Chapter 9

Invariance as a Big Idea

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Understanding big ideas in mathematics allows us to see the subject as a coherent and connected discipline. In this chapter, we shall focus on a single big idea, namely invariance, which threads a number of topics horizontally across content strands and vertically across levels.

1 Introduction

According to Charles (2005),

> a Big Idea is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole. (p. 10)

He asserts that big ideas should be the "foundation for one's mathematics content knowledge, for one's teaching practices, and for the mathematics curriculum", and understanding big ideas allows one to view mathematics as a "coherent set of ideas" and not a collection of "disconnected concepts, skills, and facts." (Charles, 2005, p. 10). Charles provides a list of 21 big ideas but states unequivocally that it is impossible to have a list that all mathematicians and mathematics educators agree on. In his conclusion, he urges teachers to develop their own lists of big ideas while keeping in mind that "Big Ideas need to remain BIG and they need to be the anchors for most everything we do." (Charles, 2005, p. 12). In this chapter, we shall focus on a single big idea, namely invariance, which threads a number of topics horizontally across content strands and vertically across levels. Our statement about invariance is as follows.

> Big Idea of Invariance: Valuable information can be obtained by studying the *invariant set* of a general class of *transformations*.

We will illustrate with several examples of invariance that occur in school mathematics, ranging from geometry at the Primary/Secondary Level to number theory at the Advanced Level (A-Level). In each example, we shall try to identify explicitly the *invariant set* and the *transformation*.

2 Invariance in Geometry

The concept of invariance is of particular importance in geometry. Many theorems in geometry are formulated as statements about certain invariants. For example, virtually every student of mathematics will know that the angle sum (*invariant set*) of a triangle in the Euclidean plane equals 180°, regardless of how the triangle is drawn (*transformation*). They will also know that the area (*invariant set*) of the triangle ABC, illustrated in Figure 1, will remain the same when the vertex C is translated (*transformation*) to C_1, C_2, or any point along the line C_1C_2 that is parallel to AB. Although it is unlikely that students know this as an example of a shear, which is an area-preserving transformation.

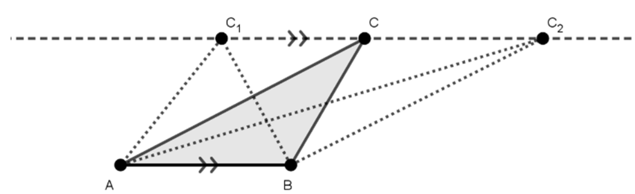

Figure 1. Area is invariant

Building on our previous example, the figure depicted on the left of Figure 2 is typically found in puzzles in geometry. Here one is supposed to calculate what fraction of the square is shaded. The standard algebraic solution is to note that if x is the length of a side of the square and h the height of one triangle, then the height of the other triangle must be $x - h$. Thus the shaded region must have area

$$\frac{1}{2}(h + x - h)x = \frac{1}{2}x^2.$$

The expression is clearly independent of the variable h and so the area of the two triangles (*invariant set*) remains the same when the point of the intersection of the two triangles is translated (*transformation*) to any other point within the square. A student who understands this fact can quickly work out the required answer by translating the point of intersection to the extreme right or even to the top right corner as seen in Figure 2. An interesting follow-up question for students is to identify precisely when the invariance breaks down if the intersection is translated to a point outside the square.

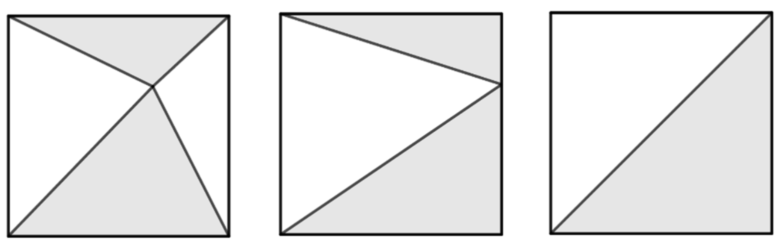

Figure 2. Finding the proportion of a shaded figure

In our next example, consider a fixed equilateral triangle ABC and any point P in its interior. If we drop a perpendicular to each of the three sides (see Figure 3), the sum of the lengths of the three perpendiculars remain constant irrespective of which interior point P was initially chosen. In this example, the transformation refers to any translation of the point P as long as it remains in the interior of the triangle ABC, while the invariant set is the sum of the lengths of the three perpendiculars.

There are many more interesting examples of invariance in geometry. In their book written for teachers, Sinclair, Pimm and Skelin (2012) articulated four big ideas in geometry. The second of which is “geometry is about working with variance and invariance, despite appearing to be about theorems." Libeskind, Stupel and Oxman (2018) gives several other geometric examples of invariance in school mathematics, including two examples on conics which might be of interest to those teaching the corresponding topic in Further Mathematics at A-Level. They advocate the approach of providing students with the opportunities to investigate invariance through dynamic geometry software such as GeoGebra.

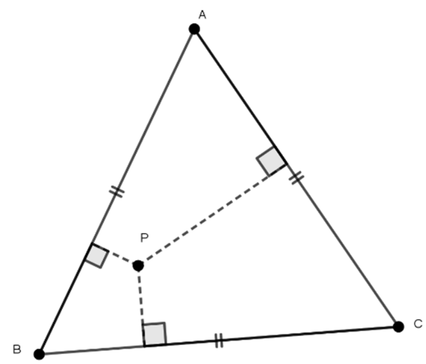

Figure 3. Sum of the lengths of the three perpendicular is invariant

3 Invariance in Numerical Iterations

The following is a simple activity to investigate invariance in numerical iterations. Using a calculator or preferably a computer, compute (in radian measure) cos(0.5). Next, continue to compute cos(x) where x is the output obtained from the previous computation. This is an example of an iteration scheme: $x_0 = 0.5$; $x_{n+1} = \cos(x_n)$, under the topic of numerical methods in Further Mathematics at A-Level. Figure 4 shows two screenshots from a graphing calculator. The left screenshot shows the first four iterations while the right screenshot shows that x_n appear to converge to an approximate value of 0.7390851332 after 60 iterations. The actual value, which we shall denote as c, is called a fixed point of cos(x) and satisfies cos(c) = c. In other words, the point c is invariant under the transformation cos(x).

cos(0.5)	cos(Ans)
.8775825619	.7390851333
cos(Ans)	cos(Ans)
.6390124942	.7390851332
cos(Ans)	cos(Ans)
.8026851007	.7390851333
cos(Ans)	cos(Ans)
.6947780268	.7390851332

Figure 4. Iterating cos(x_n)

In our example, the initial value of x_0 can be replaced by any value in the interval (0,1) and the iteration process would still appear to converge to the same fixed point after a sufficient number of iterations. It is also possible to replace the function f(x) = cos(x). Iterations using

$$f(x) = e^{-x} \text{ or } f(x) = 1 + \frac{2x - x^2}{2}$$

would also converge to the respective fixed points of these functions. It would take us too far afield to describe the conditions for such iterations to converge, but it is comparatively simple to show the existence of fixed points.

The set of functions that are continuous on the closed interval [0,1] contains most of the functions that students have been introduced to in school mathematics. These are, for example, the polynomial functions, the absolute value function f(x) = |x|, the trigonometric functions sin(x) and cos(x), and the exponential function e^x. It is thus somewhat surprising that one can extract useful information from such a general class of functions.

Theorem 1. If f(x) is continuous on [0,1], f(0) > 0 and f(1) < 1, then f(x) has a fixed point c in (0,1). That is, f(c) = c.

A rigorous proof of the above theorem is beyond the scope of this chapter but the following explanation is usually sufficient to convince most students. Let us consider the graph of f(x) and $y = x$ on the same interval [0,1]. Since f(0) > 0 and f(1) < 1, the left and right endpoints of the graph of f(x) are lying respectively 'above' and 'below' the graph of $y = x$. Although we are unable to actually sketch the graph of f(x) as we do not have any additional information about this function other than it being continuous, we can argue that any continuous graph joining the left and right endpoints must intersect the graph of $y = x$. If we use c to denote the x-coordinate of this point of intersection, we must have $f(c) = c$.

There are a number of fixed point theorems in mathematics and what we have sketched above is a special case of what is usually known as Brouwer's fixed point theorem (Munkres, 2000). Fixed points theorems illustrate very well the big idea of invariance where useful information can be extracted from very general situations. Munkres' book contains a proof of how Brouwer's theorem can be used to prove that any 3×3 matrix with positive real entries has a positive real eigenvalue. It is no coincidence that the concept of eigenvectors in linear algebra is yet another instance of invariance. In this case, the space spanned by an eigenvector of a linear transformation is a one-dimensional invariant subspace. Finally, we note that fixed point theorems are more than mere intellectual curiosities. These theorems have found applications in economics and game theory (Border, 1985).

4 Invariance in Number Theory

We focus on two examples of invariance in number theory in this section. The first involves simple arithmetic and can be viewed as an instance of a fixed point theorem analogous to those discussed in the previous section. We describe it in the form of a 'magic trick' that can be used in the classroom. The teacher first asks a student to think of a three-digit positive integer with distinct digits. He then pretends to read the student's mind, writes a number down on a piece of paper and folds it in half to prevent anyone from seeing the number. Next, he hands the folded piece of paper to another student for safekeeping. The teacher now asks the first student to rearrange the digits of his chosen number to form the largest possible number and write it on the board. For example, if the student chose 197, the digits would be rearranged to form 971. Next, he asks the student to rearrange the same three digits to form the smallest possible number and subtract this smallest number from the largest. Continuing with our numerical example, the student would compute 971 - 179 = 792. The teacher then asks the student to repeat the calculations with this new number. That is, subtracting the smallest possible number from the largest possible number, both of which are formed by rearranging the digits of the new number. The calculations are repeated until the student arrives at the number 495, whereupon the teacher asks the second student to reveal to the class that the number previously written on the piece of paper is also 495.

The 'magic trick' that we just described can be used in classrooms ranging from Upper Primary to A-Level. The transformation is the operation of subtracting the smallest possible number from the largest possible number formed by rearranging the digits, while the invariant set or fixed point is the number 495. In other words, one can begin with any three-digit number (with distinct digits) and after at most six iterations of the operation arrive at 495. For more advanced students, the same exercise can be carried out with four-digit numbers, in which case, the fixed point is the number 6174. This is known as Kaprekar's constant (Kaprekar, 1955). Let us see why this works in the three-digit case, where we may assume $a > b > c$, are the three distinct digits. Carrying out the required operation will result in $100a+10b+c - (100c+10b+a) = 99(a - c)$. We obtain one of eight possible multiples of 99, because $a - c$ can range from 2 to 9 inclusive. It is then not difficult to observe that every subsequent iteration will result in another multiple of 99 where the difference between the largest digit (which is always 9) and the smallest digit converges to 5, giving 495. When used in appropriate situations, the 'magic trick' just described creates an element of surprise and is an interesting and accessible piece of mathematics for most students. More advanced students may be tasked to investigate the situation of numbers with more digits or to find a different transformation that has an analogous invariant property.

Our second example is more advanced and requires background knowledge of number theory. Number theory is a recent addition to the Higher 3 (H3) Mathematics at A-Level. While not explicitly stated in the syllabus document, we can infer from the specimen examination paper and subsequent actual examination papers that students are expected to know one particular proof of Fermat's Little Theorem. This proof is an exemplar of the big idea of invariance and we will describe it below. We first define the concept of congruence in number theory. If n is a positive integer, we say that a is congruent to b modulo n and write $a \equiv b \pmod{n}$ if $a - b$ is an integer multiple of n. In other words, both a and b have the same remainder when divided by n. A useful way to understand the congruence $a \equiv b \pmod{n}$ is to think of it as a generalization of $a = b$. (More precisely, it is an equivalence relation.) We can now state Fermat's Little Theorem.

Theorem 2. Suppose p is prime number and a is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$.

This means that if we take any integer and raise it to the exponent $p - 1$ for some prime number p, the resulting integer always leaves a remainder of 1 when divided by p. The only exception is when the original integer was already a multiple of p, in which case the remainder is clearly 0. This is a remarkable result because it works for any prime number and any integer that is not a multiple of that prime number. Consider this numerical example. 495^{12} = 216,402,556,571,320,625,160,840,087,890,625 has remainder 1 when divided by 13. Equally surprising is the fact that we can actually prove that the theorem holds. To understand the proof, we require the following result.

Theorem 3. Suppose p is prime number and a is not a multiple of p, then the congruence $ax \equiv 1 \pmod{p}$ has a solution.

Theorem 3 is a fundamental result in number theory because it recovers the concept of a multiplicative inverse, which is missing in the integers. For example, the equation $7x = 1$ does not have any solutions in integers but the congruence $7x \equiv 1 \pmod{p}$ has a solution for every prime number p other than 7. Taking $p = 17$, a solution for $7x \equiv 1 \pmod{17}$ is $x = 5$.

All the pieces are now in place for us to use invariance to prove Fermat's Little Theorem. Given a fixed prime number p, the invariant set S, is the set of all possible non-zero remainders after dividing by p. In other words, $S = \{1, 2, \ldots, p - 1\}$. The set of transformations is that of multiplying an element by a, with the condition that a is not a multiple of p, followed by taking the remainder after dividing by p. For example, when $a = 4$ and $p = 7$, then the transformation sends 6 to 3 because $4 \times 6 = 24 \equiv 3 \pmod 7$. The action of the transformation on all the elements of S are given below:

$$1 \mapsto 4,\ 2 \mapsto 1,\ 3 \mapsto 5,\ 4 \mapsto 2,\ 5 \mapsto 6, \text{ and } 6 \mapsto 3.$$

We can also represent the above as

$$\{4 \times 1, 4 \times 2, 4 \times 3, 4 \times 4, 4 \times 5, 4 \times 6\} \equiv \{4, 1, 5, 2, 6, 3\} \pmod 7.$$

Let us now prove that the set S is invariant under the general transformation by showing that

$$\{a \times 1, a \times 2, \ldots, a \times (p - 1)\} \equiv \{1, 2, \ldots, p - 1\} \pmod p. \qquad (**)$$

Consider any pair of distinct elements from the set on the left of (**), namely $a \times j$ and $a \times k$ for some distinct j, k between 1 and $p - 1$. Suppose we have $a \times j \equiv a \times k \pmod p$, then we can multiply to both sides the solution x that is guaranteed to exist by Theorem 3. Replacing xa by 1 would mean $j \equiv k \pmod p$, or simply $j = k$ which contradicts our assumption that they are distinct. We have just shown that none of the elements from the left side of (**) are congruent to each other. Since there are exactly $p - 1$ elements, this can only mean that these elements are simply $\{1, 2, \ldots, p - 1\}$ in some order. This completes the proof of (**). We now multiply all the elements from both sides of (**) to obtain

$$\begin{aligned} (a \times 1)(a \times 2) \ldots (a \times (p - 1)) &\equiv (1)(2)\ldots(p - 1) \pmod p \\ \Rightarrow a^{p-1} (p - 1)! &\equiv (p - 1)! \pmod p \\ \Rightarrow \quad a^{p-1} &\equiv 1 \pmod p. \end{aligned}$$

In the above, we again used Theorem 3 to cancel the factor $(p - 1)!$ from both sides. Although the proof that we have presented is slightly technical, it illustrates perfectly the big idea of obtaining useful information by studying the invariant set of general transformations. The proof can be further adapted to prove a number of important results in number theory, namely Wilson's Theorem, Euler's Criterion for quadratic residues, and

Gauss' Lemma. Fermat's Little Theorem itself is commonly used as a test to see if a large integer is a prime number, and thus has important applications to the area of modern cryptography.

5 Conclusion

Understanding the big ideas in mathematics and translating them into one's teaching practice can appear to be a daunting challenge. It requires both the breadth of content knowledge and the depth of insight to identify the underlying principles behind mathematical topics or results. In this chapter, we focused on the big idea of invariance and illustrated its connections with examples from the curriculum. We hope that teachers can use these examples as a starting point to develop their own understanding of invariance and other big ideas in mathematics.

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