A variation of the Andrews-Stanley partition function and two interesting q-series identities

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Abstract. Stanley introduced a partition statistic $\text{srank}(\pi) = \mathcal{O}(\pi) - \mathcal{O}(\pi')$, where $\mathcal{O}(\pi)$ denote the number of odd parts of the partition π, and π' is the conjugate of π. Let $p_i(n)$ denote the number of partitions of n with srank $\equiv i \pmod 4$. Andrews proved the following refinement of Ramanujan's partition congruence modulo 5:

$$p_0(5n+4) \equiv p_2(5n+4) \equiv 0 \pmod 5.$$

In this paper, we consider an analogous partition statistic

$$\text{lrank}(\pi) = \mathcal{O}(\pi) + \mathcal{O}(\pi').$$

Let $p_i^+(n)$ denote the number of partitions of n with lrank $\equiv i \pmod 4$. We will establish the generating functions of $p_0^+(n)$ and $p_2^+(n)$ and show that they satisfy similar properties to $p_i(n)$. We also utilize a pair of interesting q-series identities to obtain a direct proof of the congruences

$$p_0^+(5n+4) \equiv p_2^+(5n+4) \equiv 0 \pmod 5.$$

Keywords: Partitions, Stanley's partition function, Ramanujan's congruences

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1 Introduction

A partition of a positive integer n is a sequence of weakly decreasing positive integers whose sum equals n. For a partition π, let π' denote its conjugate and let $\mathcal{O}(\pi)$ denote the number of odd parts in π. If π is a partition of n, then the number of odd parts must have the same parity as n. Thus $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 2$. Stanley [11, 12] initiated a study on the number of partitions π of n for which

$$\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 4.$$

Following [3], we define the partition statistic

$$\text{srank}(\pi) = \mathcal{O}(\pi) - \mathcal{O}(\pi')$$

and let $p_i(n)$ denote the number of partitions of n with srank $\equiv i \pmod 4$. Since srank(π) is always even, we see that

$$p_0(n) + p_2(n) = p(n),$$

where $p(n)$ is the usual partition function. Stanley [12] established the following generating function:

$$\sum_{n=0}^{\infty}(p_0(n) - p_2(n))q^n = \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}. \tag{1.1}$$

Here we use the standard notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}) \quad \text{and} \quad (a_1, \ldots, a_m; q)_\infty = (a_1;q)_\infty \ldots (a_m;q)_\infty.$$

We remark that Stanley used the notation $t(n)$ for $p_0(n)$ and $f(n)$ for $p_0(n) - p_2(n)$.

Andrews [1] subsequently obtained the generating function for $p_0(n)$:

$$\sum_{n=0}^{\infty} p_0(n)q^n = \frac{(q^2;q^2)_\infty^2(q^{16};q^{16})_\infty^5}{(q;q)_\infty(q^4;q^4)_\infty^5(q^{32};q^{32})_\infty^2}. \tag{1.2}$$

A few years later, Chen, Ji and Zhu [7] obtained the generating function for $p_2(n)$:

$$\sum_{n=0}^{\infty} p_2(n)q^n = \frac{2q^2(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2(q^{32};q^{32})_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty^5(q^{16};q^{16})_\infty}. \tag{1.3}$$

They also provided combinatorial interpretations of $p_0(n)$ and $p_2(n)$ in terms of hook lengths.

By studying the coefficients of q^{5n+4} in (1.1) and using Ramanujan's famous congruence $p(5n+4) \equiv 0 \pmod 5$, Andrews proved the remarkable congruence

$$p_0(5n+4) \equiv 0 \pmod 5.$$

Swisher [13] subsequently showed that there are infinitely many arithmetic progressions $An + B$ such that

$$p_0(An+B) \equiv p(An+B) \equiv 0 \pmod{\ell^j}$$

where $\ell \geq 5$ is prime and $j \geq 1$.

In this paper, we shall study a variation of the srank. Define the partition statistic

$$\text{lrank}(\pi) = \mathcal{O}(\pi) + \mathcal{O}(\pi')$$

and let $p_i^+(n)$ denote the number of partitions of n with lrank $\equiv i \pmod 4$, where $i = 0, 2$. It also follows that

$$p_0^+(n) + p_2^+(n) = p(n).$$

In the next two sections, we will derive the generating functions for $p_0^+(n)$ and $p_2^+(n)$ and show that they satisfy similar properties to $p_i(n)$. For example,

$$p_0^+(5n+4) \equiv p_2^+(5n+4) \equiv 0 \pmod 5, \tag{1.4}$$

which gives a new refinement of the Ramanujan's congruence for $p(5n+4)$. Finally, in Section 4, we utilize two q-series identities to give a direct proof of (1.4) which is independent of Ramanujan's congruence.

2 Generating functions for $p_0^+(n)$ and $p_2^+(n)$

Let $S_\infty(n,r,s)$ be the number of partitions π of n such that $\mathcal{O}(\pi)=r, \mathcal{O}(\pi')=s$. Andrews [1] found the following generating function

$$\sum_{n,r,s\geq 0} S_\infty(n,r,s)q^n y^r z^s = \frac{(-yzq;q^2)_\infty}{(q^4;q^4)_\infty(y^2q^2;q^4)_\infty(z^2q^2;q^4)_\infty}. \tag{2.1}$$

Combinatorial proofs of identity (2.1) were independently found by Sills [10], Yee [15], and Boulet [6]. With (2.1) in hand, we are in a position to prove the generating functions for $p_0^+(n)$ and $p_2^+(n)$.

Theorem 2.1.

$$\sum_{n=0}^{\infty} p_0^+(n)q^n = \frac{(-q^3;q^8)_\infty^2(-q^5;q^8)_\infty^2(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^5}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} p_2^+(n)q^n = \frac{q(-q;q^8)_\infty^2(-q^7;q^8)_\infty^2(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^5}. \tag{2.3}$$

Proof. Recall that $\mathcal{O}(\pi)$ and $\mathcal{O}(\pi')$ are congruent modulo 2 to the number being partitioned, thus $\mathcal{O}(\pi)\equiv\mathcal{O}(\pi') \pmod 2$. Hence, using i to denote the square root of -1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_0^+(n)q^n &= \sum_{\substack{n,r,s\geq 0\\ 4|(r+s)}} S_\infty(n,r,s)q^n \\
&= \frac{1}{2}\sum_{n,r,s\geq 0} S_\infty(n,r,s)(1+i^{r+s})q^n \\
&= \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(q^2;q^4)_\infty^2} + \frac{(q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}\right) \qquad (2.4)\\
&= \frac{(-q;q^2)_\infty(-q^2;q^4)_\infty^2 + (q;q^2)_\infty(q^2;q^4)_\infty^2}{2(q^4;q^4)_\infty(q^4;q^8)_\infty^2} \\
&= \frac{(-q;q^2)_\infty(q^8;q^8)_\infty^2}{2(q^4;q^4)_\infty^4}\left((-q^2;q^4)_\infty^2(q^4;q^4)_\infty + (q;q^2)_\infty^2(q^2;q^2)_\infty\right).
\end{aligned}$$

By [4, p. 51, Example (iv)], we see that

$$(-q^2;q^4)_\infty^2(q^4;q^4)_\infty + (q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{2(-q^3;q^8)_\infty^2(-q^5;q^8)_\infty^2(q^8;q^8)_\infty^2(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} p_0^+(n)q^n &= \frac{(-q;q^2)_\infty(q^8;q^8)_\infty^2}{2(q^4;q^4)_\infty^4} \times \frac{2(-q^3;q^8)_\infty^2(-q^5;q^8)_\infty^2(q^8;q^8)_\infty^2(q;q^2)_\infty}{(q^2;q^2)_\infty} \\
&= \frac{(-q^3;q^8)_\infty^2(-q^5;q^8)_\infty^2(q^8;q^8)_\infty^4(q^2;q^4)_\infty}{(q^4;q^4)_\infty^4(q^2;q^2)_\infty}
\end{aligned}$$

$$= \frac{(-q^3;q^8)_\infty^2(-q^5;q^8)_\infty^2(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^5}.$$

Similarly, we have

$$\sum_{n=0}^{\infty} p_2^+(n)q^n = \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(q^2;q^4)_\infty^2} - \frac{(q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}\right) \tag{2.5}$$

$$= \frac{(-q;q^2)_\infty(q^8;q^8)_\infty^2}{2(q^4;q^4)_\infty^4}\left((-q^2;q^4)_\infty^2(q^4;q^4)_\infty - (q;q^2)_\infty^2(q^2;q^2)_\infty\right).$$

By [4, p. 51, Example (iv)], we see that

$$(-q^2;q^4)_\infty^2(q^4;q^4)_\infty - (q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{2q(-q;q^8)_\infty^2(-q^7;q^8)_\infty^2(q^8;q^8)_\infty^2(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Combining the above two identities together, we get (2.3). This completes the proof. ∎

3 Congruences and Inequalities for $p_0^+(n)$ and $p_2^+(n)$

Theorem 3.1. *For all $n \geq 0$,*

$$p_0^+(5n+4) \equiv p_2^+(5n+4) \equiv 0 \pmod 5. \tag{3.1}$$

Proof. From (2.4) and (2.5), we see that

$$\sum_{n=0}^{\infty}(p_0^+(n) - p_2^+(n))q^n = \frac{(q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)_\infty^2}.$$

Comparing with (1.1), we observe that

$$\sum_{n=0}^{\infty}(p_0^+(n) - p_2^+(n))q^n = \sum_{n=0}^{\infty}(p_0(n) - p_2(n))(-q)^n.$$

Equating the coefficients of q^n, we obtain

$$p_0^+(n) - p_2^+(n) = (-1)^n(p_0(n) - p_2(n)). \tag{3.2}$$

Recall that in [1], Andrews proved $p_0(5n+4) - p_2(5n+4) \equiv 0 \pmod 5$ and used Ramanujan's congruence to deduce $p_0(5n+4) \equiv 0 \pmod 5$. Consequently $p_2(5n+4) \equiv 0 \pmod 5$. In the same way, after establishing (3.2), together with the fact

$$p_0^+(5n+4) + p_2^+(5n+4) = p(5n+4) \equiv 0 \pmod 5,$$

we can conclude that $p_0^+(5n+4) \equiv p_2^+(5n+4) \equiv 0 \pmod 5$. ∎

In fact, there are many congruences satisfied by $p_i^+(n)$.

Theorem 3.2. *There are infinitely many arithmetic progressions $An + B$, such that for all $n \geq 0$,*

$$p_0^+(An + B) \equiv p_2^+(An + B) \equiv p(An + B) \equiv 0 \pmod{\ell^j},$$

where $\ell \geq 5$ is prime and $j \geq 1$.

Proof. Swisher [13] proved that there are infinitely many arithmetic progressions $An + B$, such that for all $n \geq 0$, $p_0(An + B) \equiv p(An + B) \equiv 0 \pmod{\ell^j}$ where $\ell \geq 5$ is prime and $j \geq 1$. For those progressions $An + B$, we see that $p_2(An + B) \equiv 0 \pmod{\ell^j}$. Recall that $p_0^+(n) - p_2^+(n) = (-1)^n(p_0(n) - p_2(n))$. This means $p_0^+(An+B) - p_2^+(An+B) \equiv 0 \pmod{\ell^j}$ and $p_0^+(An + B) \equiv p_2^+(An + B) \equiv 0 \pmod{\ell^j}$. ∎

There is an easier way to prove the previous two theorems. Recall that $p_0^+(2n)$ counts partitions of $2n$ where

$$\operatorname{lrank}(\pi) = \mathcal{O}(\pi) + \mathcal{O}(\pi') \equiv 0 \pmod 4.$$

Coupled with the fact that we now have $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \equiv 0 \pmod 2$, we can deduce that

$$\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 4.$$

In other words, $\operatorname{srank}(\pi) \equiv 0 \pmod 4$, thus $p_0^+(2n) = p_0(2n)$. Depending on the parity of n, similar arguments can be used to derive other relations between $p_i^+(n)$ and $p_i(n)$. We record these relations as the next result.

Lemma 3.1. *For all $n \geq 0$,*

$$p_0^+(2n) = p_0(2n), \tag{3.3}$$
$$p_0^+(2n + 1) = p_2(2n + 1), \tag{3.4}$$
$$p_2^+(2n) = p_2(2n), \tag{3.5}$$
$$p_2^+(2n + 1) = p_0(2n + 1). \tag{3.6}$$

From (1.3), we see that $p_2(n)$ is always even.

Corollary 3.1. *For $n \geq 0$, we have*

$$p_0^+(2n + 1) \equiv 0 \pmod 2, \tag{3.7}$$
$$p_2^+(2n) \equiv 0 \pmod 2. \tag{3.8}$$

Andrews [1] and Chen et. al. [7] provided 4-dissections of $p_0(n)$ and $p_2(n)$ respectively. Using their results and Lemma 3.1, it is straightforward to write down 4-dissections of $p_0^+(n)$ and $p_2^+(n)$. In another related paper, Berkovich and Garvan proved several inequalities, including the surprising result [2, Eq. (1.17)]

$$|p_0(2n) - p_2(2n)| > |p_0(2n + 1) - p_2(2n + 1)|,$$

which holds for all $n \geq 1$. They also proved that [2, p. 281] for $n \geq 0$,

$$p_0(n) > \frac{p(n)}{2}, \text{ if } n \equiv 0, 1 \pmod 4, \tag{3.9}$$

$$p_0(n) < \frac{p(n)}{2}, \text{ if } n \equiv 2,3 \pmod 4. \tag{3.10}$$

By combining Lemma 3.1 and their results, we can obtain the following analogues.

Corollary 3.2. *For $n \geq 0$, we have*

$$p_0^+(n) > p_2^+(n), \text{ if } n \equiv 0,3 \pmod 4, \tag{3.11}$$
$$p_0^+(n) < p_2^+(n), \text{ if } n \equiv 1,2 \pmod 4. \tag{3.12}$$

Corollary 3.3. *For $n \geq 1$, we have*

$$|p_0^+(2n) - p_2^+(2n)| > |p_2^+(2n+1) - p_0^+(2n+1)|. \tag{3.13}$$

4 Two interesting q-series identities

Our previous proof of Theorem 3.1 relied on the known congruences for $p_i(5n+4)$ and $p(5n+4)$. We can actually give an independent proof directly from the generating functions of $p_0^+(n)$ and $p_2^+(n)$ which leads to a new refinement of $p(5n+4) \equiv 0 \pmod 5$. To this end, we define $a(n)$ and $b(n)$ as follows,

$$\sum_{n=0}^{\infty} a(n)q^n = (-q^3;q^8)_\infty^2(-q^5;q^8)_\infty^2(q^8;q^8)_\infty^4, \tag{4.1}$$
$$\sum_{n=0}^{\infty} b(n)q^n = q(-q;q^8)_\infty^2(-q^7;q^8)_\infty^2(q^8;q^8)_\infty^4. \tag{4.2}$$

Since

$$\sum_{n=0}^{\infty} p_0^+(n)q^n \equiv \frac{1}{(q^{20};q^{20})_\infty} \times \sum_{n=0}^{\infty} a(n)q^n \pmod 5, \tag{4.3}$$

we have $p_0^+(5n+4) \equiv 0 \pmod 5$ if $a(5n+4) \equiv 0 \pmod 5$. Similarly, if $b(5n+4) \equiv 0 \pmod 5$ then $p_2^+(5n+4) \equiv 0 \pmod 5$. In fact, we have the following stronger result.

Theorem 4.1.

$$\sum_{n=0}^{\infty} a(5n+4)q^n = -5\sum_{n=0}^{\infty} b(n)q^{5n+3}, \tag{4.4}$$
$$\sum_{n=0}^{\infty} b(5n+4)q^n = -5\sum_{n=0}^{\infty} a(n)q^{5n+3}. \tag{4.5}$$

The coefficients of $a(n)$ and $b(n)$ are interlinked in a way that is analogous to some recent investigations by Hirschhorn [8]. Just as Hirschhorn's results were generalized from the prime 5 to infinitely many primes in [9, 14], the same holds for Theorem 4.1.

Theorem 4.2. *Suppose $n \geq 0$ and $p \equiv 5 \pmod 6$ is prime. If $p \equiv \pm 3 \pmod 8$, then*

$$a\left(p^2 n + \frac{19(p^2-1)}{24}\right) = -pb(n), \tag{4.6}$$

$$b\left(p^2 n + \frac{19(p^2-1)}{24}\right) = -pa(n). \tag{4.7}$$

If $p \equiv \pm 1 \pmod 8$, then

$$a\left(p^2 n + \frac{19(p^2-1)}{24}\right) = -pa(n), \tag{4.8}$$

$$b\left(p^2 n + \frac{19(p^2-1)}{24}\right) = -pb(n). \tag{4.9}$$

Proof. Combining Entries 30(v) and 30(vi) in [4, p. 46], we have

$$f(a,b)^2 = f(a^2,b^2)\varphi(ab) + 2af(b/a, a^3b)\psi(a^2b^2).$$

(Definitions of $f(a,b)$, $\varphi(q)$ and $\psi(q)$ can be found in [4, pp. 34–36].) Applying the above identity with $a \mapsto q^3, b \mapsto q^5$, we get

$$(-q^3,-q^5,q^8;q^8)_\infty^2 = (-q^6,-q^{10},q^{16};q^{16})_\infty \frac{(q^{16};q^{16})_\infty^5}{(q^8;q^8)_\infty^2 (q^{32};q^{32})_\infty^2} + 2q^3(-q^2,-q^{14},q^{16};q^{16})_\infty \frac{(q^{32};q^{32})_\infty^2}{(q^{16};q^{16})_\infty}$$

Multiplying both sides by $(q^8;q^8)_\infty^2$, we conclude that

$$\sum_{n=0}^{\infty} a(n)q^n = (-q^6,-q^{10},q^{16};q^{16})_\infty \frac{(q^{16};q^{16})_\infty^5}{(q^{32};q^{32})_\infty^2} + 2q^3(-q^2,-q^{14},q^{16};q^{16})_\infty \frac{(q^8;q^8)_\infty^2(q^{32};q^{32})_\infty^2}{(q^{16};q^{16})_\infty}.$$

From the following identities [5, Cor. 1.3.21 and 1.3.22]

$$\frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2} = \sum_{m=-\infty}^{\infty} (6m+1)q^{3m^2+m},$$

$$\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{m=-\infty}^{\infty} (3m+1)q^{3m^2+2m},$$

and the Jacobi triple product identity [5, Th. 1.3.3], we have

$$\sum_{n=0}^{\infty} a(n)q^n = \sum_{n=-\infty}^{\infty} q^{8n^2+2n} \sum_{m=-\infty}^{\infty} (6m+1)q^{8(3m^2+m)} + 2q^3 \sum_{n=-\infty}^{\infty} q^{8n^2+6n} \sum_{m=-\infty}^{\infty} (3m+1)q^{8(3m^2+2m)}.$$

The above series representation can be rewritten as

$$\sum_{n=0}^{\infty} a(n)q^n = \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 1 \pmod 8}} xq^{\frac{16x^2+3y^2-19}{24}} + \sum_{\substack{x\equiv 2 \pmod 6\\ y\equiv 3 \pmod 8}} xq^{\frac{16x^2+3y^2-19}{24}}. \tag{4.10}$$

Similarly,

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 3 \pmod 8}} xq^{\frac{16x^2+3y^2-19}{24}} + \sum_{\substack{x\equiv 2 \pmod 6\\ y\equiv 1 \pmod 8}} xq^{\frac{16x^2+3y^2-19}{24}}. \tag{4.11}$$

Now given a prime $p \equiv 5 \pmod 6$, when

$$\frac{16x^2+3y^2-19}{24} = pm + \frac{19(p^2-1)}{24},$$

the expression is equivalent to

$$(4x)^2 + 3y^2 = 24pm + 19p^2 \equiv 0 \pmod p.$$

Since $p \equiv 2 \pmod 3$, we conclude that the above congruence holds only when $p \mid x$ and $p \mid y$. We write $x = -px_1$ where $x \equiv x_1 \pmod 6$ and $y = \pm py_1$. We further assume $p \equiv \pm 3 \pmod 8$ which means if $y \equiv 1, 3 \pmod 8$, then $y_1 \equiv 3, 1 \pmod 8$. Returning to the previous equation,

$$16x^2 + 3y^2 = p^2(16x_1^2 + 3y_1^2) = 24pm + 19p^2.$$

In other words,

$$\frac{16x_1^2 + 3y_1^2 - 19}{24} = \frac{m}{p}.$$

Now extracting the coefficients from (4.10), we have

$$\begin{aligned} a\left(pn + \frac{19(p^2-1)}{24}\right) &= \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 1 \pmod 8\\ 16x^2+3y^2=24pn+19p^2}} x + \sum_{\substack{x\equiv 2 \pmod 6\\ y\equiv 3 \pmod 8\\ 16x^2+3y^2=24pn+19p^2}} x \\ &= \sum_{\substack{x_1\equiv 1 \pmod 6\\ y_1\equiv 3 \pmod 8\\ 16x_1^2+3y_1^2=24n/p+19}} -px_1 + \sum_{\substack{x_1\equiv 2 \pmod 6\\ y_1\equiv 1 \pmod 8\\ 16x_1^2+3y_1^2=24n/p+19}} -px_1 \\ &= -pb(n/p). \end{aligned}$$

This proves (4.6). The proofs of the other cases are analogous. ∎

Finally, we remark that if we define

$$c(n) = a(n) + b(n),$$

then it follows from (2.4) and (2.5) that

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty}. \tag{4.12}$$

With some calculations, the above fact can also be deduced directly from summing (4.10) and (4.11). Theorem 4.2 means that for every prime $p \equiv 5 \pmod 6$,

$$\begin{aligned} c\left(p^2n + \frac{19(p^2-1)}{24}\right) &= a\left(p^2n + \frac{19(p^2-1)}{24}\right) + b\left(p^2n + \frac{19(p^2-1)}{24}\right) \\ &= -p\big(a(n) + b(n)\big) \\ &= -pc(n). \end{aligned}$$

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