On a certain vector crank modulo 7

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Abstract

We define a vector crank to provide a combinatorial interpretation for a certain Ramanujan type congruence modulo 7.

Keywords: partitions; congruences; crank

1 Introduction

In [7], one of the authors established several new Ramanujan type identities and congruences modulo 3, 5 and 7 for certain types of partition functions. For example, define $Q_{po,\overline{p}}(n)$ as the number of partitions of n into two colors, where the red colored parts form a partition into odd parts and the blue

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colored parts form an overpartition. Using the standard notation

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j),$$
$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$
$$(a_1,\ldots,a_m;q)_\infty = (a_1;q)_\infty\cdots(a_m;q)_\infty,$$

for $|q|<1$ and $a, a_1, \ldots, a_m \neq 0$, we can write the generating function of $Q_{po,\overline{p}}(n)$ as

$$\sum_{n=0}^{\infty} Q_{po,\overline{p}}(n)q^n = \frac{1}{(q;q^2)_\infty}\times\frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(-q,-q;q)_\infty}{(q;q)_\infty}.$$

Toh [7] proved that

$$\sum_{n=0}^{\infty} Q_{po,\overline{p}}(7n+2)q^n \equiv 0 \pmod 7. \tag{1}$$

Zhou [9] subsequently provided alternative proofs of all of the congruences in [7] with the exception of (1). She re-interpreted these partition functions as partitions into multi-colors, introduced what she termed as *multiranks* – which are essentially vector cranks as defined by Garvan [4] – and proved that these vector cranks divided the partitions into equinumerous parts. The aim of this article is to define a vector crank that will explain (1) combinatorially.

2 A vector crank

If λ is a partition, we define $\sigma(\lambda)$ and $n(\lambda)$ as the sum of the parts and the number of parts of λ respectively. Let $\mathcal{D}, \mathcal{O}, \mathcal{P}$ denote the sets of partitions into distinct parts, partitions into odd parts, and unrestricted partitions respectively. Define the cartesian product

$$\mathcal{V} = \mathcal{D}\times\mathcal{D}\times\mathcal{O}\times\mathcal{O}\times\mathcal{P}\times\mathcal{P}.$$

For a vector partition $\vec{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)\in\mathcal{V}$ define a sum of parts s, a weight w and a crank r by

$$s(\vec{\lambda}) = 2\sigma(\lambda_1)+\sigma(\lambda_2)+\sigma(\lambda_3)+\sigma(\lambda_4)+2\sigma(\lambda_5)+2\sigma(\lambda_6), \tag{2a}$$
$$w(\vec{\lambda}) = (-1)^{n(\lambda_1)}, \tag{2b}$$
$$r(\vec{\lambda}) = 2n(\lambda_3)-2n(\lambda_4)+n(\lambda_5)-n(\lambda_6). \tag{2c}$$

The weighted count of vector partitions of n with crank m, denoted by $N_{\mathcal{V}}(m,n)$, is given by

$$N_{\mathcal{V}}(m,n) = \sum_{\substack{\vec{\lambda}\in\mathcal{V} \\ s(\vec{\lambda})=n \\ r(\vec{\lambda})=m}} w(\vec{\lambda}). \tag{3}$$

We also define the weighted count of vector partitions of n with crank congruent to k modulo t by

$$N_{\mathcal{V}}(k,t,n) = \sum_{m=-\infty}^{\infty} N_{\mathcal{V}}(mt+k,n) = \sum_{\substack{\vec{\lambda}\in\mathcal{V} \\ s(\vec{\lambda})=n \\ r(\vec{\lambda})\equiv k \pmod{t}}} w(\vec{\lambda}). \tag{4}$$

Finally, we have the following generating function for $N_{\mathcal{V}}(m,n)$,

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N_{\mathcal{V}}(m,n)z^m q^n = \frac{(q^2;q^2)_\infty(-q;q)_\infty}{(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}. \tag{5}$$

Theorem 1. *The following equation holds for all nonnegative integers n.*

$$N_{\mathcal{V}}(0,7,7n+2) = N_{\mathcal{V}}(1,7,7n+2) = \cdots = N_{\mathcal{V}}(6,7,7n+2) = \frac{Q_{po,\overline{p}}(7n+2)}{7}.$$

The main ingredient in the proof of the theorem is Winquist's identity [8], which is a variant of the B_2 case of the Macdonald identities [5]. We state the identity in the following symmetric form [6, Eq. (3.1)]. If we define

$$F_1(x) = \sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2}(x^{3j}+x^{-3j}), \tag{6a}$$

$$F_2(x) = \sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2+2k}(x^{3k+1}+x^{-3k-1}), \tag{6b}$$

we have

$$F_1(x)F_2(y) - F_1(y)F_2(x) = -\frac{2}{x}\left(xq, \frac{q}{x}, yq, \frac{q}{y}, xy, \frac{q^2}{xy}, \frac{x}{y}, \frac{yq^2}{x}, q^2, q^2; q^2\right)_\infty. \tag{6c}$$

Proof of Theorem 1. If we set $\zeta = \exp(2\pi i/7)$ in (5), we obtain

$$\begin{aligned}
&\sum_{t=0}^{6} \zeta^t \sum_{n=0}^{\infty} N_{\mathcal{V}}(t,7,n) q^n \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_{\mathcal{V}}(m,n) \zeta^m q^n \\
&= \frac{(q^2;q^2)_\infty}{(q, \zeta^2 q, q/\zeta^2, \zeta q^2, q^2/\zeta; q^2)_\infty} \\
&= \frac{(\zeta q, q/\zeta, \zeta^3 q, q/\zeta^3; q^2)_\infty}{(q^7;q^{14})_\infty} \times \frac{(q^2, q^2, \zeta^2 q^2, q^2/\zeta^2, \zeta^3 q^2, q^2/\zeta^3; q^2)_\infty}{(q^{14};q^{14})_\infty} \\
&= \frac{F_1(\zeta^3)F_2(\zeta) - F_1(\zeta)F_2(\zeta^3)}{2\zeta(1-\zeta^2)(1-\zeta^3)(q^7;q^7)_\infty},
\end{aligned}$$

where we used (6) with $x = \zeta^3$ and $y = \zeta$.

Since $3j^2 \equiv 0,3,5,6 \pmod 7$ and $3k^2+2k \equiv 0,1,2,5 \pmod 7$, the power of q in $q^{3j^2+3k^2+2k}$ is congruent to 2 modulo 7 exactly when $j \equiv 0 \pmod 7$ and $k \equiv 2 \pmod 7$. This means that the coefficient of q^{7n+2} in

$$F_1(\zeta^3)F_2(\zeta) - F_1(\zeta)F_2(\zeta^3)$$

is zero since

$$(-1)^{j+k}(\zeta^{9j}+\zeta^{-9j})(\zeta^{3k+1}+\zeta^{-3k-1}) - (-1)^{j+k}(\zeta^{3j}+\zeta^{-3j})(\zeta^{9k+3}+\zeta^{-9k-3}) = 0$$

when $j \equiv 0 \pmod 7$ and $k \equiv 2 \pmod 7$. Thus

$$\sum_{t=0}^{6} N_{\mathcal{V}}(t,7,7n+2)\zeta^t = 0. \tag{7}$$

Since the minimal polynomial for ζ over the rational numbers is

$$p(x) = 1 + x + x^2 + \cdots + x^6,$$

we conclude that

$$N_{\mathcal{V}}(0,7,7n+2) = N_{\mathcal{V}}(1,7,7n+2) = \cdots = N_{\mathcal{V}}(6,7,7n+2). \qquad \square$$

We end by indicating how one may prove (1) directly as the details were omitted in [7]. This can be done by observing that

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{po,\overline{p}}(n)q^n &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3} \\ &\equiv \frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^3} \times \frac{1}{(q^{14};q^{14})_\infty} \pmod 7. \end{aligned} \tag{8}$$

Thus (1) is equivalent to proving the coefficients of q^{7n+2} in

$$\frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^3}$$

are all divisible by 7. We offer three alternative ways of doing this. The easiest way is to appeal directly to [3, Th. 2]. Alternatively, we can use one of the Macdonald identities associated with the $C_2^\vee$ root system [5, p. 137] or [6, Eq. 3.12], to express

$$\frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^3} = \sum_{\substack{\alpha\equiv 1 \pmod 8 \\ \beta\equiv 3 \pmod 8}} \frac{1}{8}(\beta^2-\alpha^2)q^{\frac{\alpha^2+\beta^2-10}{16}}.$$

If the exponent of q is congruent to 2 modulo 7, we have

$$\alpha^2+\beta^2 \equiv 16(2)+10 \equiv 0 \pmod 7.$$

Since -1 is a quadratic nonresidue modulo 7, 7 must divide both α and β. The third way is to apply the Hecke operator T_7 to $\frac{\eta(16\tau)^9}{\eta(8\tau)^3}$, a weight 3 cusp form of level 128. One can refer to [1] for examples of how this may be done.

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Proposition 2. *If* $|q|, |t| < 1$ *then*

$$\frac{(at;q)_\infty}{(a;q)_\infty(t;q)_\infty} = \frac{1}{(a;q)_\infty} + \sum_{n=1}^{\infty} \frac{t^n}{(aq^n;q)_\infty(q;q)_n}.$$

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