)}+q^{8(33m+11)}}{1-q^{8(40m+15)}}\right.$$

$$\left.+\frac{q^{8(7m+3)}+q^{8(23m+13)}}{1-q^{8(40m+25)}}-\frac{q^{8(21m+17)}+q^{8(29m+24)}}{1-q^{8(40m+35)}}\right).$$

Replacing q^{8n+11} by q^n gives us Conjecture 1.1. We remark that a slightly different formula for $T_{(1,10)}(n)$ containing 16 instead of the above 8 terms has been given by Sun [19].

2. The First 21 Conjectures

Melham's 11 conjectures for $T_{(a,b)}(n)$ [15, eq.(6) to (16)] can be summarized as follows.

Theorem 2.1. *Let $p = 5, 13$ or 37, then*

$$T_{(1,p)}(n)=\frac{1}{2}\sum_{d|4n+\frac{p+1}{2}}\left(\frac{-4p}{d}\right) \tag{2.1a}$$

$$=\frac{1}{2}\sum_{d|4n+\frac{p+1}{2}}\left(\frac{-4p}{d}\right)\left(1-\left(\frac{p}{d}\right)\right). \tag{2.1b}$$

Theorem 2.2. *Let $(p, m) = (3, 1), (5, 2), (11, 1)$ or $(29, 2)$, then*

$$T_{(1,2p)}(n)=\frac{1}{2}\sum_{d|8n+2p+1}\left(\frac{-8p}{d}\right) \tag{2.2a}$$

$$=\frac{1}{2}\sum_{d|8n+2p+1}\left(\frac{-8p}{d}\right)\left(1-\left(\frac{mp}{d}\right)\right) \tag{2.2b}$$

and

$$T_{(2,p)}(n)=\frac{1}{2}\sum_{d|8n+p+2}\left(\frac{-8p}{d}\right) \tag{2.3a}$$

$$=\frac{1}{2}\sum_{d|8n+p+2}\left(\frac{-8p}{d}\right)\left(1-\left(\frac{mp}{d}\right)\right). \tag{2.3b}$$

The first expression in each case, i.e. (2.1a), (2.2a) and (2.3a), can be found in [18, 19]. The second expression allows us to recover the explicit representations given by Melham. Note that the $p = 5$ case of (2.2b) is precisely the first statement in Theorem 1.4. We also remark that Hirschhorn [10] has independently proved the conjectures for $T_{(1,5)}(n)$ and $T_{(1,6)}(n)$.

For positive integers a, b and nonnegative n, we further define

$$P_{(a,b)}(n) = \left|\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : n = a\left(\frac{3x^2 - x}{2}\right) + b\left(\frac{3y^2 - y}{2}\right)\right\}\right|. \tag{2.4}$$

Melham's 9 conjectures for $P_{(a,b)}(n)$ [15, eq.(17) to (25)] are restated below.

Theorem 2.3.

$$P_{(1,5)}(n) = \sum_{d|4n+1} \left(\frac{-180}{d}\right). \tag{2.5}$$

Theorem 2.4. *Let $p = 13$ or 37, then*

$$P_{(1,p)}(n) = \frac{1}{2} \sum_{d|12n+\frac{p+1}{2}} \left(\frac{-4p}{d}\right) \tag{2.6a}$$

$$= \frac{1}{2} \sum_{d|12n+\frac{p+1}{2}} \left(\frac{-36p}{d}\right)\left(1 - \left(\frac{9p}{d}\right)\right). \tag{2.6b}$$

Theorem 2.5. *Let $(p,m) = (5,2), (11,1)$ or $(29,2)$, then*

$$P_{(1,2p)}(n) = \frac{1}{2} \sum_{d|24n+2p+1} \left(\frac{-8p}{d}\right) \tag{2.7a}$$

$$= \frac{1}{2} \sum_{d|24n+2p+1} \left(\frac{-72p}{d}\right)\left(1 - \left(\frac{9mp}{d}\right)\right) \tag{2.7b}$$

and

$$P_{(2,p)}(n) = \frac{1}{2} \sum_{d|24n+p+2} \left(\frac{-8p}{d}\right) \tag{2.8a}$$

$$= \frac{1}{2} \sum_{d|24n+p+2} \left(\frac{-72p}{d}\right)\left(1 - \left(\frac{9mp}{d}\right)\right). \tag{2.8b}$$

Finally, if we let

$$H_{(a,b)}(n) = \left|\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : n = a\left(\frac{5x^2 - 3x}{2}\right) + b\left(\frac{5y^2 - 3y}{2}\right)\right\}\right|, \tag{2.9}$$

we can restate [15, eq.(26)] as the next theorem.

Theorem 2.6.

$$H_{(1,6)}(n) = \frac{1}{2}\sum_{d|40n+63}\left(\frac{-24}{d}\right) \tag{2.10a}$$

$$= \frac{1}{2}\sum_{d|40n+63}\left(\frac{-600}{d}\right)\left(1-\left(\frac{75}{d}\right)\right). \tag{2.10b}$$

With the exception of $P_{(1,5)}(n)$ which has been proved independently by Hirschhorn [10] and Sun [17], we are not aware of any other existing proofs of Theorems 2.4 to 2.6.

In [21], we used the theory of genus characters to derive Lambert series representations for theta series associated to binary quadratic forms. For imaginary quadratic fields with class number 2 [21, p. 232], one may deduce $R_{(a,b)}(n)$ for 11 pairs of (a,b). We record the exact statement as the following theorem.

Theorem 2.7. *If $p = 5, 13$ or 37, then*

$$R_{(1,p)}(n) = \sum_{d|n}\left(\frac{-4p}{d}\right) + \sum_{d|n}\left(\frac{p}{d}\right)\left(\frac{-4}{n/d}\right). \tag{2.11}$$

If $p = 3, 5, 11$ or 29, then

$$R_{(1,2p)}(n) = \sum_{d|n}\left(\frac{-8p}{d}\right) + \sum_{d|n}\left(\frac{d}{p}\right)\left(\frac{-\left(\frac{-1}{p}\right)2}{n/d}\right) \tag{2.12}$$

and

$$R_{(2,p)}(n) = \sum_{d|n}\left(\frac{-8p}{d}\right) - \sum_{d|n}\left(\frac{d}{p}\right)\left(\frac{-\left(\frac{-1}{p}\right)2}{n/d}\right). \tag{2.13}$$

It is no coincidence that these 11 formulas for $R_{(a,b)}(n)$ correspond to those conjectured by Melham for $T_{(a,b)}(n)$. We shall demonstrate how to prove Theorems 2.1 to 2.6 with Theorem 2.7. We remark that formulas equivalent to Theorem 2.7 can also be found in the works of Hall [7], as well as Sun and Williams [16].

We first state two lemmas involving character sums.

Lemma 2.8. *The following identities hold for all nonnegative integers n.*

$$\sum_{d|4n+3}\left(\frac{-1}{d}\right) = 0, \tag{2.14}$$

$$\sum_{d|8n+5}\left(\frac{2}{d}\right) = \sum_{d|8n+3}\left(\frac{2}{d}\right) = 0, \tag{2.15}$$

$$\sum_{d|8n+5}\left(\frac{-2}{d}\right) = \sum_{d|8n+7}\left(\frac{-2}{d}\right) = 0. \tag{2.16}$$

Proof. In each case, let χ represent the character being summed. The set of divisors d can be partitioned into two equinumerous sets, according to the value of $\chi(d)$. $\square$

Lemma 2.9. *Let α be a positive integer and p a prime such that* $\gcd(p,m) = 1$ *and* $\gcd(p,n) = 1$. *We have*

$$\sum_{d|p^\alpha n} \left(\frac{mp^2}{d}\right) = \sum_{d|n} \left(\frac{mp^2}{d}\right) = \sum_{d|n} \left(\frac{m}{d}\right).$$

Proof.

$$\begin{aligned}\sum_{d|p^\alpha n} \left(\frac{mp^2}{d}\right) &= \sum_{\substack{d|p^\alpha n\\ p|d}} \left(\frac{mp^2}{d}\right) + \sum_{\substack{d|p^\alpha n\\ p\nmid d}} \left(\frac{mp^2}{d}\right)\\ &= 0 + \sum_{d|n} \left(\frac{m}{d}\right)\left(\frac{p}{d}\right)^2 = \sum_{d|n} \left(\frac{m}{d}\right).\end{aligned}$$ $\square$

Now by considering odd and even values of x and y, we can write

$$\begin{aligned}\sum_{x,y=-\infty}^{\infty} q^{ax^2+by^2} = &\sum_{x,y=-\infty}^{\infty} q^{4ax^2+4by^2} + \sum_{x,y=-\infty}^{\infty} q^{4ax^2+8b(\frac{y^2-y}{2})+b} \\ &+ \sum_{x,y=-\infty}^{\infty} q^{8a(\frac{x^2-x}{2})+4by^2+a} + \sum_{x,y=-\infty}^{\infty} q^{8a(\frac{x^2-x}{2})+8b(\frac{y^2-y}{2})+a+b}.\end{aligned} \tag{2.17}$$

Thus for certain values of a and b, we are able to sieve out exactly those terms that appear in the fourth sum.

Lemma 2.10. *Let a and b be positive integers. If*
i) a, b are both odd and $a + b \equiv 2 \pmod 4$ or
ii) a is odd, $b \equiv 2 \pmod 4$, then

$$R_{(a,b)}(8n + a + b) = 4T_{(a,b)}(n).$$

Proof. In case i) only the first and fourth sum of (2.17) have even exponents. However, the exponents of the first sum are divisible by 4. For case ii) only the third and fourth sum have odd exponents. But in this case, the exponents in the third sum are congruent to $a \pmod 8$. $\square$

We remark that a general discussion on the relation between sums of squares and sums of triangular numbers can be found in [2] and [1]. We are now ready to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $p = 5, 13$ or 37 and note that $\left(\frac{p}{2}\right) = \left(\frac{-4}{(p+1)/2}\right) = -1$. Lemma 2.10, together with Theorem 2.7, gives us

$$\begin{aligned}
4T_{(1,p)}(n) &= R_{(1,p)}(8n+p+1) \\
&= \sum_{d|8n+p+1} \left(\frac{-4p}{d}\right) + \sum_{d|8n+p+1} \left(\frac{p}{d}\right) \left(\frac{-4}{\frac{8n+p+1}{d}}\right).
\end{aligned}$$

Note that $8n+p+1$ is even and $4n+\frac{p+1}{2}$ is odd. If d is odd, $(8n+p+1)/d$ would be even. Hence we can simplify the sum to

$$\begin{aligned}
&\sum_{\substack{d|8n+p+1 \\ d \text{ odd}}} \left(\frac{-4p}{d}\right) + \sum_{\substack{d|8n+p+1 \\ d=2d_1}} \left(\frac{p}{d}\right) \left(\frac{-4}{\frac{8n+p+1}{d}}\right) \\
&= \sum_{d|4n+\frac{p+1}{2}} \left(\frac{-4p}{d}\right) + \sum_{d_1|4n+\frac{p+1}{2}} \left(\frac{p}{2}\right) \left(\frac{p}{d_1}\right) \left(\frac{-4}{d_1}\right)^2 \left(\frac{-4}{\frac{4n+\frac{p+1}{2}}{d_1}}\right) \\
&= \sum_{d|4n+\frac{p+1}{2}} \left(\frac{-4p}{d}\right) - \sum_{d_1|4n+\frac{p+1}{2}} \left(\frac{-4p}{d_1}\right) \left(\frac{-4}{4n+\frac{p+1}{2}}\right) \\
&= 2 \sum_{d|4n+\frac{p+1}{2}} \left(\frac{-4p}{d}\right).
\end{aligned}$$

Dividing throughout by 4 proves (2.1a). For the second expression, we set $4n+\frac{p+1}{2} = p^\alpha m$ where $\gcd(p,m)=1$. Since $p \equiv 1 \pmod 4$, we have $m \equiv 3 \pmod 4$ and

$$\sum_{d|4n+\frac{p+1}{2}} \left(\frac{-4p^2}{d}\right) = \sum_{d|m} \left(\frac{-1}{d}\right) = 0$$

by Lemmas 2.8 and 2.9. Hence we can include the above into our expression for $T_{(1,p)}(n)$ to obtain (2.1b). $\square$

Proof of Theorem 2.2. Let $p = 3$ or 11. Lemma 2.10 and Theorem 2.7 give us

$$\begin{aligned}
4T_{(1,2p)}(n) &= R_{(1,2p)}(8n+2p+1) \\
&= \sum_{d|8n+2p+1} \left(\frac{-8p}{d}\right) + \sum_{d|8n+2p+1} \left(\frac{d}{p}\right)\left(\frac{2}{\frac{8n+2p+1}{d}}\right) \\
&= \sum_{d|8n+2p+1} \left(\frac{-8p}{d}\right) + \sum_{d|8n+2p+1} \left(\frac{-p}{d}\right)\left(\frac{2}{d}\right)^2\left(\frac{2}{\frac{8n+2p+1}{d}}\right) \\
&= \sum_{d|8n+2p+1} \left(\frac{-8p}{d}\right) + \sum_{d|8n+2p+1} \left(\frac{-2p}{d}\right)\left(\frac{2}{8n+2p+1}\right) \\
&= \sum_{d|8n+2p+1} \left(\frac{-8p}{d}\right)\left(1 + \left(\frac{2}{8n+2p+1}\right)\right) \\
&= 2\sum_{d|8n+2p+1} \left(\frac{-8p}{d}\right).
\end{aligned}$$

In the third equality we used quadratic reciprocity plus the fact that d is always odd. And in the penultimate equality, we used the fact that $8n + 2p + 1 \equiv 7 \pmod 8$. This proves (2.2a). For (2.2b) we note that if $p \equiv 3 \pmod 8$, then $(8n + 2p + 1)/p^\alpha \equiv 5$ or $7 \pmod 8$. Hence Lemmas 2.8 and 2.9 combine to show that

$$\sum_{d|8n+2p+1} \left(\frac{-8p^2}{d}\right) = 0.$$

Likewise (2.3a) holds since

$$\begin{aligned}
4T_{(2,p)}(n) &= R_{(2,p)}(8n+p+2) \\
&= \sum_{d|8n+p+2} \left(\frac{-8p}{d}\right) - \sum_{d|8n+p+2} \left(\frac{d}{p}\right)\left(\frac{2}{d}\right)^2\left(\frac{2}{\frac{8n+p+2}{d}}\right) \\
&= \sum_{d|8n+p+2} \left(\frac{-8p}{d}\right) - \sum_{d|8n+p+2} \left(\frac{-2p}{d}\right)\left(\frac{2}{8n+p+2}\right) \\
&= \sum_{d|8n+p+2} \left(\frac{-8p}{d}\right)\left(1 - \left(\frac{2}{8n+p+2}\right)\right) \\
&= 2\sum_{d|8n+p+2} \left(\frac{-8p}{d}\right)
\end{aligned}$$

and (2.3b) follows from

$$\sum_{d|8n+p+2} \left(\frac{-8p^2}{d}\right) = 0.$$

The cases for $p = 5$ and 29 follow in the same manner. $\square$

For Theorems 2.3 to 2.5, we first observe by considering residues modulo 6 that

$$\sum_{x=-\infty}^{\infty} q^{x^2} = \sum_{x=-\infty}^{\infty} q^{36x^2} + \sum_{x=-\infty}^{\infty} q^{72(\frac{x^2-x}{2})+9} + 2\sum_{x=-\infty}^{\infty} q^{24(\frac{3x^2-x}{2})+1} + 2\sum_{x=-\infty}^{\infty} q^{12(3x^2-2x)+4}. \tag{2.18}$$

For certain values of (a, b), extracting the terms with exponents congruent to $a+b \pmod{24}$ yields only those involving pentagonal numbers, giving us following relation

$$R_{(a,b)}(24n + a + b) = 4P_{(a,b)}(n). \tag{2.19}$$

Proof of Theorems 2.4 and 2.5. The relation (2.19) holds for $(a, b) = (1, 13)$, $(1, 37)$, $(1, 10)$, $(2, 5)$, $(1, 22)$, $(2, 11)$, $(1, 58)$ and $(2, 29)$. The proof for each case mirrors that of Theorems 2.1 and 2.2. To obtain (2.6b), (2.7b) and (2.8b), we again use Lemmas 2.8 and 2.9. Since $\gcd(d, 3) = 1$ in all cases, we multiplied the character $\left(\frac{9}{d}\right)$ to each of the sum without changing its value. $\square$

To prove Theorem 2.3, we first establish the following lemma.

Lemma 2.11.

$$\sum_{n\geq 0}\sum_{d|12n+3}\left(\frac{-20}{d}\right)q^n = \sum_{n\geq 0}\sum_{d|4n+3}\left(\frac{-20}{d}\right)q^{3n+2} + 2\sum_{n\geq 0}\sum_{d|4n+1}\left(\frac{-180}{d}\right)q^n.$$

Proof. Rewrite the left hand side of the identity as

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{\substack{d|12n+3\\ d=3d'}}\left(\frac{-20}{d}\right)q^n + \sum_{n\geq 0}\sum_{\substack{d|12n+3\\ 3\nmid d}}\left(\frac{-20}{d}\right)q^n \\
&\quad = \sum_{n\geq 0}\sum_{d'|4n+1}\left(\frac{-20}{d'}\right)\left(\frac{-20}{3}\right)q^n + \sum_{n\geq 0}\sum_{d|4n+1}\left(\frac{-20}{d}\right)\left(\frac{3}{d}\right)^2 q^n \\
&\quad = \sum_{n\geq 0}\sum_{\substack{d'|4n+1\\ 3|d'}}\left(\frac{-20}{d'}\right)q^n + \sum_{n\geq 0}\sum_{\substack{d'|4n+1\\ 3\nmid d'}}\left(\frac{-20}{d'}\right)q^n + \sum_{n\geq 0}\sum_{d|4n+1}\left(\frac{-180}{d}\right)q^n \\
&\quad = \sum_{n\geq 0}\sum_{3d''|4n+1}\left(\frac{-20}{d''}\right)q^n + 2\sum_{n\geq 0}\sum_{d|4n+1}\left(\frac{-180}{d}\right)q^n.
\end{aligned}$$

If $3d'' \mid 4n+1$ we have $n+1 \equiv 0 \pmod 3$ so we may write n as $3m+2$. The sum becomes

$$\sum_{m\geq 0}\sum_{3d''|12m+9}\left(\frac{-20}{d''}\right)q^{3m+2}+2\sum_{n\geq 0}\sum_{d|4n+1}\left(\frac{-180}{d}\right)q^{n}$$

which completes the proof. □

Proof of Theorem 2.3. From (2.18), we have

$$\sum_{x,y=-\infty}^{\infty}q^{x^2+5y^2}=4\sum_{x,y=0}^{\infty}q^{72\left((\frac{x^2+x}{2})+5(\frac{y^2+y}{2})\right)+54}$$
$$+4\sum_{x,y=-\infty}^{\infty}q^{24\left((\frac{3x^2-x}{2})+5(\frac{3y^2-y}{2})\right)+6}+\cdots,$$

where we only listed those terms with exponents congruent to 6 (mod 24). This yields the relation

$$R_{(1,5)}(24n+6)=4T_{(3,15)}(n-2)+4P_{(1,5)}(n). \tag{2.20}$$

Using Theorems 2.7 and 2.1, this becomes

$$2\sum_{n\geq 0}\sum_{d|12n+3}\left(\frac{-20}{d}\right)q^{n}=2\sum_{n\geq 0}\sum_{d|4n+3}\left(\frac{-20}{d}\right)q^{3n+2}+4P_{(1,5)}(n).$$

Comparing this with Lemma 2.11 results in the expression for $P_{(1,5)}(n)$. □

Proof of Theorem 2.6. By considering x and y modulo 10, we have

$$\sum_{x,y=-\infty}^{\infty}q^{x^2+6y^2}=4\sum_{x,y=-\infty}^{\infty}q^{40\left((\frac{5x^2-3x}{2})+6(\frac{5y^2-3y}{2})\right)+63}+\cdots,$$

where we only listed those terms with exponents congruent to 23 (mod 40). Hence we have the relation

$$R_{(1,6)}(40n+63)=4H_{(1,6)}(n). \tag{2.21}$$

The proof then follows along the lines of Theorem 2.2, with the additional step of introducing the character $\left(\frac{25}{d}\right)$ for (2.10b). □

Since representations by Lambert series are non-unique, extra calculations may be necessary for some of the conjectures listed in [15]. We illustrate with the following example.

$$\sum_{n\geq 0}T_{(1,22)}(n)q^{8n+23}=\frac{1}{2}\sum_{n\geq 0}\sum_{d|8n+23}\left(\frac{-88}{d}\right)\left(1-\left(\frac{11}{d}\right)\right)q^{8n+23}.$$

We note that the character $\chi(d) = \left(\frac{-88}{d}\right)\left(1 - \left(\frac{11}{d}\right)\right)$ is nonzero only for 20 values modulo 88, namely

$$\begin{aligned}
L_1 &= \{15, 23, 31, 47, 71\},\\
L_2 &= \{13, 21, 29, 61, 85\},\\
L_3 &= \{3, 27, 59, 67, 75\},\\
L_4 &= \{17, 41, 57, 65, 73\},
\end{aligned}$$

with $\chi(d) = 2$ if $d \in L_1 \cup L_2$ and $\chi(d) = -2$ if $d \in L_3 \cup L_4$. So

$$\begin{aligned}
&\sum_{n\geq 0} T_{(1,22)}(n) q^{8n+23} \qquad (2.22)\\
&\quad = \sum_{k,m\geq 0}\left(\sum_{d\in L_1} q^{(88k+d)(8m+1)} + \sum_{d\in L_2} q^{(88k+d)(8m+3)}\right.\\
&\qquad\qquad \left. - \sum_{d\in L_3} q^{(88k+d)(8m+5)} - \sum_{d\in L_4} q^{(88k+d)(8m+7)}\right).
\end{aligned}$$

To obtain the representation given by Melham [15, eq.(12)], we further define

$$\begin{aligned}
N_2 &= \{5, 37, 45, 53, 69, 77\},\\
N_3 &= \{11, 19, 35, 43, 51, 83\}.
\end{aligned}$$

Now all integers of the form $8m+5$ can be written as $88k+d$ with $d \in L_2 \cup N_2$ and integers of the form $8m+3$ can be written as $88k+d$ with $d \in L_3 \cup N_3$. So the second and third sum in (2.22) can be written as

$$\begin{aligned}
&\sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_2\\ k\equiv 3 \ (\mathrm{mod}\ 8)}} q^{mk} - \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_3\\ k\equiv 5 \ (\mathrm{mod}\ 8)}} q^{mk}\\
&\quad = \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_2\\ k\equiv e \ (\mathrm{mod}\ 88), e\in L_3\cup N_3}} q^{mk} - \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_3\\ k\equiv e \ (\mathrm{mod}\ 88), e\in L_2\cup N_2}} q^{mk}\\
&\quad = \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_2\\ k\equiv e \ (\mathrm{mod}\ 88), e\in N_3}} q^{mk} - \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_3\\ k\equiv e \ (\mathrm{mod}\ 88), e\in N_2}} q^{mk}\\
&\quad = \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_2\cup N_2\\ k\equiv e \ (\mathrm{mod}\ 88), e\in N_3}} q^{mk} - \sum_{\substack{m\equiv d \ (\mathrm{mod}\ 88), d\in L_3\cup N_3\\ k\equiv e \ (\mathrm{mod}\ 88), e\in N_2}} q^{mk}\\
&\quad = \sum_{\substack{m\equiv 5 \ (\mathrm{mod}\ 8),\\ k\equiv e \ (\mathrm{mod}\ 88), e\in N_3}} q^{mk} - \sum_{\substack{m\equiv 3 \ (\mathrm{mod}\ 8),\\ k\equiv e \ (\mathrm{mod}\ 88), e\in N_2}} q^{mk}.
\end{aligned}$$

Thus

$$\sum_{n\geq 0} T_{(1,22)}(n)q^{8n+23} = \sum_{k,m\geq 0}\left(\sum_{d\in L_1} q^{(88k+d)(8m+1)} - \sum_{d\in N_2} q^{(88k+d)(8m+3)} + \sum_{d\in N_3} q^{(88k+d)(8m+5)} - \sum_{d\in L_4} q^{(88k+d)(8m+7)}\right).$$

The above representation would be identical to that given by Melham provided we write $q^{(88k+77)(8m+3)}$ as $q^{(8k+7)(88m+33)}$ which brings one term from the second sum to the fourth sum. Rewriting in terms of Lambert series and replacing q^{8n+23} by q^n yields the explicit representation of [15, eq.(12)].

3. The Remaining 277 Conjectures

We have seen in the previous section that Melham's first 21 conjectures depended on 11 formulas for $R_{(a,b)}(n)$ associated with imaginary quadratic fields or positive definite binary quadratic forms of class number 2. There are other formulas for $R_{(a,b)}(n)$ associated with higher class numbers, and these are the key to the remaining 277 conjectures.

While Melham chose to classify his discoveries in terms of the type of figurate number being represented, it is perhaps more instructive to classify them according to their period. By period, we mean the exponent of q^n that appears in the denominator of terms in the Lambert series. For example, Conjectures 1.1 to 1.3 are respectively of periods 40, 84 and 1848. Although this definition of period is somewhat imprecise, it provides us with a clue as to which binary quadratic form may be used to prove the conjectures. The three conjectures above are associated with x^2+10y^2, $3x^2+7y^2$ and $7x^2+66y^2$, with respective discriminants -40, -84 and -1848. A check on the remaining 277 conjectured formulas in [14] showed that most of the identities have periods coinciding with one of the 101 known discriminants with one class per genus (see [21]). The rest of them (32 in total) have periods that are multiples of 2, 3 or 5 times of one of these 101 known discriminants. For example, Theorem 2.6 ([15, eq.(26)]) has period 120 and is associated with x^2+6y^2 which has discriminant -24. In this case, the extra factor of 5 arises from the shape of heptagonal numbers, i.e. $\frac{5x^2-3x}{2}$.

Let us now define

$$TS_{(a,b)}(n) = \left|\left\{(x,y)\in\mathbb{N}\times\mathbb{Z} : n = a\left(\frac{x^2+x}{2}\right)+by^2\right\}\right|. \tag{3.1}$$

The four pairs of conjectures [14, eq.(25) to (32)] for $TS_{(a,b)}(n)$ have periods 24, 40, 88 and 232. It is easy to deduce that these are related to (2.12) and (2.13). Specifically, the relations we want are

$$R_{(2,p)}(8n+p) = 2TS_{(p,1)}(n), \tag{3.2}$$

$$R_{(1,2p)}(8n+1) = 2TS_{(1,p)}(n), \tag{3.3}$$

for $p = 3, 5, 11, 29$, which can be deduced from (2.17). With these relations in hand, it is a matter of calculations similar to that of the previous sections to prove these eight conjectures of Melham. We remark that many formulas for $TS_{(a,b)}(n)$ have been discussed in [20].

We further illustrate our general technique by proving Conjectures 1.2 and 1.3.

Proof of Conjecture 1.2. This identity has period 84. From the tables in [21], the discriminant -84 is of class number 4 and associated with these four reduced forms x^2+21y^2, $2x^2+2xy+11y^2$, $3x^2+7y^2$ and $5x^2+4xy+5y^2$. It is reasonable to guess that $PN_{(1,1)}(n)$ is related to $R_{(3,7)}(n)$ due to the shape of pentagonal and nonagonal numbers. By considering residues modulo 6 and 14 respectively for x and y, we have

$$\sum_{x,y=-\infty}^{\infty} q^{ax^2+by^2} = 4\sum_{x,y=-\infty}^{\infty} q^{24a\left(\frac{3x^2-x}{2}\right)+56b\left(\frac{7y^2-5y}{2}\right)+a+25b} + \cdots, \tag{3.4}$$

and we can deduce that

$$R_{(3,7)}(168n+82) = 4PN_{(1,1)}(n). \tag{3.5}$$

Now from [21, p. 234], we have

$$\begin{aligned} 2R_{(3,7)}(m) = &\sum_{d|m}\left(\frac{-84}{d}\right) + \sum_{d|m}\left(\frac{28}{d}\right)\left(\frac{-3}{m/d}\right) \\ &- \sum_{d|m}\left(\frac{12}{d}\right)\left(\frac{-7}{m/d}\right) - \sum_{d|m}\left(\frac{-4}{d}\right)\left(\frac{21}{m/d}\right). \end{aligned}$$

If we set $m = 168n+82$, we can check that $\left(\frac{-3}{m}\right) = 1$ and $\left(\frac{-7}{m}\right) = -1$, and so

$$\begin{aligned} 2R_{(3,7)}(m) &= \sum_{d|m}\left(\frac{-84}{d}\right) + \sum_{d|m}\left(\frac{28}{d}\right)\left(\frac{-3}{d}\right)^2\left(\frac{-3}{m/d}\right) \\ &\quad - \sum_{d|m}\left(\frac{12}{d}\right)\left(\frac{-7}{d}\right)^2\left(\frac{-7}{m/d}\right) - \sum_{d|m}\left(\frac{-4}{d}\right)\left(\frac{21}{d}\right)^2\left(\frac{21}{m/d}\right) \\ &= 4\sum_{d|m}\left(\frac{-84}{d}\right) \\ &= 4\sum_{d|m}\left(\frac{-84}{d}\right)\left(1-\left(\frac{-7}{d}\right)\right). \end{aligned}$$

This proves (1.6). Now

$$\sum_{n\geq 0} PN_{(1,1)}(n)q^{168n+82} = \frac{1}{2}\sum_{n\geq 0}\sum_{d|168n+82}\left(\frac{-84}{d}\right)\left(1-\left(\frac{-7}{d}\right)\right)q^{168n+82}$$

where

$$\left(\frac{-84}{d}\right)\left(1-\left(\frac{-7}{d}\right)\right)=\begin{cases}2, & \text{if } d\equiv 5,17,19,31,41,55 \pmod{84};\\ -2, & \text{if } d\equiv 13,47,59,61,73,83 \pmod{84};\\ 0, & \text{otherwise.}\end{cases}$$

Thus we can write this sum as

$$\begin{aligned}&\sum_{n\geq 0} PN_{(1,1)}(n)q^{168n+82}\\ &=\sum_{k,m\geq 0}\Big(q^{2(84k+5)(84m+25)}+q^{2(84k+17)(84m+37)}+q^{2(84k+19)(84m+11)}\\ &\qquad +q^{2(84k+31)(84m+23)}+q^{2(84k+41)(84m+1)}+q^{2(84k+55)(84m+71)}\\ &\qquad -q^{2(84k+13)(84m+29)}-q^{2(84k+47)(84m+67)}-q^{2(84k+59)(84m+79)}\\ &\qquad -q^{2(84k+61)(84m+53)}-q^{2(84k+73)(84m+65)}-q^{2(84k+83)(84m+43)}\Big).\end{aligned}$$

The above identity when rewritten as Lambert series will be identical to Conjecture 1.2 after replacing $q^{168n+82}$ by q^n. □

Among the 298 conjectures, the largest period is 1848 and it appears in six of the conjectures [14, eq.(226) to (229) and (284) to (285)]. The first four concern $PN_{(a,b)}(n)$ and we shall examine them in more detail. Now the discriminant -1848 is of class number 8, and there are 8 possible choices of reduced forms [21, p. 234]. Using (3.4) we can work out

$$R_{(7,66)}(168n+1657)=4PN_{(1,22)}(n), \tag{3.6}$$
$$R_{(3,154)}(168n+229)=4PN_{(22,1)}(n), \tag{3.7}$$
$$R_{(14,33)}(168n+839)=4PN_{(2,11)}(n), \tag{3.8}$$
$$R_{(6,77)}(168n+227)=4PN_{(11,2)}(n). \tag{3.9}$$

We shall concentrate on (3.6) which is the key to proving Conjecture 1.3.

Proof of Conjecture 1.3. Following the method described in [21], we can work out

$$\begin{aligned}4R_{(7,66)}(m) = &\sum_{d|m}\left(\frac{-1848}{d}\right)+\sum_{d|m}\left(\frac{616}{d}\right)\left(\frac{-3}{m/d}\right)\\ &-\sum_{d|m}\left(\frac{264}{d}\right)\left(\frac{-7}{m/d}\right)-\sum_{d|m}\left(\frac{-11}{d}\right)\left(\frac{168}{m/d}\right)\\ &-\sum_{d|m}\left(\frac{-88}{d}\right)\left(\frac{21}{m/d}\right)-\sum_{d|m}\left(\frac{33}{d}\right)\left(\frac{-56}{m/d}\right)\\ &+\sum_{d|m}\left(\frac{77}{d}\right)\left(\frac{-24}{m/d}\right)+\sum_{d|m}\left(\frac{-231}{d}\right)\left(\frac{8}{m/d}\right).\end{aligned}$$

Let $m = 168n+1657$ and for $d \mid m$, we may introduce $\left(\frac{-3}{d}\right)^2$, $\left(\frac{-7}{d}\right)^2$ or $\left(\frac{2}{d}\right)^2$ as appropriate to simplify the above formula to

$$\begin{aligned}4R_{(7,66)}(168n+1657) &= 8\sum_{d|168n+1657}\left(\frac{-1848}{d}\right)\\ &= 8\sum_{d|168n+1657}\left(\frac{-1848}{d}\right)\left(1-\left(\frac{-11}{d}\right)\right),\end{aligned}$$

which together with (3.6) establishes (1.7). One may check that

$$\left(\frac{-1848}{d}\right)\left(1-\left(\frac{-11}{d}\right)\right)=\begin{cases}2, & \text{if } d \equiv 227, 395, 563, 899, 1403, \dots \pmod{1848};\\ -2, & \text{if } d \equiv 29, 197, 365, 701, 1205, \dots \pmod{1848}.\end{cases}$$

Hence

$$\begin{aligned}&\sum_{n\geq 0} PN_{(1,22)}(n)q^{168n+1657}\\ &=\sum_{k,m\geq 0}\Bigg(q^{(1848k+227)(168m+11)}+q^{(1848k+395)(168m+11)}+\cdots\\ &\qquad\qquad -q^{(1848k+29)(168m+5)}-q^{2(1848k+197)(168m+5)}+\cdots\Bigg)\end{aligned}$$

which can be converted into the Lambert series representation given in Conjecture 1.3. □

4. Concluding Remarks

In this article, we proved all the 21 conjectures published in [15] on representations by figurate numbers. We also explained how our technique may be used to prove the remaining 277 conjectures in [14]. Each conjecture turned out to be a consequence of a known sum of squares formula, and their relation may be deduced from the period and the shape of the figurate numbers being represented.

Acknowledgements

We would like to thank Bruce Berndt, Heng Huat Chan, Shaun Cooper, Mike Hirschhorn and Ray Melham for their helpful comments.

References

[1] C. Adiga, S. Cooper and J. H. Han, *A general relation between sums of squares and sums of triangular numbers,* Int. J. Number Theory, **1** (2005) 175–182.

[2] P. Barrucand, S. Cooper and M. Hirschhorn, *Relations between squares and triangles,* Discrete Math., **248** (2002), 245–247.

[3] N. D. Baruah and B. K. Sarmah, *The number of representations of a number as sums of various polygonal numbers,* Integers, **12** (2012), #A54.

[4] A. Berkovich and H. Yesilyurt, *Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms,* Ramanujan J., **20** (2009), 375–408.

[5] B. C. Berndt, *Number theory in the spirit of Ramanujan*, Amer. Math. Soc., Providence, Rhode Island, 2006.

[6] S. H. Chan, *Generalized Lambert series identities,* Proc. London Math. Soc., **91** (2005), 598–622.

[7] N. A. Hall, *The number of representations function for binary quadratic forms,* Amer. J. Math. **62** (1940), 589-598.

[8] M. D. Hirschhorn, *The number of representations of a number by various forms,* Discrete Math., **298** (2005) 205–211.

[9] M. D. Hirschhorn, *The number of representations of a number by various forms involving triangles, squares, pentagons and octagons,* Proceedings of Ramanujan Rediscovered: A Conference on Elliptic Functions, Partitions, and q-Series in memory of K. Venkatachaliengar: Bangalore, (1-5 June, 2009), RMS Lecture Note Series **14** (2010), 113–124.

[10] M. D. Hirschhorn, personal communication.

[11] K. Ireland and M. Rosen, *A classical introduction to modern number theory,* Graduate Texts in Mathematics 84, 2nd edition, Springer-Verlag, 1990.

[12] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum,* Borntrager, Regiomonti, 1829.

[13] H. Y. Lam, *The number of representations by sums of squares and triangular numbers,* Integers, **7** (2007) #A28.

[14] R. S. Melham, *Analogues of Jacobi's two-square theorem,* preprint, August 16, 2007. `http://maths.science.uts.edu.au/maths/wiki/RayMelham`.

[15] R. S. Melham, *Analogues of Jacobi's two-square theorem: an informal account,* Integers, **10** (2010) 83–100.

[16] Z.-H. Sun and K. S. Williams, *On the number of representations of* n *by* $ax^2+bxy+cy^2$*,* Acta Arith., **122** (2006), 101-171.

[17] Z.-H. Sun, *The expansion of* $\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})$*,* Acta. Arith., **134** (2008), 11–29.

[18] Z.-H. Sun, *On the number of representations of* n *by* $ax(x-1)/2+by(y-1)/2$*,* J. Number Theory, **129** (2009), 971–989.

[19] Z.-H. Sun, *Binary quadratic forms and sums of triangular numbers,* Acta. Arith., **146** (2011), 257–297.

[20] Z.-H. Sun, *On the number of representations of* n *by* $ax^2+by(y-1)/2$*,* $ax^2+by(3y-1)/2$ *and* $ax(x-1)/2+by(3y-1)/2$*,* Acta. Arith., **147** (2011), 81–100.

[21] P. C. Toh, *Representations of certain binary quadratic forms as Lambert series,* Acta. Arith., **143** (2010), 227–237.

Mathematics & Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616
E-mail address: peechoon.toh@nie.edu.sg