ON ANGLES FORMED BY LINES ON A SQUARE LATTICE

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Consider two arbitrary intersecting line segments drawn on a square lattice, i.e. with the endpoints of the segments on lattice points. One may ask what values can arise as the angle of intersection. Clearly, there are too many possibilities, so we ask a simpler question. Ignoring the trivial cases of perpendicular lines and $45°$ formed by diagonals of a square, we would like to know what "nice" values can arise as the angle of intersection?

Let us rephrase the question more precisely. Let $a, b, c, d \in \mathbb{Z}$ and consider vectors (a, b) and (c, d) in $\mathbb{R}^2$. For what values of a, b, c and d, does the angle between the vectors (a, b) and (c, d) assume the values of $30°, 45°$ or $60°$? Utilizing the dot product of two vectors, the two theorems below provide a fairly complete answer. The values of $30°$ and $60°$ never appear. There are some implications. For example, it is impossible to draw an equilateral triangle or a $30° - 60° - 90°$ triangle where all three vertices lie on lattice points. On the other hand, for every line segment, one can always find another segment that intersects it at $45°$ degrees. However, up to scaling, there are essentially only two ways to do this.

Theorem 1. *There does not exist integers $a, b, c, d \in \mathbb{Z}$, such that*

$$\frac{ac + bd}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}} = \pm\frac{\sqrt{3}}{2} \text{ or } \pm\frac{1}{2}.$$

Theorem 2. *Let $a, b, c, d \in \mathbb{Z}$, such that $\gcd(a, b) = \gcd(c, d) = 1$ and*

$$\frac{ac + bd}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}} = \frac{\sqrt{2}}{2}.$$

Then if $ad - bc > 0$, $(c, d) = (a - b, a + b)$ or $\left(\frac{a-b}{2}, \frac{a+b}{2}\right)$. Otherwise, if $ad - bc < 0$, $(c, d) = (a + b, b - a)$ or $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$.

We shall prove both theorems together. First we recall the sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ad - bc)^2 + (ac + bd)^2.$$

One can easily verify the above algebraically but it is more instructive to view it as the multiplicative property of the norm in complex numbers. Suppose we have

$$\frac{ac + bd}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}} = \pm\frac{\sqrt{m}}{2},$$

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where $m = 1, 2$ or 3. By squaring and clearing denominators, we have

$$\begin{aligned} 4(ac+bd)^2 &= m(a^2+b^2)(c^2+d^2) \\ &= m(ad-bc)^2 + m(ac+bd)^2 \end{aligned}$$

which means

$$(4-m)(ac+bd)^2 = m(ad-bc)^2.$$

When $m = 1$ or 3, we have $3x^2 = y^2$ for some integers x and y which violates the Fundamental Theorem of Arithmetic. This proves Theorem 1. To continue with the proof of Theorem 2, we further assume $m = 2$ and $\gcd(a,b) = \gcd(c,d) = 1$. We have

$$ac + bd = \pm(ad - bc).$$

Recall that in the theorem statement the left side of the above is positive, so we first consider $ad - bc > 0$. This means $ac + bd = ad - bc$ and thus

$$b(d+c) = a(d-c). \tag{1}$$

As $\gcd(a,b) = 1$, Euclid's Lemma means that $b \mid d - c$. In other words, $d - c = kb$ for some $k \in \mathbb{Z}$. Likewise, $a \mid d + c$ which means $d + c = ja$. for some $j \in \mathbb{Z}$. Substituting both into (1), we can conclude $j = k$. Taking sum and differences then yield

$$2c = k(a-b) \qquad \text{and} \qquad 2d = k(a+b). \tag{2}$$

We can combine (2) into

$$2(ad-bc) = k(a^2+b^2).$$

Since $ad - bc > 0$, k must be positive. Recall that $\gcd(a,b) = 1$, so if either one of a or b is even, in order to satisfy $\gcd(c,d) = 1$, we must have $k = 2$ which leads to $(c,d) = (a-b, a+b)$. On the other hand, if both a and b are odd, then $k = 1$ which means $(c,d) = \left(\frac{a-b}{2}, \frac{a+b}{2}\right)$.

Now in the case $ad - bc < 0$, we have

$$ac + bd = bc - ad. \tag{3}$$

We can then reason as before to obtain $(c,d) = (a+b, b-a)$ or $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$.

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