Mathematics Education Research Group of Australasia

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National Institute of Education
Nanyang Technological University
Singapore

Excellence in Mathematics Education: Foundations & Pathways

Proceedings of the 43rd Annual Conference of the Mathematics Education Research Group of Australasia

Edited by Yew Hoong Leong, Berinderjeet Kaur,
Ban Heng Choy, Joseph Boon Wooi Yeo, & Sze Looi Chin

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Inc. GPO Box 2747

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Singapore MERGA 43 Local Organising Committee

Berinderjeet Kaur (Conference Convenor)

Yew Hoong Leong

Ban Heng Choy

Eng Guan Tay

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Preface

This is a record of the Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia (MERGA). The conference was hosted by colleagues at the National Institute of Education, Nanyang Technological University, Singapore. It was an online conference as there remained restrictions in travel due to the COVID-19 pandemic in 2021. The proceedings were published online at the MERGA website www.merga.net.au

The theme of the conference was *Excellence in Mathematics Education: Foundations and Pathways*. This theme was chosen by the conference organising committee to engage the research community in deliberations on the foundations and pathways through which facets of excellence in mathematics education may be actualised. Two plenary lectures were delivered on the theme. The opening and first plenary lecture by Professor Anna Sfard focussed on the invisible pitfalls when teaching-learning events are conceptualised as inter-discursive encounters. Anna concluded with guidance of how teachers and their students can benefit from such communicational gaps. The second lecture by Professor Tin Lam Toh presented a snapshot of Singapore's journey towards excellence in mathematics education by examining the role of the traditional notion of mathematics competition and other competitive activities. The Clements/Foyster lecture was delivered by Professor Vince Geiger. The lecture was devoted to the theme of *becoming* a researcher in mathematics education – a fundamental focus for MERGA. The theme of the conference was also deliberated on by four panelists during a plenary session. They shared their perspectives on excellence in mathematics education and described research they had been involved in related to some aspect(s) of excellence in mathematics education.

In addition, the conference included presentations of symposia, research papers, short communications, and a round table that covered a wide range of topics related to mathematics education in Australasia and other countries. All symposia and research papers were double-blind reviewed by panels of mathematics educators with expertise in the field and accepted for publication and presentation or presentation only. All the short communications were also reviewed by the organising committee and were either accepted for presentation or rejected if they were not research oriented. The published proceedings include the plenary papers, symposia papers, research papers, and abstracts of research presentations, short communications, and a round table.

The Editorial Team would like to thank the Review Panel Chairs and all the reviewers for their professionalism and effort in reviewing the papers and providing constructive feedback. The review process ensured that the high academic standards of the MERGA community are upheld. Delegates from Australia, Canada, Fiji, Ireland, Israel, Japan, New Zealand, Singapore, South Africa, South Korea, Taiwan, and United Kingdom participated in the online conference. This was the first MERGA virtual conference held in the new normal brought about by the COVID-19 pandemic that hit the world in December 2019.

Berinderjeet Kaur (Conference Convenor & Editor)

Yew Hoong Leong, Ban Heng Choy, Joseph Boon Wooi Yeo, & Sze Looi Chin (Editors)

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The devil in details: Mathematics teaching and learning as managing inter-discursive gaps

Anna Sfard
University of Haifa
<sfard@netvision.net.il>

Once teaching-learning events are conceptualised as inter-discursive encounters, it becomes clear that mathematics classroom talk is rife with invisible pitfalls. There are many types of unacknowledged discursive gaps, some of them necessary for learning, and some potentially harmful. Such gaps may exist also between the teacher's intentions and her own habitual moves, most of which are too brief and automatic to be controlled. Unknown to the teacher, her basic communicational routines may constitute invisible crevices through which the prejudice enters the conversation on mathematical objects. In this talk, I argue that if the devil is in the finest detail of classroom communication, it is the detail that must be considered in the attempts to exorcise the devil. I begin with illustrations of these claims and conclude with a reflection on how mathematics teachers may sensitise themselves to discursive pitfalls, how they and their students can benefit from those communicational gaps that are likely to generate learning, and how they can cope with those divides that hinder the process or infect it with unwanted messages.

Humans, unlike most other species, can exist only as a part of a society. But while our very survival may depend on effective interpersonal exchanges, our communication is only too prone to failure. Some go so far as to claim that within this context, failure constitutes the default option, whereas success should be regarded as almost a miracle (Reddy, 1979).

Perhaps the most challenging aspect of communicational breakdowns is that they often go unnoticed. Paraphrasing Hamlet, one can say that there are more communicational pitfalls in heaven and earth than are dreamt of by philosophers or suspected by ordinary people. These pitfalls tend to hide in unnoticeable details of interlocutors' actions. Obviously, people trying to reach one another across a hidden communicational gap risk falling to the bottom. As blind to the fall as they were to the pitfall, they are likely to leave the exchange with unhelpful interpretations of each other's intentions. At home, it may hurt their relationships; in the classroom, it may stymie their learning. In the words of George Bertrand Shaw, "The single biggest problem in communication is the illusion that it has taken place". This paper is about guarding ourselves against this illusion by becoming alert to communicational pitfalls.

Some may claim that the existence of certain communicational gaps is inherent to learning and thus little can be done against them. Yet, I wish to argue that even when a gap is necessary for the further development of mathematical discourse, the importance of our awareness to its existence cannot be overstated. Indeed, exposing the gaps is a critical step in turning them from obstacles into opportunities for learning. Clearly, being constantly on the watch for hidden communicational hurdles will also help in guarding ourselves against the adverse impact of those gaps that could be avoided.

In what follows, I illustrate the claim about the omnipresence of communication gaps with examples from mathematics classrooms. With the help of specially designed conceptual apparatus, evolving around the vision of learning as a process of routinisation of our actions, I zoom into the data and identify seemingly negligible details that may constitute, for better or worse, powerful shapers of students' learning.

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Communicational gaps

In the classrooms, the presence of invisible communication pitfall may signal itself by puzzling occurrences, for which neither the teacher nor an external observer can provide an immediate explanation. The danger of the *illusion* of communication, however, is at its worst when nothing seems unusual and the communicational glitch, although quite real, does not manifest itself in a palpable way.

Consider, for example, the exchange between a teacher and her student, presented in Table 1. What happens in this brief episode is so familiar that the claim about the student's initial difficulty as due to any communicational issue is likely to be met with scepticism. Indeed, nothing seems surprising that the child who is evidently quite new to the topic of fractions has difficulty multiplying a fraction by a whole number. It is also not startling that after the teacher's additional probing (see turns [3] and [5]) and with some effort on the part of the student, the proper answer is finally produced ([6]). The teacher summarised saying that a bit of effort was all the boy needed to succeed ([7]). In making this statement, she implied that the learner was already acquainted with the necessary procedure, but was not yet quite proficient in its application and performance.

Table 1
Example I: Multiplying by Fraction

#	Speaker	What is said	What is done
1	Teacher:	So, what is?	Writes ⅓ · 12
2	Student:	.....	
3	Teacher:	Try again, one third times twelve	
4	Student:	I think.... Don't know...	
5	Teacher:	Once again, one third of twelve	
6	Student:	Ahm..... It's four	
7	Teacher:	Great. See, when you think about it, you know how to do it!	

As unproblematic as this simple account seems to be, at a closer look it leaves an important question unanswered. Yes, the child did seem to make an effort. Yet, although he clearly tried hard already the first time round, he was able to produce an answer only after the teacher's third attempt. What was it about this third question ([5]) that brought the sudden insight? How was this query different from the previous ones ([1], [3])? Some scrutiny of the three instances may suffice to realise that each of the three utterances referred to the required operation in its own way:

1. with the help of the written expression '$^1/_3 \cdot 12$' ([1])
2. orally, with the expression "one third times twelve" ([3])
3. orally, with "one third of twelve" ([5]).

The first two of these renditions that make use of distinctly mathematical symbol '·' and the word "times", that belong to the formal discourse on numbers. The last utterance, which speaks about "one third *of* twelve", may be a part of the child's everyday talk and can belong to the repertoire also of a person with no access to formal mathematics. The first two utterances directed the child to as-yet unfamiliar numerical operation, whereas the third one required the everyday action of identifying a familiar part of a whole.

The difference between this account and the one offered by the teacher is subtle, may even appear negligible, but it is highly consequential. With her vision of the current state of the student's learning, the teacher will likely emphasise the need for fostering the child's procedural proficiency. In contrast, the realisation that the student might had participated in a discourse different from her own and that, in result, the task he tried to perform was not the one she had in mind will turn her attention to the conceptual side of the story. Building on the resulting conceptual interpretation, she may decide to focus on helping the learner to see connections between his everyday talk and the mathematical discourse of multiplication.

In this analysis, I exemplified the way in which we can make ourselves aware of subtle communicational issues that, if unrecognised, may lead the teacher to unhelpful pedagogical decisions, but if noticed, are likely to give rise to opportunities for significant learning. The terms such as 'discourse' or 'task' have been used in this analysis freely, without a proper introduction. The next section provides what is missing. After defining the terms as they are to be understood within a discursive theory of learning, I will be able to operationalise the notion of communicational gap and instantiate ways in which the risks of such gap can be significantly reduced and its potential as an opportunity for learning considerably increased.

Operationalising the construct of communicational gap

Mathematics as discourse

In this paper, the word *discourse* is used as referring to the special form of communication, characteristic of a particular community. The community may be that of scientists, chess players or of art theorists. Most relevantly for our present context, it may be a community of mathematicians or of mathematics classrooms. Whereas each such community is unified by its members common interest, activity or cultural practice, its discourse is designed specifically to tell stories with which this activity or practice can be usefully mediated.

Thus, the first characteristic of a discourse that sets this discourse apart from any other is its collection of *endorsed narratives* about this discourse's focal objects. The adjective 'endorsed' indicates that these narratives are considered by its participants as faithful accounts of the state of affairs in the world and thus, as reliable guides for future actions. In mathematics, endorsed narratives are about such abstract objects as numbers, sets, geometric figures, functions, etc. The communicational tools with the help of which these stories are forged and substantiated constitute additional set of characteristics that make the discourse distinguishable from other ones. Thus, there is the set of special-purpose *keywords* pertaining to the focal objects and actions of the discourse. In mathematics, these are words such as 'number', 'function', 'triangle', 'adding', 'differentiating', etc. Although many of these words may be known also from everyday talk, in specialised discourse their use is different and defined more strictly. Another special feature of a discourse is the set of special *visual mediators* that help in ensuring the effectiveness of communication. Algebraic symbols and graphs are among the most useful mediators of mathematical discourse. Finally, discourses are made distinct by their *routines*, the recurrent ways of performing different kinds of tasks, such as, in mathematics, calculating, proving or performing geometric constructions with the help of ruler and compass. Some of the routines are algorithmic, some are more of a 'rule of a thumb'. This last characteristic, routine, being particularly relevant to the topic of communicational gaps, requires some elaboration.

More about routines

Routine, far from being just an optional way of acting (and a rather boring one, some may say, because of its repetitive nature), is what makes us able to act in the first place. Indeed, it is thanks to routines that we know how to act whenever we feel expected to do something, which is most of the time. In such situation, to react to the prompt in an immediate way, the best we can do is to turn to those familiar ways of acting that worked for us in the past in a similar situation (or what we consider as such). This, indeed, was what the student in Example II was able to do when he eventually found the way to answer the interviewer's question: he recalled what was done when somebody asked the question of the form "What is one third of X?", with X being a set of a certain size (12 items, in this case).

To operationalise the construct of routine, there is a need for some auxiliary notions. Thus, the situation in which a person feel she is obliged to act will be called *task-situation*. Such situation may arise of itself, as is the case when one feels cold or hungry. Task-situation may also be created by asking questions. In Example II, this is what the teacher did three times, in turns [1], [3] and [5]. Once a person finds herself in a task-situation, she needs to decide about her *task*, that is, about what needs to be done, and about a *procedure* that suits that task. Deliberately or instinctively, this person will probably try to do this by recalling precedents. *Precedent* is any previous task-situations that appears to a person as sufficiently similar to the present one to justify doing now what was done then. Given suitable precedents, she will see it as her task to act in such a way as to ensure the reoccurrence of specific aspects of the precedent task-situation. For instance, while feeling hungry, she will probably see it as her task to make the sense of hunger disappear. Her procedure will be the prescription for action that, according to her interpretation, guided the previous task performer. In hunger instigated task-situation, the procedure may be a walk to a fridge and helping herself to some food.

Once the search for task and procedure is successfully completed, the person is ready to act. Note that in most daily task-situations, especially in those with which we are intimately familiar, this initial step is intuitive rather than conscious and deliberate, and rarely makes us slow down for reflection. We may say that the *task-procedure pair* resulting from one's search, being a prescription for an emerging pattern, is this person's *routine* for dealing with the given task-situation. Learning can now be seen as a process of routinisation of our action (Lavie et al. 2019).

Discursive gaps and their sources

In the light of the above definition, routine is not a free-floating, context-free phenomenon. I will now argue that routines depend on task-situations and on their interpreters. To put it differently, different people may interpret the same task-situation in different ways, ending up with different tasks, to be performed with the help of different procedures. To show this, I need to take a closer look at how people decide about tasks and procedures.

On the face of it, the search for routines that would fit particular task-situations appears so demanding, it is more likely to fail than succeed. Indeed, we would have little chance to succeed in interpreting task-situations if we were to search precedents among *all* past events, from all times and all locations. Fortunately, search spaces tend to shrink considerably the moment we enter a specific task-situation. Imperceptibly to ourselves, we react to such a situation with a choice of a discourse in which to think about this situation. The subsequent search for precedents will be restricted to past situations in which people had recourse to this

discourse. With different discourses come different routines, that is, different ways of acting. Thus, more often than not, a task-situation created by the mathematics teacher automatically directs the students to the discourse of this teacher's classroom, and to routines that were employed there, preferably in the most recent past. And vice versa: task-situation created in out-of-school context is likely to direct potential performers to everyday discourse, barring them from any other. Indeed, we tend to close ourselves in discourses we associate with a given situation and this tendency may account for the phenomenon known as *situativity* of learning (Brown et al., 1989; Lave, 1988), that is, for the fact that most people do not usually apply in one context routines they have learned in another. In particular, this maybe the reason why mathematics learned in school is, in most cases, practically absent from our daily lives.

It is this tendency for associating situations with discourses that may be responsible for the event presented in Table 1, in which the student reacted in different ways to what seemed to the teacher as mere repetitions of "the same" question. More generally, considering the dependence of our discursive choices on our past experience, it is only understandable that people participating in the same conversation would often turn to different discourses. In the next section, we use the former example, as well as some other ones, to show that the resulting communicational disparities carry both risk and promises, and that making them visible may help the teacher to turn the gaps from pitfalls into learning opportunities for her students.

Discursive gaps as opportunities for learning

The two examples to be presented in this section illustrate the thesis that discursive gaps, while constituting a treat to the process of learning, may also be indispensable for the development of mathematical discourse. In both these examples, a close analysis will show that two people engaged in a conversation with one another may, in fact, be participating in different discourses.

Example I: Opportunity for developing routines by bonding them with other ones

Back to the example presented in Table 1, I can now present the results of the former analysis with the help of the conceptual tools introduced above. Here is the new description: the three *task-situations* created by the teacher's questions [1], [3], and [5], although identical in the eyes of the teacher, were seen as different by the student. More specifically, questions [1] and [3] probably sent the child *searching for precedents* among past classroom situations in which a formal algorithm for multiplying fraction by a whole number was used. Question [5], on the other hand, might have brought to his mind everyday situations in which a conversation was about sharing a certain amount of cookies fairly between three friends. The *tasks* envisioned by the child as a result of these differing choices of discourses and precedents were also different: In the first case, he saw it as his job to perform the symbolic manipulation he learned in school. In the second case, his task was to find out what would be the share of one person if twelve items were distributed evenly between three people. This interpretation is summarised in Table 2.

An important insight about development of routines can be gained from this example. At a close look, these two tasks, as well as the resulting procedures, have little in common with one another. Yet, those who are well versed in multiplying by fractions and perform this operation almost automatically are usually oblivious to the difference between the sequences of actions required in these two cases. The long experience with the respective

procedures might have blinded them to an interesting phenomenon that transpired very clearly from an ongoing PhD research on the development of the discourse on rational numbers[1]. Indeed, oldtimers to that discourse typically do not remember that they were probably well acquainted with words such as *half*, *quarter*, (*one*)-*third* or *three-quarters* well before they knew anything about the formal discourse on fractions. If so, they have also forgotten that once upon a time, these basic fraction words did not function for them as names of numbers, but were rather labels for some special routines. At that time, "finding a third of a pizza" meant not much more than a physical action of cutting the pizza into three parts, whereas "giving each of three children a third of the twelve cookies" meant the circular action of handing a single cookie to each of the children (usually while saying "one for you, and one for you…"), and repeating the action until none of the twelve cookies was left. At that time, the expression "$^1/_3 \cdot 12$" was meaningless. In other words, different rational numbers corresponded in the beginning to different procedures used in execution of different tasks. It took time until the different tasks consolidated into one, and the different procedures became alternative branches of a single algorithm.

Table 2
Discourses and routines in Example I

	The teacher's interpretation of the task-situations created by all her utterances	The student's interpretation of task-situation created by the teacher's utterance [5]
discourse	numerical	of parts and wholes
task	perform the formal numerical calculation $^1/_3 \cdot 12$	find one third of a set of 12
procedure	Apply the algorithm for multiplication of rational numbers	1. Divide the whole into the number of equal parts indicated by the name of the part 2. Take one part

As argued by Lavie et al. (2019), such *bonding*[2] of several routines and turning them into a single one constitutes one of the central mechanisms in the development of discourses. In the present case, many other routines that in the eyes of the beginner have little to do with the school discourse on fractions will yet be bonded with the formal operation "$^1/_3 \cdot 12$" before the full-fledged routine for multiplying rational numbers emerges. The process of gradual bonding will lead to successive extensions in the applications of the resulting super-routine known as multiplication of rational numbers. These developments will greatly increase the usefulness of the multiplication routine, and with it, that of the whole discourse of rational numbers.

[1] The research, titled "Development of the discourse on rational numbers" is being conducted these days at the University of Haifa by Aya Steiner. Its partial results have been published in Steiner (2018).

[2] This type of bonding, one that happens between different procedures, is sometimes qualified with the adjectives *horizontal* or *external* so as to be distinguished from the bonding that occurs inside the procedure, and is thus known as *vertical* or *inner*.

Example II: Opportunity for meta-level learning

In this example, taken from a study on a 7th grade class learning about negative numbers (Sfard, 2007), a different type of discursive gap comes to the fore. Before explaining its nature and source, let us take a look at classroom events that signalled its existence.

At the time the event took place, the class has already discussed the multiplication of negative numbers by positive numbers, but some students were still questioning the claim that the result should be negative. The relevant episode began when the teacher declared that she was going to "explain" this fact in a new way. On this occasion, she would also show how the product of two negative numbers should be defined. As can be seen in the episode presented in Table 3, she decided to derive all this from the multiplication of natural numbers, with which the children were already well acquainted.

Table 3
Example II: Teacher demonstrates derives multiplication of integers

#	Speaker	What is said	What is done
1556a	Teacher:	Well, I wish to explain this now in a different way.	Points to [2 · (–3) = –6]
1556b			Writes on the blackboard the following column of equalities: 2 · 3 = 6 2 · 2 = 4 2 · 1 = 2 2 · 0 = 0 2 · (–1) = –2 2 · (–2) = –4 2 · (–3) = –6 While writing, she stops at each line and asks the children about the result before actually writing it down and stressing that the decrease of 1 in the multiplied number decreases the result by 2
1556c	Teacher:	Let us now compute (–2) times (–3) in a similar way.	As before, writes on the blackboard the following column of equalities, stopping at each line and asking the children about the result before actually writing it down and noting that the decrease of 1 in the multiplied number increases the result by 3; this rule, she says, must be preserved all along: 3 · (–3) = –9 2 · (–3) = –6 1 · (–3) = –3 0 · (–3) = 0 (–1) · (–3) = 3 (–2) · (–3) = 6

Table 4 shows objections raised by some students in reaction to the teacher's argument.

Table 4
Children's reactions to teacher's derivation of the laws of multiplication

#	Speaker	What is said
1557	Shai:	I don't understand why we need all this mess. Is there no simpler rule?
1559	Sophie:	And if they ask you, for example, how much is (–25) · (–3), will you start from zero, do 0 · (–3), and then keep going till you reach (–25) · (–3)?

The students seem to have misinterpreted the teacher's intentions. The teacher saw it as her task to justify the definition of integer multiplication by deriving it from operations on natural numbers[3]. In contrast, the children interpreted the teacher's performance as a presentation of a new algorithm for multiplication, which they then criticized as a rather cumbersome method for producing simple endorsed narratives such as (–2) · (–3) = 6 or (–25) · (–3) = 75. The nature of the resulting discursive gap is detailed in Table 5.

Table 5
Discourses and routines in Example II

	The teacher's interpretation of her own performance	Children's interpretation of the teacher's performance
discourse	of unsigned numbers	of integers
task	define "plus times minus"	calculate a product of a positive and negative number
procedure	build a list that leads form a known operation (multiplication of two natural numbers) to the desired ones ("plus times minus" and "minus times minus")	

Why this difference in the teacher's and students' interpretation of the task-situation? One explanation is that the children were still captive of the discourse of unsigned numbers. In that familiar discourse, numbers and numerical operations constituted a part of the external, mind-independent world. Indeed, so far, it was the world that dictated the result of all numerical operations, such as 2·3 or 5·½. In the discourse of signed numbers, in contrast, the nature of numeric operations seems to be established in the act of defining, as if by fiat. This change is tantamount to passing the power of deciding about what exists and what happens in mathematical universe from the external, natural powers – or maybe from the God – to humans. As such, it is difficult to accept, and even before that, to conceive.

Two discourses that differ in their routines for forging and endorsing narratives have been called *incommensurable* (Sfard 2007).[4] The transition from the discourse of natural

[3] Here, the set of natural numbers is regarded as including zero. The unspoken principle underlying the teacher's argument was that the definition of multiplication of integers would preserve some basic numerical laws that held in the realm of numbers so far.

[4] This difference means a change in meta-rules, that is, in the rules that govern the activity of mathematizing. Such meta-level change can lead to seemingly contradicting endorsements. And, indeed, the narrative "There is a number that is smaller than any other" that held in the discourse on natural numbers is one of the many that will have to be abandoned once this discourse is extended to the one on integers.

numbers to that of integers is one of several passages to an incommensurable discourse that the student will have to make in the process of learning. The learning that takes place during these passages has been described as *meta-level*, so as to signal that in this case, the learning involves not just an addition of new narratives, but also a change in how such narratives are created and endorsed. A successful meta-level learning closes the discursive gap that spurred this learning. This closure does not mean the disappearance of the former discourse – of the discourse of natural numbers in the present case. Rather, this old discourse is subsumed in the new one and subjected to its differing meta-rules.

Summary and conclusions: Implications for teaching

The two cases of discursive gap shown in this section shed much light on processes of discourse development. The first of them tells us something about the growth of routines: such growth involves turning a number of hitherto unrelated procedures into special cases of a single procedure for the execution of different variations of the same task. This means that task-situations seen by discursive oldtimers as "the same" (equivalent), may be seen by newcomers as different. The second example shows the inevitability of discursive gaps as those that spur learning (*meta-level* learning, in this case) in the first place. Indeed, every so often, further development of mathematical discourse will remain stymied until the students confront and overcome a discursive gap: until they face, and reconcile themselves with a discourse incommensurable with the one in which they participated so far. In sum, in both cases, the gap, far from being just a nuisance, is what spurs the development in the first place. As such, it is indispensable for learning.

Obviously, in cases such as those presented in this section, avoiding the gaps would preclude the possibility of learning. As such, it is not an option. Instead, one should try to minimize the risks of the gap and optimize its potential benefits. Yet, not only the students, but also teachers are rarely aware of discursive gaps such as those described in the two examples. It is by making them visible that the teacher may turn potential pitfalls into opportunities for learning. The question of how to do this must be left for another article.

Discursive gaps as a danger to teaching

Unlike in the case of discursive gaps that are necessary for students' learning and thus cannot be prevented, the two examples in this section show avoidable gaps that, if left unattended, are likely to distort teaching. In both cases, these gaps stem from the teacher's inadvertent participation in a discourse that clashes with her intentions.

Example III: Involuntary engagement in constructing students' identities

While in mathematics classroom, the students and the teachers are supposed to mathematize, that is, to participate in a discourse on mathematical objects. Yet, mathematical discourse, even when predominant, is rarely the only one. All along the mathematical conversation, participants also make statements about themselves and others. Although the *subjectifying* narratives (narratives about people, as opposed to those about mathematical objects) produced in the process may not be getting a direct attention, in a longer run that may have a considerable impact on the participants' identities, that is, on the stories they believe true about themselves and about others. When it comes to students' identities, particularly influential is the subjectifying activity of the teacher. Although in most cases the teacher would probably readily admit that she bears a major responsibility for how her students see themselves as learners, she may not be sufficiently alert to those aspects of her

classroom performances that constitute the most powerful identity-builders. Indeed, as I will now show with the help of an example, the devil may hide in tiniest details of the teacher's actions. The most powerful may be those brief moves that the teacher performs automatically, without planning in advance, without explicitly monitoring them at the time of performance and without remembering afterwards.

The example that follows comes from a study devoted to middle school students' extracurricular mathematical activities organized and led by one of the researchers (Heyd-Metzuyanim & Sfard, 2012). In the case under consideration, a group of four students described by their regular mathematics teacher as "good" (having a history of above average achievement) attempted to solve a non-standard mathematical problem. After a brief period of individual grappling, the participant whom the researchers called Ziv declared that he had answered the question, and that he did it in more than one way. Encouraged by the instructor, the boy presented one of the solutions. Yet, although Ziv's account appeared to the researchers clear and helpful, it was rejected by his classmates as incomprehensible. Explanations by another student, Dan, who also claimed to have a solution, appeared confusing and inconclusive. In spite of this, the students who previously complained about "not understanding Ziv", listened to Dan carefully and later claimed to have benefitted from his account. This event left the instructor perplexed. She was not able to figure out the reason why the students refused to learn from a knowledgeable classmate, but were eager to seek help of the one who clearly experienced difficulties not much different from their own. At that day, she left the following note in her journal:

> Although nobody seemed to doubt the correctness of Ziv's solution, no visible effort was made to find out what his proposal was all about. Nothing indicated an interest in Ziv's explanation… On the other hand, the students seemed eager to learn from Dan, who himself was struggling for understanding, and who offered ideas that seemed too blurred to be truly helpful… Unimpressed by [Ziv's] solution .... the students let the obvious opportunity for learning slip away."

It was only in later analyses that the researchers were able to account for what happened. While scrutinizing the classroom talk, they noticed a feature of which they were previously unaware: an undercurrent of intensive subjectifying was going on within what might appear to be just a regular mathematical conversation. If we remained unaware of this fact, it was because subjectifying utterances, when interjected into strenuous mathematical debates, tend to be ignored. If we were able to do some work on them now, it was because prior to the analysis, we systematically extracted them from their context and collected them together in a single table. Here, they were segregated according to their authors and to the persons about whom they spoke.

The result was startling. The majority of subjectifying utterances turned out to be about Ziv. Whether addressed to him or to another group member, whether made by himself or by another participant, these utterances were evidently evoked by the teacher's decisions and moves. Indeed, acting as the conversation coordinator, she never missed an opportunity to show her confidence in Ziv's ability to enlighten his classmates. The teacher expressed this belief in many different ways: by repeatedly urging Ziv to present his solutions ("Until now, you haven't told us what you have understood from this question" [266]), by exhorting others students to listen ("Dan listen to Ziv now" [383]), and by explicitly assessing Ziv's superior ability to understand the problem ("[Y]ou're the only one who understood [the question]"[99]). Through these and similar subjectifying actions the teacher, imperceptibly to herself, was gradually building Ziv's identity as mathematically versed and as the discourse leader. In an indirect way, these subjectifying moves identified the rest of the group as somehow inferior. Not surprisingly, Ziv's classmate reacted hostilely, trying to deny the

power evidently ceded to Ziv by the instructor. Beginning with angry claims about not understanding what he was saying ("You're never understood" [556]), through objections to his alleged intention to show his advantage and act as their teacher (see Dan's exclamation "Ziv, you won't be a teacher" [678], and one girl's complaint to the teacher/researcher: "He just… he talks to me like I'm his [little] girl!" [704]). Ziv reciprocated with explicit reinforcement for the story of his superiority (see his utterance directed at one of the girls: "I'm smarter than you, Idit" [471]). With this mutually aggravated subjectifying ping-pong going on and on, and with the identity-building activity high on everybody's agenda, Ziv evidently stood little chance to play the role of the leader.

The analysis opened the teacher's eyes to these "identity struggles" and made her aware of her own central role in the plot. In hindsight, she expressed her regret:

> [T]he conundrum of the children's tendency to learn from a less competent classmate ... seems to have been solved: the student who could [deal with] the problem was denied the identity of discourse leader… I am [now] able to see things of which, in real time, I was [unaware]. Above all, I realized that my role in the students' learning was more harmful than helpful. [I] took part in [constructing Ziv's identity] just like anybody else in this classroom. In fact, my role in this process was probably most central .... It is therefore even more regrettable that I acted the way I did, constructing students' identities unreflectively, rarely giving my [utterances] a second thought.

Were this insight gained in real time, the teacher would have probably curbed this subjectifying discourse. If the latter did not happen, it was mainly because she clearly remained oblivious to the fact that while trying to advance the mathematizing and repeatedly encouraging Ziv to share his solutions with the classmates, she was also constructing the boy's first- and third-person identities. She saw herself as preoccupied exclusively with the mathematizing discourse, whereas the students perceived her as performing the task of telling them who they were, and thus as engaged in subjectifying discourse. These two differing visions and the resulting discursive gap are summarized in Table 6.

Table 6
Discourses and routines in Example III

	Teacher (performer)	Students (interpreters)
discourse	mathematizing	subjectifying
task	scaffolding students' problem solving "by proxy"	building Ziv's (and other students') identity
procedure	inviting Ziv to present his solutions, exhorting the class to listen to Ziv, evaluating Ziv's understanding	

Example IV: The danger of modelling a discourse other than intended

The last example has shown how a gap between the teacher's own and her students' perception of her discourse may result in the teacher's involuntary participation in a harmful subjectifying activity. In the next example, we will see how a similar discursive gap can lead to the teacher's unconscious support for a wrong type of mathematizing.

While saying "the wrong type of mathematizing" I mean mathematical discourse different in its character from the one the teacher herself intended. Thus, for instance, the teacher may believe she is trying to usher her students to *explorative* mathematizing while, in fact, the way she teaches supports *ritualistic* participation. Indeed, most teachers are likely to wish their students to see themselves as engaged in mathematical *explorations*, that is, in the activity of telling potentially useful stories about mathematical objects. As it often

happens, however, the teachers' own way of acting may push their students toward rituals, that is, can make the learners believe their task is merely to show a mastery of mathematical procedures. In this later case, they feel exempted from worrying about the question of what the outcomes of their performances may be good for.

These differing views of the purpose of mathematizing are rarely introduced to the students in the direct manner. Rather, they are signaled by the teacher's discursive moves, especially those finest ones, which are also least noticeable. Among the most effective shapers of the students' interpretations is the teacher's language. Let me illustrate this claim with the example presented in Table 7, in which the teacher who participated in a recent study on teaching algebra in high school (Adler & Sfard, 2018) introduces his class to the process of solving the quadratic equation $(\mathrm{x}-2)(\mathrm{x}+2)=0$.

Table 7
Example IV: Solving $(x-2)(x+2)=0$

#	Speaker	What was said	What was done
1	Teacher:	We want to solve for x. What is our x equal to?	Writes: $(x-2)(x+2)=0$
2	Learners:	……	The learners remain silent
3	Teacher:	We are saying any of these brackets is equal to 0.	
4	Teacher:	So we are saying $x-2$ is equal to 0… OR… $x+2$ is equal to 0	While saying this, I would be writing on the board: "$x-2=0$ or $x+2=0$"
5	Teacher:	And then we transpose them. x is equal to?	
6	Learners:	2… or x is equal to -2	As the learners are saying this, the teacher writes on the board: "$x=2$ or $x=-2$"

Let us scrutinize the teacher's utterances for the objects he is talking about. Note, in particular, that the sentences "We want to solve for x" ([1]), "We are saying any of these brackets is equal to 0" ([3]), "And then we transpose them" ([5]) speak about people's actions (*solve*, *transpose*) with symbols (*x*, *brackets*). Within this context, it is justified to claim that also numerals such as '2' and propositions such as 'x=2' are considered as mere symbols, standing for nothing but themselves. This way of speaking supports ritualization, if only because of the fact that the result of symbolic manipulations seems to be of no further use and thus the performance is the only thing that counts.

To create a proper opportunity for the kind of learning that the teacher believed himself to be promoting, he should have exposed the students to explorative discourse. He would have done better if he reduced talking in terms of symbolic operations and spoke as much as possible in terms of mathematical objects, such as numbers or functions.[5] Thus, in utterance [1], instead of talking about "solving for x", he could have asked about the relevant relations between numbers: "What are the numbers x that, if substituted for x will make the product

[5] The difference between symbols and the mathematical objects is that the objects may remain the same while symbols change. Thus, the number two remains the same whether we refer to it with the symbol '2' (Arabic numeral) or II (Roman numeral), or $^{16}/_{8}$.

of x+2 and x-2 equal to 0?" Alternatively, he could have inquired about a property of a function: "For which numbers x the value of the function y=(x+2)(x-2) is equal to 0?" Utterance [3] that speaks about brackets might have been replaced with a proposition on numbers: "Any of the numbers x+2 and x-2 must be equal to 0". Finally, rather than using the cryptic verb "transpose", implying a physical action, such as rearranging symbols, he could have said, "We subtract 2 from [the numbers/functions on] both sides of the equation". The common feature of all these replacements is that they define the task by specifying the required properties of the outcome. Clearly, this stress on the product signals the legitimacy of *any* procedure that would lead to the required result and as such, ushers the problem solver into explorative discourse.

Many other properties of teachers' discursive actions are likely to encourage students' ritualistic participation[6], but in the present context, I chose to focus on those of them that hide in moves so tiny as to being imperceptible either to the students or to the teacher himself. The differences between the routines of the explorative discourse the teacher saw himself as performing and those of the ritualized discourse his students were likely to perceive are summarized in Table 8.

Table 8
Discourses and routines in Example IV

	The teacher performs	The students see
discourse	explorative mathematizing	ritualized mathematizing
task	demonstrate how to attain mathematical outcomes	demonstrate how to perform mathematical procedures
procedure	discuss the required outcome and perform a number of procedures that lead to this outcome	perform a single procedure repeatedly, giving tips for remembering how it should be done

Summary and conclusions: Why teachers should remain alert to the possibility of communicational gaps

Both examples in this section make a strong case for the teacher's awareness of the possibility of a gap between what she thinks she is doing and what her students actually see. This awareness is important because such gaps may mean that what her students learn is not what she tried to teach them. More specifically, the teacher may find herself collaborating in shaping unwanted, potentially harmful identities, while also introducing the students to mathematical discourse she herself does not appreciate. While in the classroom, therefore, the teacher must keep in mind that any of her moves may be read by the learners as saying something about themselves, if only implicitly; and she has to remember that when it comes to the question of what kind of mathematics the learner experiences, the answer is not so much in general didactic principles or even in detailed lesson plans, as in the finest details of the implementation (Sfard, 2018, p. 124).

[6] For instance, the learner's ideas about the source of mathematical narratives depend, to considerable extent, on *what* the teachers say, and to an even greater extent, on *how* they say it. Thus, the teacher who frequently appeals to the students' memory, who accepts his role as the ultimate judge of correctness and who rarely has recourse to a careful deductive derivation is likely to give rise to the students' conviction about an arbitrary nature of mathematical discourse and of its products.

Discursive gaps as the researcher's opportunities for learning about learning

Whereas both teachers and students have good reasons to be apprehensive of discursive gaps, researchers are more likely to see those gaps as gates to hidden treasures. As could already been understood from the first two examples, valuable insights about learning can be gained from close analyses of the nature of different discursive gaps and of the circumstances that occasion their appearance. In this section, I look at yet another case, in which the occurrence of a gap becomes an opportunity for learning about ways in which people match task-situations with discourses.

Example V: Opportunity to learn about student's ways to choose precedent

The example to be presented now may help researchers in identifying those aspects of task-situations that can be held responsible for students' choices of discourses in which to react to given task-situations. Some relevant insights could already be gained from Example I, where the learner was primed by the formulation of the problem, and more specifically, by words and symbols such as 'times', 'of' or multiplication sign. The new example will show again that two task-situations considered by one person as defining the same task may be seen by another as calling for different routines. This time, however, with the wording of the task-generating question remaining constant, the role of precedent-indicators will be played by contextual factors.

The data to be considered now come from a study conducted in two 7th-grade classes, of 36 students each. The students were presented with the mathematical problem: "Four children shared 14 balloons. How many balloons did every child get?" The two classes could be considered as indistinguishable in terms of the history of their mathematical learning and their achievement, and the only difference between them was that one was asked to solve the Balloons problem during mathematics lesson and the other – during a language lesson. The results can be seen in Table 9.

Table 9
Example V: Students' responses to the Balloons task

Response	Frequency	
	Mathematics lesson (N=36)	Language lesson (N=36)
"3.5"	46%	14%
"The children got 4 and two others got 3 balloons" "Each child got 3 balloons and 2 were left"	50%	80%
NA	4%	6%

As can be seen, the results obtained in the two classes are quite different. During mathematics lesson, almost half of the students responded with the non-integer number 3.5 that could not possibly constitute an answer to the question of the number of balloons. These participants clearly identified the task as a "word problem", the type of problem frequently encountered by every mathematics learner. The procedure they used was the one they often used in this context: finding and implementing the arithmetic operation that seemed to fit the question. In the present case, the division was probably chosen because of the word "sharing" appearing in the statement of the problem. In the other class, this improbable response was given by the mere 14% of the students. The majority of answers seemed to indicate that here,

just like in Example I, the children saw it as their task to perform the everyday routine of fair sharing that they often had to perform in their everyday life. Thus, whereas in Example I the difference in the choice of discourse and, in result, in the solution routine stemmed from lexical differences, in this example the decisive factor was the context in which the question was stated. For a summary of this analysis see Table 10.

Table 10
Discourses and routines in Example V

	In mathematics lesson	In language lesson
discourse	everyday	of school mathematics
task	sharing the balloons fairly between children	perform a learned operation that fits the situation
procedure	1. Give a balloon to each child 2. Repeat as long as you can	1. Find the most appropriate operation ("share" → division) 2. Perform the operation

To sum the insight that can be gained from this example, our ability to act in most situations in which we find ourselves stems from our tendency to automatically associate each such situation with a certain discourse and with its routines. What prompts these association are such characteristics of the situation as the physical components of the given space (e.g., a typical classroom arrangement) or the identity of the individuals who populate the scene (e.g., mathematics teacher). The very exposure to these identifiers may suffice to push us into the discourse we encountered under the same or similar circumstances in the past. In Example V, the association with mathematical discourse learned at school was brought by the students' awareness of their being in mathematics lesson, maybe even by the very presence of the mathematics teacher. If the language lesson did not lead to a similar choice, it was simply because mathematical discourse had never been used in this context.

Example VI: Opportunity for replacing the "deficit model"

The example that follows shows how the researcher's unawareness of a discursive gap between her and participants of her study may stymie her ability to tell a truly useful story of the phenomena she tries to fathom.

Let us consider the conversation between 4-year old Roni, 4 years and 7 months old Eynat, and Roni's mother, as presented in Table 11. The excerpt is taken from a study on children's numerical thinking conducted years ago by Roni's mother, who was also the beginning researcher, and myself (Sfard & Lavie, 2005). The conversation was held in Hebrew (in its English version, presented here, we tried to preserve idiosyncrasies of the children's language). At the time of our investigations, Roni and Eynat were already quite proficient in counting and were routinely answering the "How many?" question without a glitch. The episode began when the mother presented the girls with two identical opaque boxes. Even though the girls they could not see the contents, they knew they boxes contained marbles. On the face of it, nothing new can be learned from this example. After all, the first thing one usually learns from books and articles about early numerical thinking is that "children who know how to count may not use counting to compare sets with respect to number" (Nunes & Bryant, 1996, p. 35). Yet, at a closer look, some of Roni's and Eynat's actions did appear puzzling. If a person was listening to the conversation without seeing the boxes, she would have been likely to conclude that the children implemented the task

properly: they gave an agreed answer and knew how to justify it in a logical way (see utterances [5], [7], [9]). But for those who could actually see what was happening, the girls' decisive responses were difficult to account for. Indeed, why did the children choose a particular box? Why did they experience no difficulty in making a joint decision? Why, in the end, were they able to respond in a seeming reasonable way to the request for substantiation, even though there was no basis for the claims they made about the size of the collections?

Table 11
Example VI: Where are there more marbles?

#	Speaker	What is said	What is done
3a	Mother:	Right, there are marbles in the boxes. I want you to tell me in which box there are more marbles	
3b	Eynat:		Points to the box which is closer to her
3c	Roni:		Points to the same box.
4	Mother:	In this one? How do you know?	
5	Roni:	Because this is the biggest than this one. It is the most.	
6	Mother:	Eynat, how do you know?	
7	Eynat:	Because… cause it is more huge than that.	
8	Mother:	Yes? Roni, what do you say?	
9	Roni:	That this is also more huge than this.	

After long deliberations and a scrutiny of children's actions in this and similar episodes, we concluded that it was the language used in the description of the case that produced our puzzlement. Indeed, while stating that children do "not use counting *to compare sets with respect to number*" (emphasis added), the researchers attribute to children their own interpretation of the question "Where are there more marbles?" If so, there is little wonder they view children's actions as suffering from a certain deficit: the girls did have the necessary skill but they were unable or unwilling to use it the way they, the researchers, would have used it themselves in the same task-situation. The story of the deficit loses grounds, however, when one realizes that Roni and Eynat did not necessarily interpret the question "Where are there more?" as requiring quantitative comparison. Indeed, having freed oneself from the assumption, one realizes that, perhaps, the children simply tried to choose the box that they *preferred*. As implied by previous studies (see e.g., Walkerdine, 1988), rather than interpreting the word 'more' as referring to quantitative advantage, they were likely to understand it as referring to whatever could count as better, for one reason or another. In sum, we understood that there was a gap between the children's and grownups' visons of the task, and thus between their respective discourses and routines. These differences are summarized in Table 12.

Table 12
Discourses and routines in Example VI

	Interviewer	4 year old children
discourse	quantitative, numerical	of choosing for oneself
task	identify the box that has more marbles	choose (together?) the box you (both?) prefer
procedure	1. Count marbles in each box 2. Compare the last number words obtained in B	Point to, or take, the one you prefer (possibly: trying to agree with your friend)

The insight gained in this event had a lasting impact on our later work. From now on, we have been avoiding telling stories on what children did not do and, instead, have been documenting what they actually did. The sentence "children who know how to count may not use counting to compare sets with respect to number" has now been reformulated in our reports as "Children who know how to count, when asked 'Where is there more?', are likely to make a choice without counting".

The importance of the lesson that can be learned from this example by both teachers and researchers cannot be overestimated. When students seem to err, we tend to assume that the error is due to their insufficient mastery of procedures. It occurs to us only rarely, if ever, that the apparent mistake may result from a difference between the task the learners try to perform and the one intended by the task-setter. Yet, what we saw in this example alerts us to the fact that when a routine develops, transformations in the students' vision of the task may be at least as significant as the gradual increase in these students' mastery of procedures. To do their job properly, those who teach and those who investigate learning must bracket their own mathematical discourse. They should always try to present the one's performance as it was seen by the performer herself. This is the only way to disrupt the long tradition of portraying the learning of mathematics as a process of overcoming lingering deficit. To begin picturing learning as a series of creative advancements towards an ever greater complexity, the researcher must always remember that the journey to full-fledged participation in historically established mathematical discourse involves traversing multiple, possibly invisible discursive gaps.

Summary and conclusions: Wariness of communicational gaps as a protection against deficit model of learning

The two latest examples as well as some of the previous ones make it abundantly clear that researchers should embrace discursive gaps as opportunities for their own learning rather than just problems to solve. The first of these examples has shown how a recognized discursive gap becomes a window to inner workings of the process of learning. Through this window we had a close-up at the way people choose precedents to task-situations, and what we saw shed light on the phenomenon known as *situativity of learning* (Brown et al., 1989; Greeno, 1997; Lave, 1988). The second example brought a message about some hitherto unrecognized pitfalls, in which we often fall as researchers. Here, we saw how our own mathematical discourse may blind us to critically important aspects of children's activity, making us oblivious to the mechanisms of discourse development. It warns the researchers against relying on their own mathematical discourse while trying to make sense of what children are doing.

Coda

In this talk, I joined Wittgenstein in his "battle against the bewitchment of our intelligence by means of our language" (Wittgenstein, 1953/1967, p. 47). Diverse ways in which language may lead us astray have been illustrated with multiple examples. These examples were also used to show how important it is that all the parties to processes of teaching and learning, whether participants or observers, are always alert to the possibility of discursive gaps. The examples illustrated the claim that some of these gaps are inevitable. I argued that these ineluctable discursive discontinuities should be embraced as opportunities for learning. Those gaps that do little more than jeopardize learning – and my examples imply that these are not any less frequent than the useful ones – can and should be prevented. In all the cases, however, the devil hides in the tiniest details of interpersonal communication and our first task is to learn how to make the gaps visible. Unknown to the teacher, her basic communicational routines may constitute invisible crevices through which the prejudice enters the conversation on mathematical objects.

It would be naïve to think that the uneasy task of detecting and preventing or utilizing discursive pitfalls could be implemented without a deliberate effort. Echoing Michael Reddy, successful exchange "cannot happen spontaneously or of its own accord" (Reddy, 1979, p. 296). Remembering that "[h]uman communication will almost always go astray unless real energy is expended." (p. 295), we need to invest as much energy as possible in minding even those discursive gaps that at the moment remain invisible.

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Many pathways towards "Excellence" in Singapore mathematics education

Tin Lam Toh
National Institute of Education, Nanyang Technological University
<tinlam.toh@nie.edu.sg>

This paper presents a snapshot of Singapore's journey towards excellence in mathematics education by examining the role of the traditional notion of mathematics competition and other competitive activities. It could be seen using the context of mathematics competition that the notion of "excellence" has evolved over time. Excellence as a high standard for individuals to achieve or as a set of obstacles for individuals to pit against the norm has been gradually broadened to include excellence as an internal goal for an individual to achieve, and even excellence as a goal for the mathematics education landscape.

The Singapore Education System

Since the independence of Singapore in 1965, developing a robust education system has been the focus of the nation. Recognising that Singapore had no hinterland or natural resources, the young nation had since been striving towards building an efficient, universal education system to fulfil the role of economic development and social cohesion in Singapore beginning with the visionary leadership of the prime minister Mr Lee Kwan Yew (NTU President, 2015). The importance of education to Singapore has continued to be emphasised by the Singapore politicians. In 2001, the then prime minister, Mr Goh Chok Tong, during a Teachers' Day Rally on 31 August acknowledged that the "skills and resourcefulness of our people" are pivotal for the nation's survival.

The education system in Singapore has been recognized as of a high quality. A speech in 1999 by the Education Minister, Teo Chee Hean, that Singapore has "no failing schools, only good schools, and very good schools…" (a full speech is provided in Ang, 2006) is a testimony to this. Further, the performance of Singapore students in the Trends in International Mathematics and Science Study (TIMSS), the largest international comparative study of student achievement in the two subjects, and the Programme for International Student Assessment (PISA) is among the top of all the participating nations. The students' performance in these two international comparative studies is usually taken among the indicators of the quality of a nation's education system.

With regard to the performance of the top elite students, Singapore has also performed very well in the International Olympiads of the three sciences (Physics, Chemistry and Biology) and Informatics (Lee, n.d.). Beginning from 2011, Singapore has also emerged in the first ten positions in the International Mathematical Olympiad (IMO), according to the information provided in the official website of the IMO.

The stellar performance of students' in TIMSS and PISA, and the prestigious Olympiads, are signs of the ongoing pursuit of excellence in the Singapore education landscape. The term "excellence" is found in the Singapore education documents. For example, the term "excellence" is found in the Singapore school management system, which is known as the *School Excellence Model*. Schools are being empowered to develop themselves into "excellent schools" based on the appraisal system of the School Excellence Model. Such a school appraisal strategy has been shown to have a significant impact on student performance (e.g., Huang et al., 2019). To achieve professional excellence among practicing teachers, centres of Teaching and Learning Excellence were set up in 2015 to provide them

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with up-to-date professional development so that practicing teachers bring back to their respective schools up-to-date teaching and learning strategies to impact the quality of teaching and learning there (Academy of Singapore Teachers, n.d.). Prior to that, within each zone in Singapore, a Centre of Excellence for mathematics had been set up as a platform for promoting the professional growth of mathematics teachers in that zone (Chua, 2009).

Singapore's pursuit of excellence can be understood by the social-cultural context of Singapore. The Singapore society has been engineered to embrace "a pragmatic and competitive national paradigm grounded in economic rationalism" (Ang, 2006, p. 1). Lessons learnt along the road to the nation's independence and the nation's vulnerability as a nation without resources are two factors that shaped the development of a competitive mindset (Cooper, 2001; Lee, 1998 cited in Ang, 2006).

Much has been done in Singapore beyond the nation's visible pursuit of "excellence" in the Singapore education landscape discussed above. The nation has traversed a long journey in shaping its own definition of excellence of education at the various levels of the society. With reference to mathematics education in particular, how the notion of "excellence" has evolved is discussed in this paper using the illustration of mathematics competition.

A two-pronged approach to excellence: Grounds up and top down approaches

The word "excellence" can be roughly understood as "exceptionally good and of superior quality" (Lierse, 2018). Based on this notion of excellence, an excellent education system refers to one that is exceptionally good and of superior quality. This vague notion of excellence in education has been operationalised. According to the European Network of Education Councils (EUNEC), excellence in education should transcend the "quality control" or even the benchmarking of education systems to *identifying, developing and intensifying talents* within the education system (EUNEC, 2012).

Two lines of effort in the pursuit of excellence in Singapore mathematics education can be discerned: the approach to excellence from (1) the grounds up; and (2) the top down. The grounds up approach towards excellence in mathematics includes the efforts by educational institutions and professional bodies to identify and develop mathematical talents; the top down approach refers to policies that impact the systemic level in achieving excellence. In this paper, we focus the discussion primarily on the grounds up approach; and briefly discuss the top down approach. A detailed discussion of the latter has been presented in Toh (in press), hence will not be further elaborated in this paper.

The notion of "excellence" in the case of mathematics competitions

In discussing "excellence" in mathematics, the idea of competitive activities as opportunities to pit against the norms (Franks, 1996) is readily forthcoming to the mind. Mathematics competition is part of the grounds up approach initiated by the local mathematics community. It is recorded that the first national level mathematics competition in Singapore emerged prior to its independence in 1956 by the Singapore Mathematical Society, which was founded at that time. Note that the first IMO was first launched three years after that in 1959 in Romania. Following the launch of the first mathematics competition in Singapore and the IMO, various other mathematics competitions at the national and school levels have started in the decades that followed. The first mathematics competitions were organised for upper secondary and high school students. This age group was the target as it was the participating age group of students for the IMO. This

corresponded to identifying and nurturing of mathematical talents, and started a systematic process of identifying and developing talents for the IMO.

Subsequently, other competitions were organized for students of younger age groups at the primary levels. Not only that, the mathematics competitive activities also scaled out from the top elite group of students to the vast majority of the student population. A full description of the emergence of various mathematics competitions and their evolution can also be found in Toh (in press). Alongside identifying, developing and nurturing mathematical talents, the pursuit of excellence in identifying and nurturing talents had also broadened to include selecting potential students from other Singapore mainstream schools. The evolution of the mathematics competitions in Singapore can be traced to at least three phases: (1) identifying and nurturing mathematical talents; (2) popularizing mathematics among a wider student population beyond potential competition contestants; and (3) aligning to the Singapore mathematics education.

Phase 1: Identifying and nurturing mathematical talents

This phase began with the first mathematics competition organized by the Singapore Mathematical Society in 1956, to around the early 1990s. In this phase, the key objective of identifying and nurturing mathematical talents could be seen as aligning to the selection of the best among the mathematical talents to represent the nation in the IMO and other prestigious international mathematics competitions. This phase corresponded to the pursuit of excellence as reaching the highest possible standard in mathematics.

Phase 2: Popularizing mathematics among a wider student population beyond potential competition contestants

This phase approximately corresponded to the period from 1990 to 2010. Starting from 1990, mathematics competition of the primary school students was launched and in 1994, the Singapore Mathematics Olympiad (SMO), the most prestigious mathematics competition at the national level, launched the Junior Section for lower secondary students in addition to the usual Senior Section (for upper secondary students) and Open Section (for the pre-university students).

Phase 2 was characterised by the effort of the mathematics community to popularize mathematics to a much wider student population, in addition to identifying and nurturing mathematical talents. In 1994, in the collection of challenging mathematics problems collated from the various interschool and national mathematics competitions published by the Singapore Mathematical Society, it was stated that the objective of the collection of problems was to "inspire in its readers the desire to learn more about mathematics" (Singapore Mathematical Society, 1994, p. ii). Various compilation of competition questions for different student levels were subsequently published with the objective to "stimulate interest and develop prowess in mathematics among students in the primary schools of Singapore" (The Chinese High School, 2003, p. ii), or to "instil a love for and to generate interest in Mathematics amongst Primary school students" (National University of Singapore High School of Math & Science, 2007, p. i). This phase showed a broadened notion of excellence as individualised; reaching an individualised peak of excellence is a worthy goal.

Phase 3: aligning to the Singapore mathematics education

The third phase began in the early 2010s, and this phase was characterised by a conscious effort of the mathematics communities in aligning the mathematics competition to the school

mathematics curriculum, in addition to the objectives of the previous two phases. In the preface of the compilation of the past year SMO questions, the compilers commented that "We align the SMO more closely to the school curriculum … there will be a considerable number of questions in Round 1 [the section that all contestants will attempt] of each section which are based on the school curriculum…" (Ku et al., 2016, 2017, 2018, p. ii). The mathematics competition questions no longer exclusively contained the extremely challenging questions which are beyond the reach of the general student population. A considerable number of the mathematics competition questions were based on the contemporary school mathematics curriculum, although many of these questions require a creative use of the mathematical techniques taught in school mathematics. The subtle difference between Phases 2 and 3 is that while both phases saw a similar effort to reach out to a wider range of students, there was a visible effort to align to the school mathematics curriculum in Phase 3, thereby possibly impacting the classroom mathematics instruction. The notion of excellence in this phase has expanded beyond individual peak of excellence, to encompass excellence in the teaching and learning processes for all teachers and students.

Mathematics competition questions beyond competition

As discussed above, in Phase 3, the link between mathematics competitions and the school mathematics curriculum has become explicit. The intention of the local mathematics communities to align the prestigious mathematics competitions to the local school mathematics syllabuses had enlarged the functions of the mathematics competition questions. More competition questions were then made accessible and were being accessed by the general student population. A larger student population had then the opportunity to challenge themselves with the mathematics competition questions which were within their capacity, and to reflect on the school mathematics content that they have learnt.

Mathematics competition questions have also been valued because of the affordances of these items in the preservation of the "old" mathematical techniques within the contemporary mathematics syllabuses. These techniques have been de-emphasised in the curriculum due to an increased emphasis on technology in the school curriculum (Toh, 2015). Many of the problems that require these "old" mathematical techniques epitomise a high degree of creativity in the use of more delicate mathematical techniques (without resorting to technology). This is still relevant to the Singapore mathematics curriculum, which emphasises mathematical problem solving. Illustrations 1 and 2 are exemplars of this category of problems, which could serve to motivate more students to acquire creative mathematical techniques for the mathematical content which is found in the current syllabuses and appreciate the nature and beauty of mathematics.

Illustration 1: Simplify $144\left(\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}\right)$.

(A modified item from a typical genre of the SMO questions on simplifying surds without the use of calculating tools)

Illustration 2: Which of the following numbers is largest?

(A) $\sqrt{10}-\sqrt{9}$
(B) $\sqrt{20}-\sqrt{19}$
(C) $\sqrt{30}-\sqrt{29}$
(D) $\sqrt{40}-\sqrt{39}$
(E) $\sqrt{50}-\sqrt{49}$

(A modified item from a typical genre of questions on comparing the magnitude of surds without the use of calculating tools)

The solution of Illustration 1 can be obtained indirectly by considering the square of the given expression. A careful application of the rules of surds will result in a perfect square, for which the square root of the square number yields the answer. Illustration 2 can be solved by considering the process of irrationalising each of the five surdic expressions, and comparing the five fractions which have equal numerator. Such problem solving strategies which lead to elegant solutions are not stressed in the mainstream curriculum, as the use of calculating tools renders such strategies unnecessary. This is further hindered by the provision of calculators for all high-stake national mathematics examinations.

Other competition questions engage the solvers to think more deeply and reflect on the usual misconceptions that students have in applying algorithmic procedures (exemplified by Illustrations 3 and 4 below). Such items are atypical of high-stake national examinations. Illustration 3 challenges the solver to re-think their usual understanding of solving an algebraic equation in relation to the process of like-terms in both sides of an equation. This makes them re-think of the equivalence of the two equations, and easily relates to the big idea of Equivalence in mathematics. Illustration 4 invites the solver to examine the common misconception that $\sqrt{a^2} = a$ for all real values of a. The preservation of such items within the existing mathematics competitions is an indicator of the effort to emphasise the metacognitive aspect of problem solving, which is stressed in the syllabuses.

Illustration 3: How many real numbers x satisfy the equation $\frac{x^2-x-6}{x^2-7x-1} = \frac{x^2-x-6}{2x^2+x+15}$?

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Illustration 4: Let $a < 0$. Find $\sqrt{a^2} + \sqrt{(1-a)^2}$.

(A) 1 (B) -1 (C) $2a - 1$ (D) $1 - 2a$ (E) None

Some mathematicians lament that the mathematics curriculum today is far from the level of difficulty of that in the 1980s (e.g., France & Andzans, 2008). The various mathematics competitions, with their unofficial "syllabuses" for the competition and the lack of provision of allowing calculating devices, serve to preserve many of the elegant mathematical content which were otherwise not emphasised in the contemporary syllabuses. With the trend of increasing student participation in the various local mathematics competitions, many of these mathematical questions with elegant solutions are kept alive but are downplayed in the mainstream school curriculum.

A further step to popularize competition-type of mathematics problems is found in the contemporary mathematics textbooks which have been approved by the Singapore Ministry of Education (MoE) for schools. Under the paradigm of differentiated instruction, the inclusion of tiered practice tasks in the textbooks has resulted in the inclusion of many of such competition-type questions. The ready availability of such questions, usually classified under the section "challenging questions" (or similar classification of tasks to the same effect), is a further step to engage all students to challenge themselves in higher level mathematical thinking. This is especially important for the students who might not participate in mathematics competitions.

The notion of excellence in mathematics competition has also expanded to influence professional development of mathematics teachers as well. From the author's first-hand experience in working directly with practicing teachers in the Singapore schools in several of the teacher professional development activities, many of the challenging mathematics

competition questions have provided opportunity for teachers to identify the "blind spots" in their own knowledge of mathematics. It is common knowledge that mathematical content knowledge which is not frequently tested in the high-stake national exams tends to be out of a teacher's attention. The occurrence of such items in the various mathematics competitions could also bring a teacher to reflect on the content essential for classroom teaching. Some of these items have been incorporated into professional development courses for teachers. We consider one example in the Singapore Additional Mathematics syllabus using illustration 5 below, which is an item adapted from a past competition question (year unidentified). This item brought out several interesting discussions among the author and some secondary school teachers about logarithms.

Illustration 5: Find the value of $9^{2\log_9 5}$ without the use of calculator
(Adapted from a past year competition question in Singapore Mathematical Olympiad)

Although the following rule of logarithm is common knowledge for most students and teachers,

$$\log_a a^x = x$$

this rule is usually understood by most teachers and students in the usual computational sense as a procedural rule:

$$\log_a a^x = x \log_a a = x.$$

The following rule, which is a counterpart of the above rule of logarithm,

$$a^{\log_a x} = x$$

is less well-known among students and teachers. Although both rules involve the composition of a function and its inverse (i.e., the exponential function and the logarithmic function), the first rule can be easily algorithmised as "shifting the power of a logarithm down" while it is recognisably more difficult to proceduralise the second rule. The occurrence of items such as Illustration 5 reminds the teachers of the importance of the notion of the composition of a function and its inverse, rather than a pure utility of logarithms as a tool for conversion to exponential function (Kenny et al., 2013). This is an important alert to teachers that the concept of function underpins most mathematical concepts in the syllabuses, although explicit knowledge of functions and their composition are not required for the national examinations in the secondary school mathematics syllabuses (MoE, 2018).

Mathematics competition questions and problem solving

A further stage in utilizing the mathematics competition questions is in adapting them for teaching mathematical problem solving to *all* secondary mathematics students (that is, problem solving is not only reserved for the elite few, but for the whole student population). As it is well-known, mathematical problem solving is the heart of the Singapore mathematics curriculum. In New Zealand, Holton (2010) introduced mathematical problem solving processes to IMO students through imparting them the mathematical content knowledge on discrete mathematics. Motivated by this approach, a similar effort in mathematics education research in Singapore emerged in the late 2000s to the early 2010s.

The new interpretation of problem solving using the science practical paradigm (i.e. problem solving to mathematics is in the same way as science practical to science) in an effort to make problem solving accessible for *all* students, and to illustrate to teachers how an authentic problem solving lesson can be enacted in the mathematics classroom. Broadly speaking, problem solving lessons in mathematics should be treated as science practical

lessons in science, and the role of teachers is to facilitate the students' experience of the entire problem solving process (Toh et al., 2008). This initiative was introduced in recognition of the fact that most school mathematics teachers might not have taught students problem solving to the true sense of its spirit as proposed by Pólya (1945). This approach to teaching problem solving is contrary to many teachers' usual classroom practice in "routinizing the problems" into exercises for the students.

A detailed discussion on the conceptualisation of the science practical paradigm, proposal on how problem solving lessons could be enacted in the mathematics classrooms, and the reports of the various experiment schools about their successes and challenges in enacting a problem solving lesson have been discussed (Leong et al., 2013; Toh et al., 2008). In the problem solving lessons, authentic problems that could highlight the various problem solving stages must be selected as the vehicles for teaching problem solving. As such, competition questions become suitable choice of questions for the teaching of problem solving. Illustrations 6 and 7 appended below are two exemplars of competition-type questions which have been used for teaching authentic problem solving.

Illustration 6: Find the last digit of 13^{77}.

Illustration 7: Find the last digit of $1962^{2009} + 2009^{1962}$.

The content of the two exemplars above is on Elementary Number Theory, which is not taught in the Singapore school mathematics curriculum. As such, these problems will be "non-routine" to most students – one of the two criteria to qualify as a "problem" (Toh et al., 2008). However, the content of these two questions are easily understandable even for a primary school student. Hence, these problems can be used as authentic problems that can serve to reinforce and illuminate the various problem solving heuristics, and can "force" students to acquire problem solving processes (in this case, looking for patterns and making conjectures for illustration 6, and, in addition, looking for sub-goals in illustration 7). In short, this type of problems is realistic enough for students to experience authentic problem solving by experiencing all the Pólya stages of problem solving.

Mathematics Competitive Activities beyond the Traditional Competition

Mathematics competitive activities have transcended the confines of the common notion of paper-and-pencil tests by the traditionalists. Some talents in mathematics and high-achieving mathematics students may be more inclined towards other forms of competitive mathematics activities, such as collaborative problem solving activities involving real world problems, or engaging in authentic mathematics research with professional mathematicians, are among the competitive activities that are designed to capture the various talents in mathematics. The biennial event of the Singapore International Mathematical Challenge is organised to provide opportunity for students to work collaboratively with their peers in solving real-world problems by making use of available technological tools and information. To develop young research mathematicians, opportunities are provided for students to work on mathematics research projects with professional mathematicians beyond their schools. The annual Singapore Mathematics Project Festival is a platform for students to showcase the fruits of their research to their contemporaries and other mathematicians. More details of alternative competitive mathematics activities are described in Toh (in press) and will not be elaborated.

In an effort to engage an even wider spectrum of students in mathematics competitive activities, the Singapore Mathematical Society has initiated a new series of mathematics essay competition, an annual event that aims to expose the participating individuals to an identified mathematical topic and to encourage the participants to articulate mathematics through the exposition on the topic (Singapore Mathematical Society, 2021). This further widens the group of students who might not be inclined to the modes of competitive mathematical activities described previously. In addition to sharpening an individual's thinking and reasoning, this activity encourages the participants to communicate mathematics precisely, clearly and logically. It is aligned to the latest emphasis in the Singapore mathematics curriculum on communication in mathematics (Kaur & Toh, 2012).

Achieving Excellence at the Systemic Level

At the systemic level, the pursuit for "excellence" has transcended the notion of a unique peak of excellence understood by the traditionalists' view. The notion of excellence has now been interpreted as the existence of many peaks, and even a peak for each student, in order to encompass excellence for every individual. The systemic effort in the pursuit of excellence can be seen to be guided by the dual objectives of enabling students of different capacities to define and reach their own peak of excellence (Shanmugaratnam, 2006) and, "lifting the bottom but not capping achievements and limiting opportunities at the top…" (Ong, 2018).

The notion of not capping achievements and limit opportunities at the top is best epitomized by the education system in identifying and nurturing talents in various way, and depicts a concerted effort by the MoE in stretching excellence to the fullest potential among an individual. The holistically talented students are identified early at the upper primary level and offered an opportunity to the Gifted Education Programme within the Singapore education programme. This specialized programme for the gifted individuals (defined as individuals who form the top 1% of the top performing students) continues to be supported by school-based gifted education programme found in selected secondary schools.

Specialized schools have been set up for students who are specifically talented in a specific discipline. In particular, the NUS High School of Science and Mathematics has been specifically set up for students who are specifically inclined towards mathematics and sciences. In this specialized school, students are not bound by the high-stake national examinations at the end of the high school as the scope of the national exams capped the learning of the students. In addition, students in this school are given the opportunity to read a subject at the undergraduate level and to even do a research project at the higher secondary levels. Under the supervision by their teachers or mathematics professors, the research work carried out by the student approximates the research work of a professional mathematician.

Another movement in the Singapore education system to move towards stretching all students' potential to the fullest is the recent introduction of subject-based banding of mathematics (and three other subjects), with the ultimate goal of pushing for subject-based banding for all subjects at the primary and secondary school education. This movement can be seen to be modelled after the pre-university education system in which the students can read each academic subject at a level that is suitable for them. Under this opportunity, all students will have the opportunity to be stretched in all disciplines according to their capacity and inclination.

Conclusion

The journey towards excellence in education is best summarized by the speech of the then Minister of Education, Mr Heng Swee Keat, during his interview with the Straits Times on 22 August 2015. Mr Heng commented that the pursuit of excellence should be "part of Singapore's DNA", but stressed the need to "broaden the definition of excellence and to recognise everyone for achieving his personal best" (The Straits Times, August 22, 2015). Even within mathematics education, it is clearly evident that Singapore is moving towards "a mountain range of excellence, not just one peak, to inspire all our young to … climb as far as they can." (Shanmugaratnam, cited in Lee et al., 2008).

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“Becoming” a researcher in mathematics education

Vince Geiger
Institute for Learning Science and Teacher Education
<vincent.geiger@acu.edu.au>

As a contribution to the legacy of the Annual Clements/Foyster Lecture, this paper will focus on the theme of becoming a researcher in mathematics education – a fundamental endeavour for MERGA from its foundation. I use the term *becoming* in the socio-cultural sense, that is, how a person develops in their role as an active member of a community. This participation led to the development of an identity – in our circumstance, as mathematics education researchers. Thus, the presentation will not be a research lecture in the traditional sense but, rather, a personal reflection that maps the lived experience of defining my own research program against important junctures of development and a growing sense of *becoming* within the MERGA, and other, communities.

When invited to present the Clements/Foyster Lecture, my first thought was to talk about my research program in the teaching and learning of applications of mathematics – numeracy, mathematical modelling, STEM and the role of digital tools in these areas. This was perhaps a go-to-first gut reaction, and really, I have already spoken about my research in these areas during conferences reaching back to MERGA-16 in 1995. My second thought was that the invitation was an incredible honour, as this lecture was initiated “to honour the foresight of Ken Clements and John Foyster in founding MERGA” (Galbraith, 2014, p. 38). So, the question became, “how could I contribute to this legacy?” Or, to quote David Byrne of the Talking Heads:

> And you may say to yourself, “My God! What have I done?”
>
> …
>
> And you may ask yourself, “well…how did I get here?”

The answers to these questions are far from simple. My background is not one that predisposes an individual to an academic career and certainly not one that would lead me to the position of being one of the few research-only academics in mathematics education. As my beloved mother often remarks, “How does a boy from the working class become a professor?” My good friend, Tom Lowrie, has described me as an outlier in terms of career pathway and academic success (even if moderate success). I am acutely aware, also, that I have not done this on my own. There have been many hands holding me up and giants’ shoulders on which I have stood. Given this background and the outcome, it occurred to me that in seeking to understand how indeed “I got here”, I might be able to provide insights that point ways forward for others.

So, to answer these questions, I had to reflect on my own development as a researcher in mathematics education, my own *becoming* in the Jean Lave sense of the word, in that “mainly, people are *becoming* kinds of persons” (Lave, 1996, p. 157). This *becoming* spans transformations from student to teacher to researcher, in my case, via a circuitous route. In developing an analytical narrative of this transformation, I will draw on the approach of others who have engaged in the self-study of their own development as mathematics educators/researchers, by reflecting on my personal history through the lens of a theoretical framework (e.g., Krainer, 2008; Tzur, 2001). Given that my formation has been influenced and supported by different communities of teachers and scholars as I stepped into and out of

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different practices, I will adopt a socio-cultural perspective in describing and analysing the development of person-in-practice-in-person (Lerman, 2000).

Conceptual Framework

Studies on the origins of consciousness and knowledge acquisition have tended to focus on individual cognition and intellectual development. The 1980s, however, saw the emergence of theoretical frameworks that placed greater emphasis on the social origins of meaning, thinking and reasoning, a movement Steve Lerman referred to as a "turn to the social" (Lerman, 2000). The origin of such social theories is generally attributed to Vygotsky's (1978) work on child intellectual development. Central to Vygotsky's perspective on the process of intellectual development is the interaction between the learner and a more experienced other working within zones of proximal development (ZPD). The ZPD can be conceptualised as a set of possibilities for development that become actualised when learners interact with more knowledgeable people, for example teachers, and their learning environment. From this perspective, there can be no strict separation of an individual from his or her social environment (Luria et al., 1979), with cognitive development as an outcome of the process of acquiring culture. Thus, the individual and the social must be regarded as complementary elements of a single interacting system (Leont'ev, 1981).

An iconic work in socially oriented theories of learning that emerged at this time was Jean Lave's *Cognition in Practice* (1988). In this book, and later work (e.g., Lave, 1996), she challenged cognitivism and transfer theory in mathematics learning by identifying mathematical *practices* that were appropriated within professions, trades and everyday activities – ways of working and modes of thinking that were far more than the mere application of mathematics acquired from formal education. From her perspective, strategies and decision making associated with the use of mathematics were *situated* in, and products of, the social milieu in which they were employed.

Building on this work, Lave and Wenger (1991) describe learning as a form of apprenticeship where novices are initiated into a learning community, or *community of practice*, through a process termed legitimate peripheral participation. Experts or more knowledgeable peers are responsible for the induction of individuals into the culture of a community, including beliefs, values, modes of discourse, and means and methods of knowledge creation. Judgments about learning are therefore based on the increased range of participation of the learner within the community. Through this participation, an individual moves from a novice towards mastery as part of who they are *becoming* within a community of practice.

> A community of practice is an intrinsic condition for the existence of knowledge, not least because it provides the interpretive support necessary for making sense of its heritage. Thus, participation in the cultural practice in which any knowledge exists is an epistemological principle of learning. The social structure of this practice, its power relations, and its conditions for legitimacy define possibilities for learning (i.e., for legitimate peripheral participation). (Lave & Wenger, 1991, p. 98)

From this perspective, knowledge must be understood relationally, between people, activity, and social contexts. *Becoming* is consequently the degree to which a participant adopts the values, modes of reasoning and discourse practices of a community of practice. This position is, therefore, a direct challenge to the notion that knowledge construction takes place by the transfer of decontextualised mental objects from one individual to another.

Building on his work with Lave, Wenger (1998) extended the notion of *becoming* to the formation of an individual's identity within a community of practice. As a consequence, an

individual's identity within a community is strongly influenced by their personal affiliation with its beliefs, values, modes reasoning and processes of knowledge production.

> Because learning transforms who we are and what we can do, it is an experience of identity. It is not just an accumulation of skills and information, but a process of becoming - to become a certain person or, conversely, to avoid becoming a certain person. Even the learning that we do entirely by ourselves contributes to making us into a specific kind of person. We accumulate skills and information, not in the abstract as ends in themselves, but in the service of an identity. (Wenger, 1998, p. 215)

Thus, the identity an individual establishes within a community of practice is dependent on how they act and interact with others – the role they play as part of a community. At the same time, this identity is influenced by the community itself and the individual's sense of belonging to a community. The relationship between the individual and the community is thus reflexive – one evolving with the other.

Lerman (2000), in offering a critique of these ideas as they relate to mathematics education, raises individuality and agency as issues not fully accommodated in the thinking of proponents of situated and social understandings of learning. How, for example, are they able to realise their own goals within an existing community of practice? In response to this dilemma, he makes the observation that a person's goals are already aligned with a community when they step into a practice because this is the reason they choose to become part of a community of practice. This means that not only is the person becoming in the practice but that the alignment of goals means that the practice is becoming in the person. Consequently, he suggests that the unit of analysis for socially orientated studies in mathematics education should be extended to person-in-practice-in-person.

Since this time, the "social-turn" in mathematics education has been extended to incorporate the role of cultural practices, institutional contexts, personal histories, beliefs and values in attempting to understand and describe interactions central to teaching and learning (Goos, 2014). Valsiner (1997), for instance, reconceptualised Vygotsky's zone of proximal development (ZPD), to include two additional *zones* that accommodate both the influence of social settings and the goals and actions of individuals – the zones of free movement (ZFM) and promoted action (ZPA). Within this new construct, the ZPD is a space that defines an individual's potential development, the ZFM was conceived as the ways in which an individual is permitted to act within a context, and the ZPA identifies the conditions within a situation that promote action.

While Valsiner's zone theory was conceived as a theory of child intellectual development, others have extended its use as a tool for understanding human development in other areas. Goos (2013), for example, has interpreted the notion of development more broadly:

> …I take ''development'' to mean more than the formation of higher mental functions in children; instead, it refers to the emergence of new domains of action and thinking and new cultural frameworks that organise a person's social and psychological functioning. (p. 523)

A broader perspective on Valsiner's zone theory has underpinned research within education including students as learners (e.g., Blanton et al., 2005), teachers as learners (e.g., Goos, 2014; Goos & Bennison, 2019; Geiger et al., 2017) and teachers' numeracy identities (e.g., Bennison, 2015). Goos and Bennison (2019) applied the principles of Valsiner's zone theory to the development of teacher educators, attempting to understand how they learn within contexts defined by opportunity and conditions. Through this work, insight was developed into how teacher educator identities develop, and how teacher educators' opportunities to learn can be improved. From this perspective, the ZPD represents the possibilities for development of teacher educators' knowledge and beliefs. This includes the

knowledge of mathematics and pedagogy for teaching, how new teaching practices are learned, and their beliefs about which teaching and learning practices are effective. Within the context of teacher educators' professional environment, the ZFM can be constructed as both external constraints and an individual's own interpretation of related limitations or affordances. Such affordance and constraints include curriculum and assessment requirements stipulated by professional accreditation authorities, access to teaching resources, and the beliefs and expectations of prospective teachers. A teacher educators' ZPD relates to how an individual's goals and actions can be promoted or inhibited by features of their environment or the actions of others, such as their peers or institutional leaders. In the case of teacher educators, a ZPA might include academic structures, recognised markers of career development and promotion, and access to accomplished mentors.

The ZPD, ZFM and ZPA form a complex that represents the dynamic interaction between possibilities and limitations. Teacher educators' learning is thus the interaction of their development potential and their interpretation of opportunities for, and constraints on, progressing professional goals.

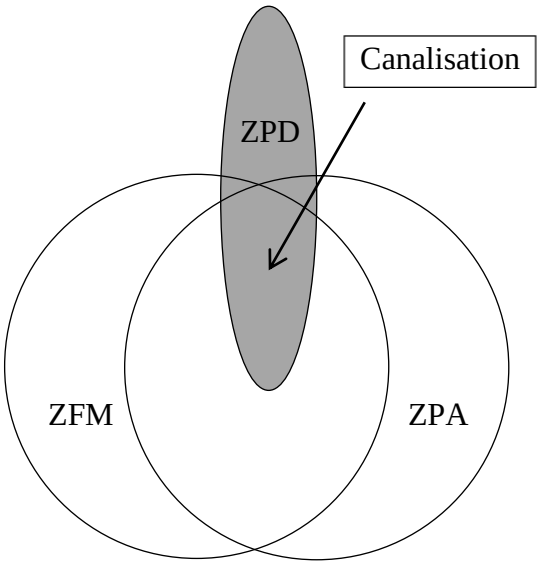

Figure 1. Canalisation of the ZPD

The influence of the intersection of the ZFM/ZPA (Figure 1), on what it is possible to promote within what is permitted, is known as the *canalisation of development* within the ZPD (Blanton et al., 2005; Oerter, 1992). Thus, canalization is how development is shaped under the dynamic influence of the ZFM & ZPA. This means that even though there are constraints individuals retain agency and are not just passive participants.

While Goos has referred to the development as teacher education researcher in her discussion of how teacher educators learn (e.g., Goos, 2008), and others have described the skills and attributes required by educational researchers (e.g., Boaler et al., 2003), there has been limited discussion specific to the development of researchers in mathematics educations across the span of a career. In the remaining part of this paper, I will attempt to provide some insight into this theory/practice gap by drawing on noteworthy junctures of my own development (or failure) as a researcher to speculate on how these contributed to

my understanding of personal and collective history, enculturation, and identity development as a researcher.

Origins

History precedes us. I was one of five children born into a working-class family in Brisbane. My father went to work as a flower boy immediately after finishing primary school, eventually finding an apprenticeship as a wood machinist, which he stuck with until retirement. My mother left school before the end of Year 6 and worked in a string of jobs until finding employment as a seamstress – something she still practices today for friends and family.

For whatever reason, I was good at school mathematics and science. Although they weren't quite sure of the implications, my parents encouraged this interest. I was the only child in the street who owned a microscope, a Christmas present during my primary school years. I'm sure I took on the role of suburb odd bod, sitting on the footpath studying whatever insects I could find to study. My parents' support came out of an understanding that the opportunities in life were afforded by education – my father was determined that all his children would complete Year 10! And so we did. I won a Commonwealth Scholarship which allowed me to go forward to Year 12, only one of five in my cohort. I was already exceeding my fathers' expectations! I remained good at mathematics, although I wasn't always the best student – there were too many other things to do, cricket, rugby and school parties! Towards the end of Year 12, my father had set up a job for me in a bank. This sounded fine to me, but a teacher contacted my parents to say I should consider going onto university. They weren't sure. No one in my extensive extended family (my maternal grandmother had 13 children) had ever done so and it was about time I started earning my keep. The teacher explained that there was a tertiary assistance scheme for those whose parents' combined income was below a particular threshold. We were well below. Always supportive, my parents sat me down and asked if I'd like to go to university. "I suppose so" I said. And so, I found myself enrolled in a Bachelor of Science in the second intake of the newly minted Griffith University.

I majored in physics and physical chemistry, with widely varying levels of achievement across the degree. I didn't quite get the game of tertiary study at that stage. But university provided me with the opportunity to take courses in the philosophy of science, something quite exotic for a boy from the working class. I found it fascinating. Seminars revolved around types of thinking I had never encountered before – especially discussions about how knowledge was generated. In *Conjectures and refutations: The growth of scientific* knowledge, Karl Poppers (1963) argued that knowledge is not simply discovered but developed through a process of conjecture and refutation. This set me back on my heels! This was followed by Thomas Kuhn's (1962) *Structure of Scientific Revolutions*, in which he outlined the paradigmatic nature of knowledge creation in science; an epiphany that set me on the path of fallibilism for life.

I had initially entered the course thinking I would complete a dual qualification – a Bachelor of Science and a Diploma of Teaching – but I had found the demands of tertiary study demanding and did not believe I had the discipline to continue for another year beyond the BSc. I completed the qualification and went looking for a job. I spent the next 18 months working on the line gangs for Telecom until I was successful with an application for a research assistant within the Department of Engineering at the University of Queensland – a dream job!

While it might be premature to think about the affordances and opportunities at this stage of my life, especially in terms of ZPD, ZPA, and ZFM, there are traces of this prism I can see from the distance of time. I had an interest in mathematics and scientific study from an early age (ZPD), which was supported by my parents, despite their limited educational opportunities (ZFM). They also provided me with resources to encourage my mathematical and scientific interests (ZPA). My opportunities in education far exceeded those of my parents as I was able to continue in school through to Year 12 and then to university (ZFM/ZPA complex). From this point, however, my belief that I could not study beyond a BSc was a self-imposed ZFM. My way forward was to create a ZPA that saw me on a trajectory from my current ZFM, working as a linesman for Telecom, to a role in a university that supported the research of others.

My identity, through this period of my life, was subject to constant change – but who was I *becoming*? I had been successful at school, but the cohort to which I belonged went their separate ways after graduation. My participation in academic life at university was not to a depth where I felt engaged enough in the community that I wished to continue. Yet, it would seem that my goals aligned with research work in universities sufficiently to step into a new community within an engineering department. Would this take hold?

Changing Course – a Teaching Career

I enjoyed the work in the Department of Engineering, most of which was focused on the building and testing of a wind tunnel. I was able to use a little of my mathematical and scientific capabilities but after 18 months and many hours sitting in front of a small, heated metal filament used to measure the characteristics of wind flow in the tunnel as part of the process of calibration, I began to think that, perhaps, there were more exciting ways to make a living. I had a number of friends who had completed their dual qualifications at Griffith University and were now teachers. They were enjoying the challenges of the profession, so I decided to join them – and enrolled in a Diploma of Education at the University of Queensland. During the course I met two people who would be very strong influences on my life, Marjorie Carss and Peter Galbraith, both of whom were teaching in the program.

I was introduced to the idea of pedagogy! I had entered the course believing that teaching was only a matter of telling or showing others how to do something. There was apparently much more to it! I was intrigued that there might be different approaches to teaching that should be implemented depending on the context – there was no single right way, an echo of fallibilism. During practicum I became aware of the complexity of the classroom and learnt that my best lessons were those that were approached as a problem to solve. I also came to understand that documenting what worked and what didn't (as stipulated by Marjorie and Peter), and reflecting on why, made a difference to the success of follow-up lessons. Then, before I thought I was ready (but whoever is), I had finished the course and been invited onto the staff at the school where I had completed my final practicum. It seems that the principal thought I had some potential.

In those first years, I experienced all the ups and downs that most early career teachers encounter. But I slowly established myself within the school as I worked on improving my teaching in mathematics, junior science, and physics. I was never left to my own devices as I could always depend on teaching colleagues for advice and there was also ongoing contact from Marjorie and Peter. It was Marjorie who convinced me to enrol in a Bachelor of Educational Studies (BEdSt). I didn't really know why this was a good thing to do, but Marjorie was so sure! After completing the BEdSt, there was a pincer movement from both

Marjorie and Peter that resulted in my enrolment in a Master of Educational Studies (MEdSt). It helped that all of these courses were free at the time.

After starting this course, however, I decided to spend some time in Europe – like many young Australians. I resigned from my position and headed to England. To make ends meet, I took up a supply teaching position at an inner-city London school at the turbulent time of school amalgamations. Schools at this place and time could be described as cheerful but violent. Students, in the main, came from low-income backgrounds, many with dysfunctional families. Few of the parents I met had aspirations for their children's education beyond finishing O-levels. Things were tough, students were difficult to manage, and staff were on occasion assaulted. Almost in contrast, the school was well-resourced with the quality and range of available teaching materials better than those I had access to in my first teaching post. These resources were aimed at developing mathematical competence alone, with little attention to how this might be applied to problems in students' own lives. Without engagement, however, little learning was possible. This experience helped me understand the outcome of disadvantage. It also convinced me of the need to teach mathematics and science in a way that connected with students' lived experience.

The European adventure concluded, and I returned to teaching in Australia. My MEdSt awaited me. I had formulated the idea for my thesis.

It would appear my ZPA was oriented towards a life related to learning and a connection to research was apparent through my employment as a research assistant in a Department of Engineering. Despite working "out-of-field" for a period of time, I was drawn to education, initially as a way of making a more interesting living, but I was open to changing the direction of my life. My ZFM was fashioned by people who became mentors. They provided advice and support that led to my development as a reflective practitioner and further study in education. I was fortunate to have the opportunity for further study without the deterrent of paying fees. My engagement with a preservice program in education, ongoing encouragement from mentors, and experience in the classroom in two different countries provided the impetus for ongoing professional learning – related to both my teaching practice and further formal education (ZPA).

I was being drawn into a community that I did not yet fully understand, but my goals seemed to be aligning. My identity had changed from that of research assistant to teacher. At the same time, a new identity was developing, that of educational researcher, evident in my enrolment in a Research Masters program. However, the identities of teacher and researcher were separate – teaching was my career and focus, while research was something I did out of interest. I was now participating, in a peripheral sense, in two communities of practice. Although there were overlaps, there were different modes of meaning making, reasoning, and knowledge generation to appropriate and reconcile. I can clearly remember being surprised at the differences between discourses as I negotiated my role in these different communities. What was my role as person-in-practice-in-person?

Teacher and Researcher

After returning to Australia, I was successful in an application for a teaching position in a significant city just outside of Brisbane – Ipswich. The school was not unlike that in which I had been worked in London, with many families suffering some form of disadvantage. A significant number of families were from various trouble spots throughout the world, with the students' parents moving to Australia to give their children a better life. The experience only confirmed my conviction that making mathematics relevant to students was key to their engagement and success. This time coincided with work on my MEdSt thesis, *A study of the*

mathematical problem-solving behaviours Year 11 students solving application problems, with my principal supervisor, Peter Galbraith. In this work I was searching for a way to provide students with the type of feedback they needed to improve the way they addressed problems in the real world through mathematics – consistent with my belief that students needed to find mathematics relevant to their lives. This was also a time of marriage and children; the thesis took an age to finish. And I thought I was done with further study!

It was around this time that Marjorie convinced me to attend my first MERGA conference, held in Brisbane in 1993. While I found some aspects of the conference interesting, it appeared to be a combative environment where egos were put on display with abandon. People argued about what I saw as minor points and few took the time to include me in discussions. I decided I would not attend again. An upside of the experience, however, was a presentation by a young researcher named Merrilyn Goos. I thought she made some sense – and she won an Early Career award as an outcome of the presentation!

Three years after returning to Australia, I was successful in securing a Head of Mathematics position at a new school. Marjorie Carss was also encouraging me to make a contribution to the work of mathematics teacher professional associations – first, editor of the Queensland Association of Mathematics Teachers (QAMT) journal and eventually president. As president of QAMT, I found myself as chair of the steering committee for a major national initiative – the National Professional Development Program (NPDP) aimed at improving teaching and learning in Australian schools - a daunting experience for someone with no experience in leading state-wide initiatives. I was also contributing to state-wide committees related to curriculum development and assessment. Marjorie continued to provide advice about how I should shape my career – and I found myself as president of the Australian Association of Mathematics Teachers! There was enough to do. Life was busy and I had a clear direction. I thought I had liberated myself from the demands of further study…but then I was dragged back again!

Peter Galbraith rang. He said there was a young researcher he thought I would enjoy talking to. At that time, I had developed a somewhat cynical attitude toward educational researchers. There had been a number of visits to see what we were doing in our school's Mathematics Department – it had gained some notoriety in the state. They had typically harvested data and left, never to be heard of again. This engagement felt like I was putting in significant effort with no return. But because it was Peter, I agreed to take a call. Some days later, the call came…"Hello, my name is Merrilyn Goos. Peter Galbraith said we should talk". So we did. Merrilyn had begun a PhD study in which she was recruiting secondary school teachers for a project in mathematical problem solving and metacognition. Merrilyn talked with such enthusiasm about her research that I was convinced (with some reluctance) to participate in the study. This began a series of nearly weekly visits from Merrilyn over a period of close to three years.

Merrilyn was different to other researchers I had encountered previously – she was genuinely interested in what I had to say, regarding research as a joint venture with teachers and not something that was done to them. After observation sessions, Merrilyn was never critical, she merely wanted to know why I had taken particular approaches to instruction. I had been a reflective practitioner for some time, but this was an extra pair of eyes that helped me go deeper into the reasons that underpinned my classroom decision-making. In these circumstances, having a researcher in the room was not a burden – it was a serious advantage! What I hadn't realised when agreeing to participate, was that the research was part of an ARC award that Peter Galbraith, Merrilyn, and others had secured. This meant there were publications to be generated! Consistent with Merrilyn's approach to

researcher/teacher collaboration, I was invited to join the writing team in instances when data had been collected from my classroom. Some of this early work (e.g., Goos et al., 2000; Goos et al., 2003), related to the affordances and constraints of technology in promoting collaborative problem solving, remains some of my most highly cited.

I have previously described a pincer movement that had saw me return to study and research in education. This time Merrilyn had established a foundation on one flank by drawing me into her research and co-authorship, while Peter made advances from the other. He had been pleased with the quality of my MEdSt and encouraged me to develop a MERGA paper and to nominate for the Practical Implications Award (PIA). Peter provided advice through rounds of drafting and redrafting, and then off it went. It was successful! Merrilyn meanwhile had insisted I co-present with her at the next MERGA conference. It seemed I had no choice by this stage, and so I had to find a way there – MERGA 18 held in Darwin in 1995. In the PIA paper (Geiger, 1995), I presented a framework for providing feedback to students engaging with applications of mathematics to real world problems. The paper I presented with Merrilyn (Goos & Geiger, 1995) reported on a case study of metacognitive activity and collaborative interactions in a mathematics classroom – my classroom. I can't say I was hooked, but I could see no way out.

It would seem my ZPD was expanding, firstly though Marjorie's encouragement to become engaged with state-wide and national initiatives through participation in teacher professional associations (peripheral participation). At the same time, Merrilyn and Peter had opened up possibilities for involvement in educational research. My ZFM was defined by access to established and promising researchers and my school was supportive of my involvement in their project. I was increasingly becoming involved in communities that engaged with national initiatives in teacher professional learning and those that conducted research in the teaching and learning of mathematics (ZPA). Marjorie's, Merrilyn's and Peter's differing influence, as knowledgeable others, was impacting on my ZFM/ZPA complex, guiding me into new ways of *becoming*. There was a flame and I was the moth.

Through this time, my identities as a teacher and researcher were being reinforced through participation in two different communities of practice. However, other identities were emerging through living life, husband and father, and by participation in a new teacher professional association community. Marjorie was shaping my ZPA through her introduction to the teacher professional development community, with Peter and Merrilyn helping to induct me into research – a different ZPA. At this stage, both were within the constraints of my ZFM. However, was I being pulled in too many different directions?

"Now It's Your Turn"

Merrilyn finished her PhD. It was a wonderful piece of work and provided the basis for articles in the best journals in mathematics education, Educational Studies in Mathematics and the Journal for Research in Mathematics Education. She was on her way! Again, I thought there was a moment when I could escape, but then Merrilyn asked me for coffee and said, "now, it is your turn". It took a little while to agree but I had very much enjoyed working with Peter and Merrilyn, and they convinced me I had something to contribute. And so it came to pass, with Peter and Merrilyn as supervisors. Merrilyn continued to come along to my classes – the extra eyes were invaluable, and the study began well. After the two years, however, things slowed down. The weight of all I had taken on, including the arrival of additional offspring, took its toll. No one was ever able to identify the malady, but I had to stop both work and study. Through this time, however, support was never far away. I was encouraged to do what I could when I could. Slowly I could do more, and eventually, I made

my way back to work and to research. I will be forever thankful for the unwavering support I received at that time from friends and colleagues. They know who they are.

The episode lasted for close to 12 months in its severest stage and for close to five years in all. I could have walked away at any stage but the connection to ideas and the community had become strong. The need to be involved in research was now a part of my identity, and so I was drawn back to the practice - person-in-practice-in person – despite the constraints of poor health (ZFM). It wasn't so easy to get away. After the worst, I began to pick up the threads of my PhD and I received support to present tentative findings at my first MERGA conference after a brief hiatus. I began to understand that involvement in research was now a part of who I was and that needed to sit with my love of working with students. These separate identities were about to reconcile. It was around this time that Peter retired (2003) and Merrilyn took on the responsibility of principal supervisor for my PhD.

Merrilyn and I have written about our work together during this phase of our collaboration (Geiger & Goos, 2006; Goos & Geiger, 2006). These publications took the form of a conversation between two different types of researchers where power and authority were shared in recognition of different types of expertise. But by now I had fully committed to completing my PhD, a very hard thing to do while working in a school – my goals had changed as had my way of thinking about mathematics education. Thus, there was a developing mismatch between my identity and that of the role of a mathematics coordinator within a school.

A friend sent me an advertisement for a Lecturer B position at Australian Catholic University – a relatively new institution that emerged during the transformation of institutions of higher education during the Dawkin's reforms of the early 1990s. I applied for the job and was interviewed by Elizabeth Warren and Tom Cooper. To my surprise, I was successful.

The decision to pursue a PhD meant that my ZPD was about to be extended. In time, my opportunity to complete was facilitated by a change in working circumstances (ZFM) and my own determination to do so (ZPA). Research was about to be part of my responsibilities, not just a "hobby", an essential component of my ZFM, although illness limited my progress for a time. Merrilyn's and Peter's support were a key influence on my ZFM/ZPA complex and identity formation, as this was guided towards further involvement in research, as was the formal requirement to conduct research within my new academic position. Co-authorship was a particularly influential factor in promoting my progress as a researcher.

There were further incremental shifts in identity. My more active participation in the educational research community was disrupting my singular engagement with my role as a teacher. I was now seeking alignment with goals that had changed over time and a new community of practice – this had implications for the person-in-practice-in-person – who was I and to which practice did I belong? Merrilyn and Peter were helping me bridge into the mathematics education community. But at this time, it was still a leap of faith.

Choosing Something Else – A Mathematics Teacher Educator/Researcher

Life had changed again. I was now responsible for the preparation of teachers, principally in mathematics but also in curriculum and assessment – nine courses as lecturer-in-charge in a year. The level of regulation was considerably higher than in a school – course outlines, advanced and detailed notice of assessments, and accreditation considerations. There was a lot to learn, and there was that PhD to finish! Life as a teacher educator/researcher was complex.

I did find a way to complete my PhD (Geiger, 2009) and to write. At first, it was mainly conference papers and book chapters – typically collaborations with more experienced researchers – but slowly I began to take the lead. It was not always smooth sailing, however, with as many rejections as successes in my attempts to publish in high quality outlets. I remember being shattered when it took nearly two years for one manuscript to be rejected! It has never been published as there were other events that overtook me.

A period of study-leave in Giessen, Germany, working with Professor Rudolf Straesser in 2010, provided the space I needed to focus on academic writing and begin to think about funding applications. This was a productive period for publication (4 journal articles and a book chapter). The visit to Giessen also established an ongoing research collaboration with Rudolf that continues to this day (e.g., Geiger & Straesser, 2015; Geiger, Delzoppo, et al. 2021).

Upon returning to Australia, I started attracting additional administrative responsibilities, secondary program coordinator and then, deputy Head of School (Research). This heightened the challenge of maintaining a research identity as teaching loads, administration and research all had to be kept in balance. I made sure that there was at least one writing day a week. This did not mean I put less effort into teaching. This was still central, but I had started to think about how these two different aspects of my identity could be better reconciled. I had begun to think more deeply about the nexus between research and teaching and worked with others on a project related to providing technology based support and resources to students while on practicum. This project, WebCT as a pedagogical resource and communicative tool for use in the professional experience program, was recognised nationally via an ALTC Citation in 2009.

About this time, my Head of School asked me to think more about how to take others forward – I think she was suggesting that I should do more than just think about myself! I took the advice to heart and applied for a number of internal grant opportunities, including others in the applications. One related to the potential of computer algebra systems with Merrilyn and Rhonda Faragher (Geiger et al., 2010) and another related to the collaborative use by teachers of video stimulated recall techniques to improve numeracy teaching practice (e.g., Geiger, Muir, et al., 2016). The first was supported by the Mathematics and Literacy Flagship at ACU, which was led by Doug Clark and the second was supported by an Education Faculty grant. I was also asked to lead a *research support team* for members of the school of Education in Queensland which provided funding for the engagement of a senior researcher in a consultancy role – I asked Robyn Jorgensen. Each of these small grants provided an opportunity to gather data, and the mentorship provided by Robyn promoted our publication capabilities.

I also decided to contribute in a more substantial way to the mathematics education community via MERGA, and successfully nominated for the role of Secretary on the executive (2009-2012). I served under two Presidents – Judy Mousley and Merrilyn Goos. My learning during this opportunity was about the scope of activity in which researchers could be involved, publication, conferences, development and, of course, leadership in the field.

Further leadership opportunities were also emerging. After a selection process, Doug invited me to take up the role of Deputy Director of the Mathematics and Literacy Flagship.

Through this period, I continued to work with Merrilyn on a series of projects related to improving numeracy teaching practice within schools based on a model for numeracy for the 21st Century. The model brought together the dimensions of context, mathematical knowledge, dispositions, tools, and an evaluative element, a critical orientation, for the first

time (e.g., Goos et al., 2014). Shelley Dole and Anne Bennison worked with us on these initial projects which led into a successful ARC Discovery application with Helen Forgasz (2012-2015). I learnt much during this time from established researchers in the field about managing large projects – how to approach schools for recruiting purposes; how to work with education systems as well as teachers in schools, effective practices in data collection and achieving as well as analysis. And there were more opportunities to write. Not many academics from ACU had been involved in ARC funded research at that time and I was determined to take every opportunity to support the project, taking the position that if I couldn't contribute as fully to the study as others on the intellectual plane at that stage, I could make up for it with sheer hard work. This program of research has led to significant publications as different perspectives on numeracy practice emerged, including the use of the numeracy model as a scaffold for planning numeracy lessons across the curriculum (Goos et al., 2014); auditing curriculum for numeracy opportunity (Goos et al., 2012), the nature of numeracy (Geiger, Goos, & Forgasz, 2015), the design of numeracy tasks (Geiger et al., 2014), numeracy readiness of pre-service teachers (Forgasz et al., 2015), role of technology in effective numeracy practice (Geiger, Goos, & Dole, 2015; Goos et al., 2013), and development of numeracy identity (Bennison, 2015). The model that underpins all of this work has received international recognition as a holistic approach to enacting numeracy. For example, it has been included as a framework that informed the development of the PIAAC Cycle 2 assessment framework: Numeracy (Tout, et al., 2021). It also received a MERGA Research Award in 2017. This work is ongoing (e.g., Bennison et al., 2020), with further opportunities to explore in this space with good friends and colleagues.

My ZPD had now changed to accommodate the demands of a mathematics teacher educator – teaching, research, and service. The challenges associated with balancing these demands provided constraints within my ZFM. Further constraints included the standard required for publication in high quality international journals. Involvement with the Numeracy Across the Curriculum program, however, as well as mentoring by established colleagues and a period of study leave, had canalised my development in research and strengthened my connection to relevant communities of practice – strong positive influences on my ZPD. The ZFM/ZPA complex, at this time, was enabled by my development as a researcher even while meeting the many demands of a teaching/research academic – two separate but interrelated communities of practice and associated ZFM/ZPA complexes. The deeper enculturation into educational research was transforming my role as novice into fuller participation in a national community of practice in mathematics education, and I was taking my first steps into the international community. I had also taken steps towards the mentoring of others in a research community of practice, shifting my identity from that of complete novice towards mastery. Each of these developments were contributing to further transformations of my identity – becoming more fully immersed in the research community and while maintaining focus on other aspects of who I was professionally. I had now developed the belief and confidence that I could be a successful researcher but doing everything well was becoming harder, there were only so many hours in a day!

A Broadening Role in the Mathematics Education Community

I had another opportunity for study leave at the beginning of 2014. This time I chose to visit Gabriele Kaiser from Hamburg University in Germany, Katja Maass in Freiburg, and Peter Freid and Jonas Arleback in Linkoping, Sweden. All were part of the mathematical modelling community and connected with my research interest in the teaching and learning of mathematical applications. These visits resulted in further publications, in the short term

or over a longer period of time (e.g., Geiger, Ärlebäck, et al., 2016; Maass et al., 2019; Cai et al., 2014).

During this time, I also wrote drafts for ARC DECRA (for early career researchers) and Discovery Awards. The former was successful and the later, while receiving encouraging reviews, was not supported by the ARC. The focus of the DECRA was an extension of work I had been doing with colleagues, this time looking at the processes teachers engaged when designing numeracy tasks for implementation across the curriculum. Through this study I developed a framework that outlines how numeracy task design takes place through the processes of identification or archiving ideas (looking, seeing, noticing), the shaping of a task to fit the classroom circumstances in which it was to be implemented, and the actualisation of a task in a classroom through a well-considered pedagogical architecture (e.g., Geiger, 2016; Goos et al., 2019). This work provoked further thinking about the role of the critical aspects of numeracy and how these could be actualised by teachers in the classroom through the structure of tasks, measured responsiveness, and forms of questioning (e.g., Geiger, 2019).

Because of the DECRA, I was now being noticed within the university, no one at ACU had been successful previously. I was invited to become a member of the newly formed Institute for Learning Science and Teacher Education, an initiative aimed squarely at establishing research at ACU as world class. I was now a research-only academic. While this provided time to think and write, it corresponded with an increasing number of invitations to collaborate with others on national projects – the *Opening Real Science* (2013-2016) project led by Joanne Mulligan and supported by a range of colleagues from very different backgrounds in mathematics, science, and education (e.g., Geiger et al., 2018), the *Building an evidence base for national best practice in mathematics education* (2015-2016) project, sponsored by the Office of the Chief Scientist and led by Rosemary Callingham, in which I worked with many good colleagues in mathematics education nationally (Geiger et al., 2017). There was further success with an international funding application to the Australian Universities-German DAAD Joint Research Cooperation Scheme (2017-2018), which provided opportunity to work with Jodie Miller and Jill Fielding-Wells as Early Career Researchers on a collaborative project with German colleagues from Darmstadt University led by Regina Bruder. And then, a revision of a previously unsuccessful ARC Discovery application with Gloria Stillman, Jill Brown, Peter Galbraith and Mogens Niss (e.g., Geiger, Galbriath, et al., 2021) was awarded funding for 2017-2019. This focus of this project was on identifying enablers of mathematical modelling from both the perspectives of instruction and learning. Each of these projects provided opportunity to extend ideas within research themes I had been working on for some time – quality teaching and learning through a focus on task design and implementation, applications of mathematics, and the role of digital tools in enhancing instruction. But there was a lot to do! I had learned, through these times that the contributions of support staff make a project work. The contributions are sometimes downplayed by researchers – at their peril! I had learned that leading research was about more than grant capture and publications (although these aspects are important) – it is also about leading people – another identity.

More recently, a collaboration with Sharon Fraser (UTas), Kim Beswick (UNSW) and members of the mathematics education community, led to a successful tender for the Principals as STEM Leaders project (2018-2020) sponsored by the Department of Education, Skills and Employment. An important aspect of the project to date has been a framework of capabilities required by principals, teachers, students, the community, and researchers to promote positive STEM learning cultures within schools. The development of

this framework drew on the dimensions of the model for 21st Century numeracy, instigated by Merrilyn and further developed through collaboration with other colleagues over 15 years. Ideas build on themselves over time.

Robyn Jorgensen continued as an informal mentor beyond her role as a consultant on the research support group in my school, inviting me to put my name forward for a role as Associate Editor of the Mathematics Education Research Journal (MERJ). Robyn was the Editor-in-Chief. I was flattered but was I good enough?! It was a steep learning curve between 2013 and 2018, with a period as Acting Editor-in-Chief. There was much more to publication when looking from the other side of the process – managing reviews and reviewers, developing consistent feedback across submissions and working on my own understanding of what is required in a quality publication. This experience and a maturing publication record led to an invitation to act on the Editorial Board of the International Journal of Science and Mathematics Education as well as three Guest Editorships of ZDM - Mathematics Education.

Other opportunities for international collaborations were now opening up. I was awarded the Giovani Prodi Guest Professorship at Wurzburg University, Germany (2018-2019) from an international field of 50 scholars. This experience has been the foundation of an ongoing collaboration with Hans-Stephan Siller and his team. My ongoing role is to collaborate with Stefan's team on the internationalisation of their research, in the first instance through publication (e.g. Siller et al., under review), leading to funding applications.

I am currently working with an international team on the *Cycle 2 of the Programme for the International Assessment of Adult Competencies* (e.g., Tout et al., 2021). This work has drawn on our numeracy research, and that of others, especially the critical aspects of what it means to be an informed and active citizen. I am also working on another international project with Iddo Gal (Haifa University, Israel) and an international team including Jill Fielding-Wells on the impact of the COVID-19 pandemic on pedagogy in mathematics. And then there is the current ARC submission that focuses on critical aspects of mathematical thinking, including the role of social justice in mathematics-based decision-making. There remain opportunities to research and learn!

My current ZPD is now one of a mature researcher. I am now a Director of a research program within ILSTE, with a focus on STEM Education, and must accommodate all of the demands required of leadership. I have a team to mentor and lead, as well as PhD students. These create demands on my time that constitute constraints within my ZFM, as well as institutional demands that require publications be submitted to only the best journals. Increasing involvement in national and international collaborations are now an important element of my ZPA. These collaborations include both formal and informal mentoring from highly esteemed colleagues (I have been published in ESM, at last, with their support). I hope that these collaborations are also having a positive impact on the ZPA of others. My ZFM/ZPA complex is now fully directed towards research, with aspirations to excellence. This complex also overlaps with those of others in my roles of leader and mentor. I hope I am viewed more as an affordance than a constraint!

I am now fully involved in two research communities of practice: one, a national and international related to mathematics education, and the other, related to my Institution. Many of the goals are the same, but there are important differences. Each has both affordances and constraints to how I participate. I think I have now moved a little beyond novice, but it is up to others to decide if I have achieved any sort of mastery. I hope I am now achieving some aspect of my goals related to generation of new knowledge and research excellence. At the same time, facilitating the fuller participation of others into the mathematics education

community is a goal of increasing focus. That is, providing guidance than canalises the ZFM/ZPA of others – another change in identity.

Conclusion

In their research, Goos and Bennison (2019) have traced the identity trajectory of teachers in mathematics education in a manner consistent with Wenger's (1998) notion of identity-as-becoming. In this paper, I have attempted to connect this thinking to that of researcher development in mathematics education. Through this narrative, I have described a transformation of identity over time as an outcome of my participation in a range of communities – student, public servant, teacher, member of teacher professional associations, and researcher. Each participation has fostered multiple identities consistent with the practices of each community (Wenger, 1998). Entering each new community required realignment and an ongoing evaluation of whether my goals remained consistent with those of the community. Eventually I have come to participate more in some communities and less and others. Crow et al. (2017) have argued that "Key to successfully negotiating our stable selves is the reconciliation of the multiple identities which are constructed in these multiple communities of practice" (p. 268). This rings true for me as I believe I have retained the essence of each of the identities I have assumed through my career in some form, although each has come to the fore at different times – a different emphasis for person-in-practice-in-person.

So what messages do I have for researchers in mathematics education having experienced these different identities? I believe there are six, which I hope are evident in the preceding narrative:

1. *Contribute to your research community* – they will challenge you to do your best work and support you when times are tough.
2. *Work with the best in the field* – they will stretch you, bringing you forward into fuller participation in the community of mathematics educators. They will also let you know when you have more work to do before the next big step. On this point I have been lucky.
3. *Lead* - don't stand back waiting to be asked, initiate conversations about potential research ventures. Do not be afraid to bring others with you.
4. *Be wary of low hanging fruit* – test yourself, aim high in terms of international publications, keep applying for funding despite the risk of rejection. Focus on quality rather than quantity.
5. *Collaborate* – be generous with your time, there will be a point when you need to depend on others.
6. *Think nationally and internationally and not just about local demands* – there are many opportunities out there.

Through this lecture, I hope I have stimulated some thinking about the notion of a "reflective" researcher. I have two additional questions:

1. We readily place the expectation of being "reflective" on teachers. Do we do the same when considering the development of researcher identity?
2. What will be my/your next transformation of identity?

I finish with another quote drawn from culture, this time Andy Warhol.

> When people are ready to, they change. They never do it before then, and sometimes they die before they get around to it. You can't make them change if they don't want to, just like when they do want to, you can't stop them.

How much you wish to change very much depends on you.

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Aspects of excellence in mathematics education

Berinderjeet Kaur
National Institute of Education, Nanyang Technological University, Singapore
<berinderjeet.kaur@nie.edu.sg>

The theme of the plenary panel is Excellence in Mathematics Education. Taking excellence to mean a commitment to bring out the best leads us to view excellence in mathematics education as a goal such that teachers, students and curriculum, the three corners of the didactical triangle, and their interactions result in the best possible outcomes. Each of the four panellists share with us a unique aspect of Excellence in Mathematics Education.

The theme of this plenary panel is Excellence in Mathematics Education. In the context of this panel discussion, excellence in mathematics education is viewed as a commitment through means to bring out the best amongst the interactions between teachers, students and curriculum, the vertices of the didactic triangle shown in Figure 1.

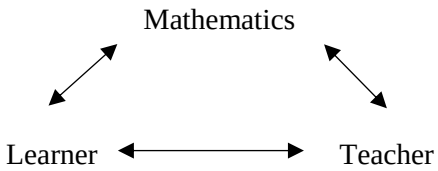

Figure 1. Didactic triangle (Straesser, 2007, p. 165)

As noted by Schoenfeld (2012), it is clear that each of the entities in the figure, each of the arrows, and the triad denote something of importance. As such excellence is mathematics education is multi-faceted. In some ways mathematically powerful classrooms encompass all the interactions between mathematics, teachers and students. This is evident in the Teaching for Robust Understanding (TRU) framework (Schoenfeld, 2016, p.10) shown in Figure 2.

The Five Dimensions of Mathematically Powerful Classrooms	
The Mathematics	The extent to which the mathematics discussed is focussed and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained.
Cognitive Demand	The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development.
Access to Mathematical Content	The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class.
Agency, Authority, and Identity	The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency and authority resulting in positive identities as doers of mathematics.
Formative Assessment	The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

Figure 2. The five dimensions of mathematically powerful classrooms

The four panelists were asked to present their perspective on excellence in mathematics education and describe research and developmental project (s) that they have been involved in related to any aspects of excellence in mathematics education. It is apparent that each of them has approached the theme in a unique way.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 47-48. Singapore: MERGA.

Choy notes that having high expectations and providing strong support to all students, a notion of equity, is a necessary constituent for achieving excellence in mathematics education (NCTM, 2000). He uses the metaphor of confluences to characterize excellence and illuminates how confluences of "Big Things' such as societal expectations, policy formulation and implementation, and 'Small Things' such as classroom practices – teachers juggling the balance between developing procedural fluency and conceptual understanding in their instructional practice whilst ensuring that students have adequate practice for examinations orchestrate in tandem in Singapore thereby resulting in excellence in mathematics education at the systemic level.

Kwon whilst unpacking the complexity of the term *excellence* draws on all the three vertices of the didactic triangle and opines that excellence in mathematics education is best described in terms of research-based curriculum development, research-based teaching practices, and professional development of mathematics educators. She draws on her research projects: Inquiry Oriented Differential Equations (IO-DE) curriculum development project; Inquiry-Oriented teacher Actions (IOTA) research-based teaching practices project; and Community-Based Teacher Professional Development Model a professional development project to illuminate the three aspects of excellence in mathematics education.

Attard notes that while we continually strive for excellence in mathematics education this strive comes with challenges. She illuminates how the current COVID-19 pandemic has highlighted the many variances in technology-infused mathematics teaching due to influences such as school context, community support, school commitment to technology use and school culture. Adopting a holistic model of technology integration she notes that clarity regarding contextual affordances and constraints may assist teachers in their planning of mathematics teaching and learning thereby facilitating pursuit of excellence in mathematics education.

Tan proposes a framework for teaching excellence in mathematics. In the context of undergraduate mathematics, the framework encompasses four aspects namely module learning outcomes, lesson plan, teaching nodes and motivational strategies. Tan notes that although the learning component rests on students' initiatives, there are several aspects of the learning process that teachers can facilitate.

It is apparent from the four panelists presentations that a framework like that of TRU by Schoenfeld could provide a more holistic lens when considering excellence in mathematics education from both the perspectives of educators and researchers. This would allow for deeper understandings of the inter-relationships of the vertices of the didactic triangle. Following the presentations by the four panelists, it is hoped that the questions posed by the conference participants will illuminate other facets of excellence in mathematics too. Lastly, we hope the panel discussion will ignite conversations that would continue beyond the session during the conference.

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Excellence in mathematics education: Influences on the effective use of technology in primary classrooms

Catherine Attard
Western Sydney University
<c.attard@westernsydney.edu.au>

The current COVID-19 pandemic has highlighted the many variances in technology-infused mathematics teaching due to influences such as school context, community support, school commitment to technology use, and school culture. These elements have a significant impact on how teachers plan to use technology in mathematics classrooms. In this brief paper I provide a snapshot of findings from a larger study to highlight some of the variances found in four case studies across three different primary schools.

While we continually strive for excellence in mathematics education, we also continue to face challenges. The forced school shutdowns experienced by many countries during 2020 caused by the COVID-19 pandemic forced many teachers to shift to more technology-infused practices. This highlighted the critical role that technology plays in contemporary mathematics education and the need to understand more about the influence of school context, culture, community, and commitment on technology use in classroom practice. In this brief paper I share some insights from four case studies conducted in primary classrooms within three Australian schools to illustrate the abovementioned influences on technology integration in mathematics classrooms. I do this through the lens of a holistic model of technology integration, the *Technology Integration Pyramid (Mathematics)* (TIP(M)). The TIP(M) emerged from a larger study conducted across 10 Australian classrooms ranging from early childhood through to senior secondary (Attard & Holmes, 2020a, 2020b). The TIP(M) considers the influences on technology integration at a school level, along with the critical considerations for effective technology use within mathematics classrooms. In this paper I provide a snapshot of the complex influences across four case studies in relation to the teachers' effective implementation of technology-infused mathematics lessons.

A Model for Technology Use in Primary Mathematics Classrooms

There are several frameworks that attempt to describe the types of knowledge required to integrate technology into teaching and learning. For example, the widely cited TPACK framework (Koehler & Mishra, 2009) provides a model of a professional knowledge construct, and according to Krauskopf et al. (2018), potentially provides a richness to teaching conversations, providing a theoretical vocabulary to help understand the required pedagogical considerations of technology integration (Koh, 2018). However, there are limitations to the TPACK framework. Although it is helpful in identifying specific knowledge domains for technology integration, TPACK is regarded as a pedagogically neutral model (Bower, 2017). The framework makes no suggestions about specific technologies and pedagogies that would be appropriate for mathematics, nor does it consider the importance of student engagement, which is a particular concern within the discipline of mathematics education. While TPACK provides an acknowledgement of school contexts, it does not provide insight into the complex contextual elements that may influence task design, teacher practice and student learning, and does not consider the variety of barriers and dilemmas that are typical to technology integration, such as a lack of technical support

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or issues of access. Arguably, these issues influence how technology-infused teaching plays out in individual classrooms.

The *Technology Integration Pyramid (Mathematics)* (TIP(M)) (Figure 1) (Attard & Holmes, 2020b) emerged from existing frameworks and the findings of the broader study from which this paper is drawn. TIP(M) is conceptualised as a three-dimensional model to illustrate the connections and inter-related elements within it that teachers should consider when planning for the use of any technology, regardless of device, software, access and school context. The purpose of TIP(M) is to assist in future-proofing technology-infused teaching and learning as new technologies continue to emerge. It presents a holistic means of understanding the parameters within which teachers operate and a recognition that student engagement with mathematics is a critical element for learning to occur in contemporary classrooms. In this paper, a sample of findings from four case studies of teachers considered to be effective users of technology in mathematics education is used to illustrate the variances and complexities that influence technology-infused mathematics teaching across different schools.

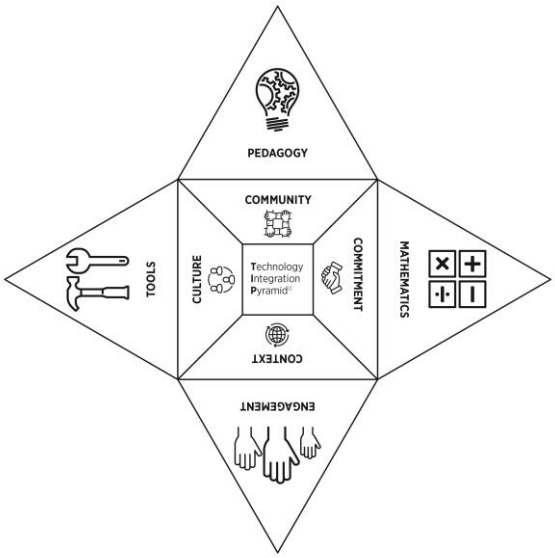

Figure 1. Technology Integration Pyramid (Mathematics) (Attard & Holmes, 2020b)

Methodology

To assist in understanding how the influences described on the base of the TIP(M) evolved, a brief overview of the methodology employed in the larger study is provided. A qualitative multiple case study approach was utilised. Each case consisted of a classroom teacher, one member of the school leadership team, and a focus group of five or six students. Cases were identified through a process of purposive sampling. The case studies were conducted in a mixture of public and private schools and represented a range of socio-economic and geographic areas.

Participants

Case study teachers were identified through professional networks as teachers who are considered by their peers as effective and innovative users of technology. While the three

schools (two case study teachers taught at the same school) were located in metropolitan areas, they differed significantly in terms of size, socio-economic status, access to technology, and school support for technology integration. School leaders were identified as those who had a formal leadership role. Students participating in focus groups were selected by their teachers as a representative sample of the case study teachers' students. Where possible, students were chosen to represent a mixture of gender, ability, and attitudes towards mathematics. Students below Grade 3 did not participate in focus groups.

Data Collection and Analysis

Data collected from the case study teacher included classroom observations, lesson plans, and interviews. Students participated in a focus group discussion and the nominated school leader participated in an interview. Data drawn from interviews and focus group discussions were audio recorded and transcribed verbatim. Observations were video recorded. Data analysis was conducted in alignment with the components of the TIP(M). To do this, all relevant data from interviews and focus group discussions from each of the case studies were collated to provide collective responses to the research question. Field notes and observations were used to support further analysis. For a more detailed description of the larger study, its methodology and findings, see Attard and Holmes (2020a, 2020b).

The Influences on Effective Technology Use

The three school settings examined in this paper varied in context with two being government schools and the other a very well-resourced independent school. The independent school (Case A) utilised a whole-school approach to technology integration, ensuring a one-to-one iPad ratio and providing professional development for teachers, largely in-situ, allowing for a highly contextualised approach. The teachers were expected to consistently reflect on the proposed purpose when thinking about using a new technological tool or app. Teachers in this school were actively encouraged to limit the number of apps used during teaching, only adding new ones when there was a clear pedagogical purpose for doing so.

In contrast, the government school in two cases (B and C) had a different approach for the early years (K-2) and the primary years (3-6). All students in Years 3 to 6 were required to have their own iPad which the school facilitated through an Apple purchase plan. Students in the lower years had a small number of iPads to share in the classroom, but the teachers of these years were perceived as being more sceptical about the value of technology for learning. Rather than taking a whole-school approach, the technology divide in this school between older and younger students was quite embedded and unlikely to change with current teaching staff. In Case D, a whole school approach was not yet in place due to the school being new, yet there was still an ethos of encouragement of technology use, albeit through a "trial and error" method, rather than through an agreed systematic approach.When the Bring Your Own Device (BYOD) iPad plan was introduced for Years 3 to 6, the school in Cases B and C faced considerable backlash from parents, concerned about how the technology might change the teaching and learning practices. The school then increased communication with parents to ensure that support for the technology was present at home as well as at school. Interestingly such concerns were not raised at the independent school where even very young learners were expected to have their own devices.

Despite significant differences in levels of support and access, the teachers at all schools saw the benefit of using technology in the mathematics classroom to shift the focus from

learning content to developing conceptual understanding and mathematical reasoning. They recognised increased opportunities for students to explore mathematics content and to communicate their mathematical understanding in a variety of modes using digital cameras, audio and video recording, and screen capture. In Cases A, B, and C, *Google Sheets* was used for learning about data and *Beebots* and/or *Spheros* were used to enhance spatial reasoning through basic programming. Cases A, B and C used a learning management system (*OneNote, SeeSaw*) as a means of tracking student progress and to share student work with parents. *Kahoot* was employed in all schools to check on student progress both from the teachers' perspectives and as a means for students to gain immediate feedback on their understanding.

In all observed lessons there was evidence of high levels of student engagement because of how teachers utilised the tools at hand. The technology was used seamlessly with few technical difficulties, regardless of constraints posed by some school contexts and communities. While the influences at each school varied, each teacher was able to find ways of using the available technologies in effective and meaningful ways.

Arguably, some of the influences such as system policies, school funding, and provision of professional development are beyond the individual teachers' control. Others, such as individual teacher beliefs about technology, their willingness to innovate and the depth of their pedagogical content knowledge can be somewhat controlled and influenced by the teacher. An understanding the four categories of influence (context, culture, community and commitment) within a teacher's school will help to understand the possibilities for effective technology-infused mathematics education within each unique and individual context. Further, clarity regarding contextual affordances and constraints will assist teachers in the planning of mathematics teaching and learning and contribute to the pursuit of excellence in mathematics education.

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Excellence in mathematics education: Multiple confluences

Ban Heng Choy
National Institute of Education, Nanyang Technological University
<banheng.choy@nie.edu.sg>

Excellence in mathematics education is often linked with high performance in international achievement tests such as TIMSS. In this short paper, I broaden the notion of excellence by considering how the different aspects of mathematics education come together instead of only focusing on what these aspects are. Using confluence as a metaphor to describe excellence, I examine Singapore's excellence in mathematics education by showing how the "big things" of education such as societal expectations, policy formulation and implementation, and how the "small things" of classroom practices—scheme of work, tasks (especially typical problems), and examinations—flow together towards the same vision of ambitious teaching articulated by the Singapore Mathematics Curriculum Framework.

Excellence—from the Latin word *excellere*, meaning surpass—is multi-faceted. In mathematics education, excellence is often associated with high performance in international achievement tests such as TIMSS and PISA. Achieving top performance in these tests has been likened to obtaining medals in the "Olympics" of education (Leung, 2014) and declining performance over the years in these achievement tests has triggered calls in various countries to reform mathematics education (Gerritsen, 2021). However, I believe most mathematics educators would see performance in these international benchmark tests as a very narrow interpretation of excellence. Examining the notion of excellence in mathematics education may require us to investigate a myriad of educational components operating together in diverse contexts. In this paper, I use the metaphor of *confluences*—where two or more rivers, each with their own flow and paths, meet to form a bigger river—to characterise excellence. I view the notion of excellence in mathematics education as the coming together or flowing together of different educational aspects at a single purpose: to *provide all* our students with *quality mathematical learning experiences* so that they are supported to *achieve* the *desired learning outcomes*.

Having high expectations and providing strong support to all students relates to the notion of equity, a necessary ingredient for achieving excellence in mathematics education (NCTM, 2000). There are two aspects of confluences here. First, there is a directed flow of policies, initiatives, and practices towards the same goal of providing high quality learning experiences for all. Second, there is a coming together of different understandings about the main elements of an excellent mathematics education, namely curriculum, teaching, learning, assessment, and technology. The idea is not to have a single understanding about what or how to teach. Rather, the aim is to achieve a balance point in which our different understandings about mathematics teaching and learning are compatible. In practical terms, this means that the educational policies, initiatives, and practices are in sync with the purpose of providing high quality learning experiences for all. Hence, finding the balance point and getting the policies, initiatives, and practices to "flow" in sync are the key levers to excellence. Seeing excellence in mathematics education as confluences therefore positions excellence as a journey and not merely a destination. In the rest of the paper, I will illustrate this idea of seeing excellence as confluences through the Singapore experience in mathematics education.

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Confluences of 'Big Things'

I begin by looking at the confluences of key elements of an excellent mathematics education. To that end, the principles for school mathematics, as proposed by the National Council of Teachers of Mathematics (NCTM), serves as a good reference point. According to NCTM (2000), the following six principles are fundamental to achieving excellence in mathematics education: equity, curriculum, teaching, learning, assessment, and technology (NCTM, 2000, pp. 12–24). On the surface, it is hard to imagine why anyone would have issues with these principles but the "math wars" in the US suggests otherwise. On one side, traditional mathematics advocates emphasise the importance of mastering procedures (back to basics) and use of more teacher-directed teaching approaches such as direct instruction; on the other side, reform mathematics advocates emphasise the importance of developing conceptual understanding via the use of more student-centric approaches such as inquiry-based teaching. These "wars" are not unique to the US and different versions of these wars are still "fought" in various countries (Chernoff, 2019; Yoon et al., 2021). I find these wars unproductive because the polarising language used in these discourses promotes a "winner takes all" notion of what excellence in mathematics education means.

Avoiding these extreme positions, excellence in mathematics education can be characterised by the confluences of societal expectations, policy formulation, and implementation. In other words, the actions of the policy makers, school leaders, teachers, students, parents, and mathematics educators should flow together towards a clearly articulated vision of mathematics education. Flowing together towards a common vision does not necessarily mean having a one-size-fits-all approach to teaching and learning. Rather, the idea is that different policies, initiatives, and practices, which may differ in their epistemological foundations, are directed at achieving the same vision. Such a notion allows for a balancing of different pedagogical and curricular positions. Singapore, widely acknowledged for its excellence in mathematics education, is an example of this confluence.

In Singapore, we place a high premium on education and there is a high expectation for every child to do their best in education. All schools are well-funded and there is a high expectation for the professionalism of teachers and their quality of teaching. The Ministry of Education in Singapore, the governing body responsible for policy formulation and implementation, are largely made up of teachers. There is one teacher training institute responsible for pre-service teacher education to ensure consistently high-quality teacher education. All these environmental factors come together to lay the groundwork for Singapore's excellence in mathematics education.

Singapore's mathematics education and assessment, from primary school to pre-university, is guided by the Singapore Mathematics Curriculum Framework (SMCF) since 1990. This framework focuses on developing students' competencies in mathematical problem solving, supported by five-interrelated components (Ministry of Education-Singapore, 2018): understanding concepts, proficiency in skills, competencies in processes, positive attitudes for mathematics, and metacognition (p. 10). It is interesting to note that most, if not all, of Singapore's curricular policies and initiatives, including the SMCF, take ideas from all over the world to be adapted to the Singapore's context. Perhaps, it is Singapore's pragmatic approach that has enabled these different ideas to come together as a coherent curricular intent (Tay et al., 2019).

As detailed by Lee et al. (2019), the SMCF guides how different national policies such as National Education, ICT Masterplan, and more recently, 21st Century Competencies are implemented through the intended mathematics curriculum. Changes in policies are appropriately integrated within the mathematics curriculum while keeping an eye on the

goals articulated by the framework. Hence, changes to the national curriculum, pedagogical approaches, assessment emphases, textbooks, curricular materials, and even school-based curricular innovations are all introduced in reference to this framework. In addition, communication on these changes is carefully orchestrated to ensure consistent and coherent messaging and schools have some autonomy to implement these ideas in different ways. This ensures that the curriculum goes beyond a collection of activities and initiatives to a more connected and coherent focus on mathematics and its implementation, which may be uneven at times, is moving in the same direction. These confluences of different policies, initiatives, and practices at the ambitious goals of mathematics teaching have improved the state of Singapore's mathematics education over the years.

Confluences of 'Small Things'

Despite the seemingly eclectic mesh of ideas for our intended curriculum, one of the keys to Singapore's excellence in mathematics education lies in the recognition that effective teaching can take a variety of forms (Kilpatrick et al., 2001). This is evident from how mathematics teachers *comprehend* and *transform* the intended curriculum into instruction (Shulman, 1987). Each school interprets the curriculum documents and translates the intended curriculum into implementable schemes of work, detailing the selection and sequencing of content as well as the pedagogical approaches tailored to their students.

Singapore teachers use a variety of teacher-centric and student-centric approaches in their teaching while juggling the balance between developing procedural fluency and conceptual understanding (Leong & Kaur, 2019). For example, the prevalent use of typical problems or textbook-type questions in mathematics classroom in Singapore, particularly how these problems are selected, adapted, and implemented deserves more attention (Cheng et al., 2021; Choy & Dindyal, 2018, 2021). In particular, Choy and Dindyal (2021) described how a competent secondary school teacher in Singapore noticed the affordances of typical problems and orchestrated a productive discussion around them, similar to the five practices proposed by Smith and Stein (2011). While Smith and Stein (2011) highlights the importance of using a rich task to orchestrate such discussions, Choy and Dindyal highlights the possibility of using typical problems for mathematically productive discussions.

Similarly, Choy (2020) described how a beginning primary mathematics teacher orchestrated a discussion around the seemingly simple question: 0.8×4. These examples amongst others (see Cheng et al., 2021) suggest there is something interesting going on at the classroom level. These teachers' practices cannot be simply classified as traditional teaching or reform-based teaching because these labels do not capture the complexity of their practices (Leong & Kaur, 2019). Instead, what these teachers have done is to create high-quality mathematical learning experiences for their students in ways that honour both conceptual and procedural fluency (Choy & Dindyal, 2021). More importantly, these practices are not unusual in Singapore. Based on a large-scale study on the enactment of the Singapore mathematics curriculum (Kaur et al., 2019), the researchers highlight that there is a prevalent and skilful use of such problems both for mastery and concept development, with many of these classrooms said to be mathematically productive.

This is despite the commonly held perception that our mathematics education is predominantly focused on high-stake examinations. What is often neglected is that these examinations do not simply test students on their procedural fluency, but they are designed to assess whether students understand and apply mathematical concepts to different problems in different contexts. Hence, teachers tend to maintain a strategic approach to

teaching mathematics, balancing the need for conceptual and procedural fluency as stipulated by the SMCF.

In this short paper, I have tried to paint a landscape of Singapore mathematics education by showing how the "big things" of education, such as societal expectations, policy formulation and implementation, and how the "small things" of classroom practices—scheme of work, tasks (especially typical problems), and examinations—flow together towards the same vision of ambitious teaching articulated by the SMCF. The picture is one of many different rivers, both big and small, coming together at different points to flow towards the sea, which forms part of the larger water cycle. It is not so much the features of mathematics education that makes it excellent. Rather, it is the confluences of these big and small pieces of mathematics education that generate the supportive environment to empower teachers in their work to enhance students' learning experiences and achievements.

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Excellence in mathematics education: Models for teacher education practices

Oh Nam Kwon
Seoul National University
<onkwon@snu.ac.kr>

This paper provides an overview of my research projects that are part of collaborative research program seeking to illustrate the complexity of excellence in mathematics education. The aim of the first two projects were to the theoretical and empirical grounding for an innovative approach in differential equations called the *Inquiry Oriented Differential Equations* (IO-DE) project. The aim of the third project was to provide a model of professional development of mathematics teachers in South Korea.

The concept of excellence in mathematics education is one of the most elusive in the educational literature. Writers often use the term *excellence* and assume their readers know what it means. Dictionaries give such definitions as "the quality of being excellent", "an excellent or valuable quality", and the "quality of being outstanding or extremely good". Thus, arriving at a simple definition is a challenging matter. However, excellence in mathematics education can be described in terms of research-based curriculum development, research-based teaching practices, and professional development. I would like to describe my research project in related to three aspects of excellence in mathematics education.

Inquiry Oriented Differential Equations (IO-DE) project

The *Inquiry-Orientated Differential Equations* (IO-DE) project is an example of a collaborative effort between mathematics educators and mathematicians that seeks to explore the prospects and possibilities for improving undergraduate mathematics education, using differential equations as a case example (Kwon, 2002). In this section, I highlight the theoretical background for the IO-DE project and a summary of quantitative and qualitative studies of the IO-DE project on student learning and how teachers create and sustain an inquiry-oriented learning environment.

While there are clear calls for inquiry in both science and mathematics classrooms, what exactly characterizes an inquiry-oriented classroom is less clear. To clarify the nature of inquiry-oriented classrooms and to provide a more comprehensive perspective on the complexity of teaching and learning, Rasmussen and Kwon (2007) characterize inquiry in terms of both student activity and teacher activity. In particular, students learn new mathematics by inquiry, which involves solving novel problems, debating mathematical solutions, posing and following up on conjectures, and explaining and justifying one's thinking. The first function that student inquiry serves is to learn new mathematics by engaging in genuine argumentation. The second function that student inquiry serves is to empower learners to see themselves as capable of re-inventing mathematics and to see mathematics itself as a human activity. On the other hand, teachers also engage in inquiry. Teacher inquiry centres on inquiring into their students' mathematical thinking and reasoning. Teacher inquiry into student thinking serves three functions. First, it enables teachers to interpret how their students build mathematical ideas. Second, it provides an opportunity for teachers to learn something new about particular mathematical ideas in light of student thinking. Third, it better positions teachers to follow up on students' thinking by posing new questions and tasks.

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Accomplishing these three goals was facilitated by conducting research in three related strands: (1) adaptation of an innovative instructional design approach at the undergraduate level; (2) systematic study of student thinking as they build ideas and teacher knowledge to support students' re-invention; and (3) careful attention to the social production of meaning and student identity. These three strands do not represent a linear progression in our research. We conducted research in these three strands concurrently and view the strands as complementary.

The implications of the IO-DE project are threefold. First, based on the results of the pre-test and the delayed post-test (Kwon, 2005; Rasmussen et al., 2006), the IO-DE students from each of the four institutions outperformed traditionally taught comparison students on the post-test. This result was true for both males and females and for high and low achieving students. This result demonstrates that this instructional approach can be applicable to university mathematics. Secondly and more importantly, the instructional methods and curriculum design approach guided by Realistic Mathematics Education (RME) framework are applicable to promoting student learning in all mathematics classrooms (Kwon, 2002). Thirdly, the IO-DE project can provide a model for how it is that teachers create and sustain inquiry-oriented learning environments in which students gain mathematical power and sophistication.

Since Rasmussen and Kwon (2007) reported their work on Inquiry-Oriented Differential Equations (IO-DE) class, Inquiry-Oriented Instruction(IOI) has been widely used in the field in which researchers applied IOI in other content areas such as linear algebra (Wawro et al., 2012), scaled up curricular materials for IOI in abstract algebra courses (Larsen et al., 2013), and theorized principles for enacting IOI in practice (Kuster et al., 2018). IO-DE project exemplify a research-driven reform in instructional practices of excellence in mathematics education that have been led by the field of research in university mathematics education

Inquiry-Oriented Teacher Actions (IOTA) Project

In the past decades the K-16 mathematics education community has strived to improve the teaching and learning of mathematics via a concerted effort to develop innovative curriculum, to train more effective and knowledgeable teachers, to better understand how students build mathematical ideas, and to better understand how teachers create and sustain mathematics classrooms in which students learn mathematics in powerful and deep ways. Much progress has been made in terms of curriculum development and building models of students' mathematical learning. Much less progress has been made, however, in understanding how it is that teachers create and sustain classroom learning environments in which students build robust relational understandings of mathematics and develop desirable dispositions and attitudes towards knowing and doing mathematics. Indeed, past research as well as our experiences with undergraduate mathematics teachers demonstrates that it is quite difficult for teachers to develop and sustain such classroom learning environments. Models of how teachers accomplish this task would contribute both theoretically to the literature on teaching and practically to professional development efforts.

The goal of the *Inquiry-Oriented Teacher Actions* (IOTA) Project is to develop a model for how it is that teachers create and sustain inquiry-oriented learning environments in which students gain mathematical power and sophistication. In particular, we focus on characterizing teachers' discursive moves in inquiry-oriented classrooms. We use an innovative approach to differential equations, referred to as the IO-DE project as a case example.

We define *inquiry-oriented learning environments* as those classrooms that have two distinguishing features. First, regarding student activity, students routinely explain and justify their thinking and listen to and attempt to make sense of others' ideas. That is, students engage in genuine argumentation as they build mathematical ideas. Regarding teacher activity, teachers routinely inquire into how it is that students are thinking about the mathematics. In other words, teachers are continually attempting to understand their students' mathematical reasoning. Such understanding contributes to their decisions about how to proceed to advance their mathematical agenda.

As a start to define *discursive moves*, we operationalize discursive moves in terms of the following three types of teacher actions: Teacher Questioning, Teacher Revoicing, and Teacher Telling. We leave open the possibility that our analysis will reveal other types of discursive moves. In addition, these discursive moves are intended to include verbal utterances as well as their kinaesthetic actions, such as gestures.

Kwon et al. (2008) detail four different functions of the teacher's revoicing in an inquiry-oriented classroom, because it is one of the discursive strategies that often occurs in the teaching of mathematics, but which has received limited attention in mathematics education research at the undergraduate level. Our analysis shows that a teacher's revoicing can constitute a major repertoire of his or her discursive moves and carries out critical functions in the context of mathematics practice in class. For example, one function of revoicing identified was that of a binder – in which the teacher's revoicing created a context for students to bring up and align themselves with diverse mathematical positions – which supported the discursive, social process of negotiating meaning. Theoretically, these pedagogical moves were related to the instructional design theory of RME (Rasmussen & Kwon, 2007) and Vygotsky's notion of culture tool. Pragmatically, these moves provide strategies for others who wish to create mathematical discursive communities to support students' evolving mathematical reasoning.

A Community-Based Teacher Professional Development Model

Kwon et al. (2014) introduced a conceptual framework and practices, yield by research, into a teacher professional development program focusing on teacher community for mathematics teachers to increase professionalism. Conceptually, it was distinguished from the other training programs in terms of the participants, curriculum and methods. The teacher communities consisting of three or four teachers from the same school, as well as a mentor and sub-mentor, master, or professional teachers with professional expertise and executive capability. The curriculum of our program includes some process practicing and reflecting of teachers' communities on their own classes. The program's structure required active participation. Through our program, the teachers improved their teaching competency. Also, the operational ability of the teacher learning communities was improved. A teaching and learning community culture had been formed in each school, which showed that the community could continue even though the PD was no longer being conducted at the school operated even after our program was over. In the past, teachers avoided opening up their classrooms for others to observe, as this was previously regarded as a form of teacher evaluation in Korean classroom culture. However, the teachers who participated in the program now offered to open up their classrooms for other teaching community members, and saw this as an opportunity to contribute to improving the teaching competency of the community.

The ultimate purpose of the community-based mathematics teachers PD program that was developed by this research is to support continuous development of teachers'

professionalism through training, where professionalism of mathematics teachers is regarded as a factor enhancing their ability to improve their lessons and help students' learning. To this end, rather than transferring all responsibilities to individual teachers, their professionalism was enhanced by growth through collaboration and reflection within the teachers' community.

The concept and procedural model of the training program developed by this research may be modified to suit the needs of course subjects other than mathematics, so that the model can be applied to the operation of PD programs for these other subjects. This systematic PD program will facilitate sustainable development of teachers' professionalism as teacher-researcher, the spread of community among teachers, and the enhancement of teachers' capability to implement the learning material, thereby creating positive change in mathematics education. In fact, inspired by these positive effects, the Korea Foundation for the Advancement of Science and Creativity (KOFAC) is implementing our PD program model in its PD program for elementary school teachers to foster mathematics classes based on storytelling.

Final Words

How can we inspire leaners to excel? To achieve excellent learning outcomes, we need excellent teachers. These projects discussed in this paper provide models towards excellence in teacher education. It is clear that these models need to be investigated in more depth, both as research topics and innovative practices.

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A framework for teaching excellence in the context of university mathematics education

Victor Tan
National University of Singapore
<mattanv@nus.edu.sg>

In this paper, I propose a framework for teaching excellence in mathematics, particularly in the context of universities. The framework encompasses four aspects: module learning outcomes, lesson plan, teaching modes, and motivational strategies. Through this framework, I will share with readers my view on various aspects that a math teacher should pay attention in order to excel in his or her teaching.

Regardless of subjects and levels, education is made up of three components: *curriculum*, *teaching* and *learning*. In the university context, the curriculum of a program is typically developed at the departmental level; the teaching is delivered by the lecturers, instructors or teaching assistants; and the learning comes from the students. Although the three components are acted upon by three distinct groups of people, they are clearly inter-related (Figure 1). This three-way structure is similar at the school level. In the Singapore local school context, the curriculum is developed by the Ministry of Education.

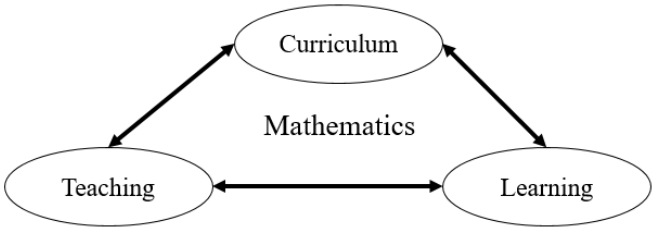

Figure 1. Three components of mathematics education

Generally speaking, as long as it aligns with the university's educational direction, the department has the liberty to develop the curriculum for its program independently. This includes the program requirement, structure and study plan. The department also looks at the syllabus and prerequisites of individual modules. The component that I will be focusing on is *Teaching*, in particular at the module level.

A Framework for Teaching Excellence

The lecturer, who is usually also the module coordinator, needs to define the *module learning outcomes* (MLO) based on the syllabus. Guided by the learning outcomes, the lecturer will proceed to design the module. This entails coming up with the lesson plan and deciding on the teaching and assessment modes. At a micro level, the lecturer and the TA can also do the same for every single class he or she conducts.

Although the learning component mainly comes from the students' own initiative, there are various aspects that the teachers can facilitate the learning process. Other than transmitting the knowledge and assessing the students, teachers can engage the students by asking questions and providing feedback to them. Another important aspect that I will elaborate is to come up with strategies to motivate student learning.

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I would like to propose a *Framework for Teaching Excellence* (Figure 2) at the module level. The framework takes the form of a pyramid. At the tip of the pyramid is the *module learning outcomes* (MLO), which is the ultimate goal that every module should thrive to achieve. The other three layers of the pyramid which contribute to reaching the MLOs are having a sound lesson plan, adopting teaching mode that will engage the students, and coming up with effective strategies that motivate student learning.

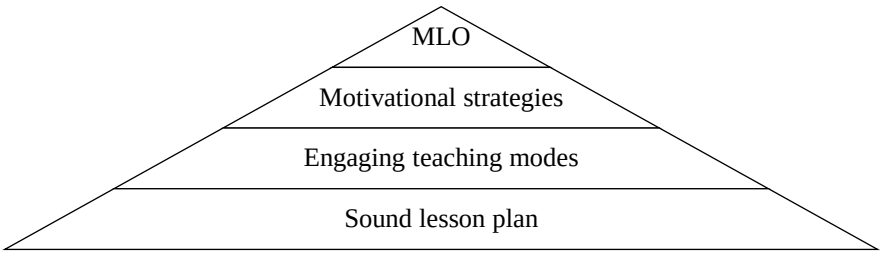

Figure 2. Framework for Teaching Excellence

Module Learning Outcomes (MLO)

All good teaching should come with a set of learning outcomes that are clearly articulated and communicated to the students. Good MLO should not just narrowly focus on the concepts within the syllabus that students are expected to learn. It should also include other higher order learning, such as applying the concepts within and outside the module, integrating the concepts within the module, and seeing connection of the concepts beyond the module. The teacher should then develop and design the module with the MLO in mind.

Lesson planning

The bottom line in the module design is to have a lesson plan. Lesson planning is generally quite straightforward – as long as it is aligned with the MLO. A *sound lesson plan* is comprehensive and has more than just a list of the topics to be covered for each lesson. It should be supplemented with in-class and outside classroom activities that enhance student learning. For example, a good teacher will pay attention to the difficulties that students have and address them adequately by allocating more time to illustrate with additional examples or the use of analogies to illuminate the concepts. On the other hand, easier concepts or topics can be left as assignment for students to read on their own. Teachers should also surface common mistakes, misconceptions, and other pitfalls among the students. Though it is more direct for the teachers to highlight them to the students, it is more effective to design some examples or problems for students to self-discover their own mistakes. Furthermore, assessments should be an integrated part of the lesson plan. Other than the traditional summative assessments, like tests and examinations, formative assessments in the form of quizzes, assignment, and group work can also be introduced to gauge students' understanding of the concepts and to make the lesson more interactive.

Teaching modes

There are several modes of teaching that teachers may choose to adopt, ranging from the traditional face-to-face (F2F) "lecture + tutorial" format, to blended-learning, to flipped-classrooms – which teaching mode to adopt depends on the nature of the classroom activities. Due to Covid-19 restrictions, many F2F classes have been converted to online classes,

typically delivered through video-conferencing platforms (e.g., Zoom, Microsoft Teams). The module coordinator should also take into consideration the students' level of understanding of mathematics, class size, and nature of the modules, together with other constraints, when choosing the appropriate mode of teaching. The question is not about whether flipped classroom is better than traditional lecture – it is about whether a teaching mode can effectively engage the students with the intended lesson plan.

The key word is "engaging". If most students in a class are highly motivated or high-ability students, the teacher may consider replacing live lectures with fully flipped classes. The students can be challenged to read the notes or textbook independently. This should be complemented with some interactive activities such as discussion or seminar-styled sessions. On the other hand, if a class mainly consists of students with weak mathematical foundation, for example a bridging course, then an interactive F2F class may be more suitable to gauge the students' understanding and to provide instant clarification. More commonly, there are students with diverse aptitude and backgrounds in the class, typically found in foundational courses like calculus or linear algebra. The lecturer can consider a hybrid mode in this case. One approach is to prepare lecture materials (can be in the form of pre-recorded videos) for the students to read or view in advance. This is then followed by F2F sessions for the lecturer to further elaborate on the more difficult concepts. Such sessions can be made optional just for the weaker students. Nevertheless, if a teacher can find the right balance to engage all the students in class regardless of their backgrounds, such sessions can also be made compulsory if they help to meet the MLO. A lecturer should also take into consideration the short attention span of the new generation of students when choosing their teaching modes.

Motivational strategies

To complete the puzzle of excellent teaching, learning must take place among the students. As much as we hope that all students will be self-motivated with their learning, the reality suggests otherwise. No matter how hard a teacher tries to explain the concepts, if the students are not motivated to take the learning seriously, the MLO will not be met. It is therefore essential for good teachers to develop some strategies to motivate student learning.

We are mainly concerned about two groups of students: the first group are those that are not motivated and typically only study near the exam date; and the second group are motivated solely by the exam grades. For the first group of students, it is definitely undesirable for them to cramp the learning of mathematics within a few days. There are diverse reasons for their behaviour. For some, this may be caused by not being able to follow the class or not seeing the relevance of the module. For others, they may be simply unimpressed with the teaching, while some are simply not interested. The teacher should identify the more common reasons and come up with appropriate strategies to address them. Giving support, encouragement, and feedback to the students will definitely help. Rewarding with points for constant work can also be an effective strategy.

The second group of students can be very hardworking. Some may even approach the teachers for more exercise problems or past year papers to practice. The concern here is superficial and rote learning. The lecturers could guide the students to see the big idea and provide them with the insights. They could also advise students to slow down and do some reflections and analysis of their own works instead of rushing through as many problems as they can. Once the students are enlightened, they will become genuine learners and will be motivated to go deeper to explore the subject.

The Framework for Teaching Excellence in Action

I shall now illustrate the Framework for Teaching Excellence in action with an example from the NUS mathematics program (Figure 3). In the university program, most modules are inter-related by prerequisite trees:

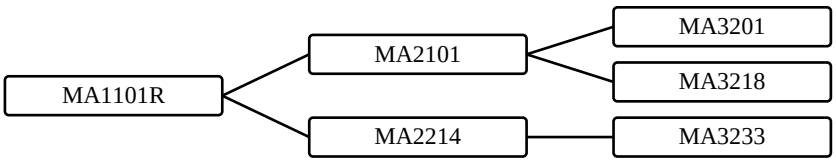

Figure 3. An example of a prerequisite tree

The module MA1101R (Linear Algebra) is the prerequisite for the two level 2000 modules MA2101 and MA2214, which in turn are prerequisites of some other level 3000 modules. In other words, the lecturer teaching MA3201 may assume his students already know the concepts taught in MA1101R and MA2101. However, it is rather common to hear colleagues lamenting about their students being clueless about concepts that they were supposed to learn in the prerequisites. We are quick to blame the students. They learned the module and they passed the exam, but they are not able to apply or connect what they have learned beyond the module. The lecturers who teach those prerequisite modules could also reflect on how to address such issues.

Using the Framework for Teaching Excellence, a lecturer can make it explicit to include in the *MLO* that require students to "apply the concepts beyond the module". This serves as a message for the students to see the larger objective of the module. But more importantly, by making this learning outcome visible, it also reminds the lecturer to design the module with this end goal in mind. Conscious effort can be made to build in some class activities or assessments in the *lesson plan*. For example, a mini group project with the task to look for some applications of the concepts that are not found in the module. Through appropriate *teaching mode*, the lecturer can convey the message in his or her instructions. In particular, to serve as a *motivation*, the lecturer may give a preview of how some of the concepts will be relevant in future courses.

The above example illustrates the importance of MLO in the framework to guide the module design. The example also suggest how to formulate higher order learning outcomes beyond the topics to be covered.

Concluding Remarks

I have briefly discussed some aspects of good practices in teaching excellence, mainly for mathematics education. An excellent math teacher needs not be someone who is charismatic and eloquent. He or she must be one who is sincere in the teaching and willing to put in time and effort in crafting the MLO, designing the module, as well as motivating and supporting student learning.

Singapore Enactment Project

Berinderjeet Kaur
National Institute of Education,
Nanyang Technological University
<berinderjeet.kaur@nie.edu.sg>

Tin Lam Toh
National Institute of Education,
Nanyang Technological University
<tinlam.toh@nie.edu.sg>

Joseph Boon Wooi Yeo
National Institute of Education,
Nanyang Technological University
<josephbw.yeo@nie.edu.sg>

Yew Hoong Leong
National Institute of Education,
Nanyang Technological University
<yewhoong.leong@nie.edu.sg>

Lu Pien Cheng
National Institute of Education,
Nanyang Technological University
<lupien.cheng@nie.edu.sg>

The *Enactment Project* is a Programmatic Research Project funded by the Ministry of Education, Singapore, and administered through the Office of Educational Research, National Institute of Education, Nanyang Technological University. The project began in 2016 and its aim is to study the enactment of the Singapore mathematics curriculum across the whole spectrum of secondary schools within the jurisdiction. There were two phases in the project: the first involved in-depth examination of 30 experienced and competent mathematics to draw out characteristics of their practices; in the second phase, we study the extent of these characteristics through a survey of 677 mathematics teachers. A symposium was organised in MERGA 42 in 2019 where the foundational elements of this project were presented; we would like to share more findings of this project in this year's conference.

Paper 1: Berinderjeet Kaur *Models of mathematics teaching practice in Singapore secondary schools*

This paper revisits the models of mathematics teaching practice that were proposed by earlier researchers of the Singapore mathematics classrooms: Traditional Instruction (TI), Direct Instruction (DI), and Teaching for Understanding (TfU). The data from the survey in this project point to hybridisation of these models.

Paper 2: Tin Lam Toh *An experienced and competent teacher's instructional practice for normal technical students: A case study*

This paper presents a case of how an experienced and competent teacher engaged mathematics "low-attainers" in the learning of mathematics in a way that was responsive to their learning needs while upholding the ambitious goal of helping them acquire relational understanding of mathematical concepts.

Paper 3: Joseph Boon Wooi Yeo *Imbuement of desired attitudes by experienced and competent Singapore secondary mathematics teachers*

One of the components of the Singapore Pentagonal curricular framework is "Attitude". This paper presents findings of a survey that point to specific strategies used by Singapore mathematics teacher to imbue positive attitude towards mathematics in their students.

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Paper 4: Yew Hoong Leong & Lu Pien Cheng *Singapore mathematics teachers' design of instructional materials*

Case studies based on the data in Phase 1 of the project revealed that the teachers crafted their own instructional materials based on modifications of reference materials. This paper summarises some of the moves teachers adopted when designing instructional materials for their lessons.

Models of mathematics teaching practice in Singapore secondary schools

Berinderjeet Kaur
National Institute of Education, Nanyang Technological University
<berinderjeet.kaur@nie.edu.sg>

A model of instruction is a set of strategies that guide teachers in their instructional practice. The purpose of this paper is to dispel the myth that mathematics teaching in Singapore schools is all about drill and practice, as perceived of many Asian systems. This paper draws on data of a large project that examined the enactment of school mathematics curriculum in Singapore secondary schools. Based on the teaching practices of 30 experienced and competent teachers, a survey was constructed and administered to 677 teachers. The data from the survey showed that teachers go well beyond traditional forms of instruction in their teaching practices in Singapore secondary schools.

Leung (2001) noted that in East Asian mathematics classrooms

> Instruction is very much teacher dominated and student involvement minimal. … [Teaching is] usually conducted in whole group settings, with relatively large class sizes. … [There is] virtually no group work or activities, and memorization of mathematics is stressed … [and] students are required to learn by rote. … [Students are] required to engage in ample practice of mathematical skills, mostly without thorough understanding. (Leung, 2001, pp. 35–36).

Hogan et al. (2013) examined the instructional practices of Grade 9 mathematics teachers and found that several models of instruction were prevalent in the practices. All of which had the goal of mastery and examination preparation. In a synthesis of past mathematics classroom studies done in Singapore, Kaur (2017) conjectured that instructional practices for mathematics in Singapore classrooms, based on the data of the study by Hogan et al. (2013) and the Learners Perspective Study carried out in Singapore (Kaur, 2009), cannot be considered either Eastern or Western but a coherent combination of both. Basis of the claim is that: i) Traditional Instruction (TI) provides the foundation of the instructional order, and ii) Direct Instruction (DI) builds on TI practices and extends and refines the instructional repertoire. While Teaching for Understanding/ Co-regulated Learning Strategies (TfU/CRLS) practices build on TI and DI practices and extend the instructional repertoire even further in ways that focus on developing student understanding and student-directed learning. The study reported in this paper further illuminates models of teaching practices of mathematics teachers in Singapore secondary schools.

The Study

The study reported in this paper is part of a larger project, details of which are available elsewhere (Kaur et al., 2018; Toh et al., 2019). A study of mathematics lessons enacted by 30 experienced and competent mathematics teachers in Singapore secondary schools revealed that teacher and student actions from three main models of instruction were guiding teachers in their instructional practice. We elaborate the models and provide examples of teacher and student actions that were observed in the lessons of the experienced and competent teachers (which are marked *) as well as those that were not but were included in the survey. For actions that are marked * we also indicate the respective courses of study which are Integrated Programme (IP), Express Course (EX), Normal (Academic) Course

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(NA) and Normal (Technical) Course (NT) where the actions were observed. The IP is for the mathematically able students and the NT is for the least able ones.

Traditional Instruction (TI)

A method of instruction that is teacher-centred, rather than learner-centred, in which the focus is on rote-learning and memorisation. In the context of Asian classrooms it is often associated with drill and practice (Biggs & Watkins, 2001; Hogan et al., 2013; Leung, 2006). There were altogether 13 TI teacher actions, and examples of two such actions are as follows:

Teacher –

- *asking students direct questions to stimulate students' recall of past knowledge / check for understanding of concepts being developed in the lesson (EX, NA)
- *providing students with sufficient questions from textbooks / workbooks / other sources to practise so as to develop procedural fluency (EX, NA, NT)

Direct Instruction (DI)

A method of instruction that involves an explicit step-by-step strategy, often teacher-centred, with checks for mastery of procedural or conceptual knowledge (Hattie, 2003; Hogan et al., 2013; Good & Brophy, 2003). There were altogether nine teacher actions and two student actions and examples of two each are as follows:

Teacher –

- *using the "I do, We do, You do" strategy, i.e.
  - Demonstrating how to apply a concept / carry out a skill on the board [I do]
  - Demonstrating again the same using another similar example but with inputs from students [We do]
  - Asking the students to do a similar question by themselves [You do] (EX, NA, NT)
- *explaining what exemplary solutions of mathematics problems must contain (logical steps and clear statements and / or how marks are given for such work during examinations) (IP, EX, NA)

Students –

- *asking questions when they do not understand (IP, EX, NA, NT)
- *practising a similar problem after the teacher has shown them how to do a similar one on the board (IP, EX, NA, NT)

Teaching for Understanding (TfU)

A method of instruction that places student learning at the core. Teacher facilitates, monitors and regulates student learning through student-centred approaches (Hogan et al., 2013; Good & Brophy, 2003; Perkins, 1993). There were 13 teacher actions and 15 student actions, and examples of two each are as follows:

Teacher –

- *focusing on mathematical vocabulary (such as equations, expressions) to help students build mathematical concepts (IP, EX, NA, NT)
- *providing collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework (IP, EX, NA, NT)

Students –

- *explaining how their solutions or how their answers are obtained (IP, EX, NA, NT)
- *discussing and helping each other while doing individual seatwork (IP, EX, NA, NT)

The Survey

The survey had three parts. The first part had 60 items (36 describing teacher actions and another 24 describing student actions). Amongst these items were the seven items on TI, 11 items on DI and 28 items on TfU. In the survey, teachers were asked to reflect on their lessons for a course (IP, EX, NA or NT) they were teaching, and respond to the items indicating the frequency of their actions on a Likert Scale of 1 (Never/Rarely) to 4 (Mostly/Always). 691 teachers completed the survey. In the preliminary screening of the data, some responses were removed as they did not meet the requirements of the survey. The data of 677 teachers were used for subsequent analyses. Forty percent of the teachers were male while 60 % were female and this was representative of the demographic of the teacher population in secondary schools which were 36 % males and 64 % females (MOE, 2018). In addition, the representation by course of study, almost 65% for the IP and EX, and 35% for the NA and NT courses was also coherent with the demographic of the student population in secondary schools which was 64% and 36% respectively for the IP and EX and NA and NT courses (MOE, 2018). Forty-five percent of the teachers had more than three but less than 10 years of mathematics teaching experience while the rest 55% had more than 10 years of the same experience.

What models of instruction guide mathematics teaching in the classrooms of mathematics teachers in Singapore secondary schools, in general?

Table 1
Means of the three models of instruction

Course of Study	Mean+ Model of Instruction		
	TI	DI	TfU
All (n=677)	2.78	3.11	2.86
Integrated Programme (IP) (n=58)	2.42	3.07	3.00
Express (EX) (n=380)	2.78	3.10	2.88
Normal (Academic) (NA) (n=151)	2.81	3.10	2.77
Normal (Technical) (NT) (n=88)	2.94	3.17	2.85

+maximum = 4; minimum = 1.

Table 1 shows that teachers appear to draw on teaching moves from all the three models of instruction, though with differing emphasis to enact their lessons. Direct Instruction appears to be the dominant model that teachers draw on in all the four courses of study. In the NA and NT classes, Direct Instruction and Traditional Instruction are apparently more prevalent whilst in the IP and EX classes Direct Instruction and Teaching for Understanding are apparently more prevalent. We next examined the survey items for each course of study that had a mean greater than 3 and a standard deviation of less than or equal to 0.7. The following teaching/learning actions were found to be common across all the four courses of study.

- Teacher providing students with sufficient questions from textbooks / workbooks / other sources to practise so as to develop procedural fluency
- Students asking questions when they do not understand
- Teacher walking around the class and providing students with between-desk instruction (i.e. help them with their difficulties) when they are doing their work at their desks
- Teacher walking around the class noting student work that teacher would draw on to provide the class feedback during whole class review
- Teacher only progressing to the next objective of the lesson when he/she is confident that students have grasped the one before
- Teacher providing feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors and improve
- Teacher providing collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework
- Teacher focusing on mathematical vocabulary (such as factorise, solve) to help students adopt the correct skills needed to work on mathematical tasks
- Students explaining how their solutions or their answers are obtained

We conclude that the model of instruction that mathematics teachers in Singapore secondary schools adopt is a hybrid one comprising TI, DI and TfU. This finding lends to strengthen our earlier conjecture that mathematics instruction in Singapore secondary schools is neither Eastern nor Western but a coherent combination of both, i.e. a hybridisation of TI, DI and TfU.

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An experienced and competent teacher's instructional practice for normal technical students: A case study

Tin Lam Toh
National Institute of Education, Nanyang Technological University
<tinlam.toh@nie.edu.sg>

This paper presents a case study of an experienced and competent mathematics teacher's classroom instructional practice in a Normal Technical Mathematics course. The topic that was observed was Volume and Surface Area of a Pyramid, a subtopic within the mensuration topic in Secondary Two syllabus. The teacher used a video clip on the Egyptian Pyramids to integrate students' prior knowledge on pyramids, which raised their attention on the topic. This was followed by engaging the students in hands-on activity to understand the formulae.

The case study is part of the larger research project on enactment of the curriculum in the mathematics classroom as reported by this symposium.

Low Attaining Students

Studies have shown that low attaining students are generally visual and kinaesthetic learners (e.g. Amir & Subramaniam, 2007; Rayneri & Gerber, 2003). The mainstream education programmes worldwide are usually more theory-based than skill-based with ample hands-on opportunity for individual learners (Glass, 2003). Therefore, it is not at all surprising that this dissonance puts the low attaining students, who usually learn best through visual and physical engagement, at a disadvantage in the education system.

Low attaining students generally have little interest in academic subjects. They lack focus during lessons, have short attention span and hence tend to be restless in classes (Lui et al., 2009). Thus, typical teacher-centric teaching approaches might not be most appropriate for them. Myron and Keith (2007) stressed that in order for teachers to be more successful in working with the low attaining students, they must be more cognizant of the various learning styles of their students and attempt different teaching approaches for different groups of students.

Normal Technical Students in Mathematics

Singapore mathematics teachers are genuinely concerned about the performance in mathematics among the Normal Technical students (Toh & Lui, 2014). This concern is not unfounded as many of the Normal Technical mathematics students exhibit many of the characteristics of low attainers (Toh & Kaur, 2019).

Studies have also shown that Singapore teachers are not passively using traditional instructional materials and resource for teaching Normal Technical students. As the students' difficulties with mathematics and reasons for their lack of interest in the subject are various, teachers' effort to reach out to this group of students is also diverse. In addition to honing their pedagogical skills in the classrooms, teachers are also actively adapting less conventional instructional approaches and developing unconventional instructional material to address the learning needs of this group of students (Toh & Lui, 2014).

To have a first-hand glimpse into how mathematics lessons are conducted by a experienced and competent teacher in a typical Normal Technical class, the author (hereafter, first person pronoun) followed through one such identified teacher's lessons for two weeks on teaching a subtopic of mensuration in a Secondary Two Normal Technical

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mathematics class in a Singapore mainstream school. A few striking observations that were made will be reported in this paper.

Method

All the lessons that were observed in this study, the teacher interview, and the student interviews were video-recorded and transcribed. The video-recording, adapting the Complementary Accounts Methodology of Clarke (1998, 2001), used three video cameras to focus on: (1) the classroom as seen from the teacher's perspective; (2) the activity of two particular students in each lesson; and (3) the classroom from the perspective of an observer at the back of the classroom.

The teacher, Lucy-Marianne (pseudonym), was identified as an experienced and competent mathematics teacher by the mathematics education community. She was a Senior Teacher in her school, in her mid-forties at the time of our study, had more than ten years of experience teaching in the school and had been teaching mathematics in Express, Normal Academic and Normal Technical stream for more than fifteen years at the time when this study was conducted. In a discussion with her during the teacher interview, she expressed her passion in teaching the group of low attaining students. According to Lucy-Marianne, this group of students "deserved our attention more". She was trained to teach both Mathematics and Computer Applications.

Observation and Discussion

In unpacking teacher Lucy-Marianne's pedagogical practices from the entire set of video-recordings of her lessons, a very skilful scaffolding sequence to facilitate her students in understanding a complex concept was visible:

1. she first elicited her students' prior knowledge related to the concept;
2. she aroused her students' interest about the concept;
3. she built on their induced interest to further develop the mathematics concept;
4. she engaged her students in hands-on activities to "derive" the formula; and
5. she gave students ample opportunity to practise the application of the formulae.

During the teacher interview, she revealed that this was the constant sequence in teaching the other mathematical topics as well as to her Normal Technical students.

Eliciting students' prior knowledge

Her teaching of the subtopic on surface area and volume of a pyramid is the focus here. She built on her students' prior knowledge selectively for her lesson development, as illustrated by a portion of the dialogue below. Letters T and S denote the teacher and student participant.

Dialogue		Commentary
(after housekeeping matter)		
T:	Now let's move on to volume of pyramid – uh no, surface of pyramid [first]. OK by the way, let me introduce the word "pyramid". **What is [a] pyramid**?	*Teacher elicited her students' prior knowledge on pyramid.*
S:	A 3D.	
T	OK It's a 3-dimensional object... A pyramid is no longer flat [tapped the table],it's no longer flat [tapped whiteboard], but it's a 3-dimensional object. But **what does it look like and how does it look like…**?	*Teacher responded to a student's use of the term 3D (3 dimensional) by distinguishing between 3D and 2D objects (prior to this lesson, the students learnt mensuration of circle – a 2D object).*
S:	Cone.	

T:	It looks something like a cone. Oh, OK... "Looks something like" doesn't mean it's exactly the same. So later we are going to learn cone, today let's talk about pyramid. **Anyone can describe pyramid?**	*Teacher was careful to acknowledge the response that a pyramid looks something like a cone, but did not want to elaborate the concept of cone to avoid confusing the students (mensuration of a cone would be the next subtopic).*
	::::::::::::	
S:	Huh? Oh triangle.	
T:	Thank you … they (two students) are right... There are triangles on pyramids. So this is a pyramid like what you see in Egypt. Now, **if I were to look from top down, what do you think is on the ground?** What shape? S: I know, a square.	*Teacher elicited the responses of "triangle" and "square" from the students about their knowledge of pyramid. However, teacher did not further elaborate that the bases of pyramids can be made of other shapes at this juncture.*

Arousing students' interest and curiosity in the concept.

Teacher Lucy-Marianne skilfully related the geometrical figure of a pyramid to the Egyptian Pyramids at Giza. She discussed the historical function of the Egyptian Pyramids after showing a short video clip selected from YouTube about the Egyptian Pyramids. The content of the video clip raised students' awareness of mathematics in the real world; this is aligned to the Ministry of Education (MOE)'s desire to "prepare its citizens for a productive life in the 21st century" (MOE, 2012, p. 2). The selected video covered the students' responses: the sides of the pyramids (consisting of triangles), the plan view of the pyramids (squares), the dimensions and the historical functions of the pyramids. The use of videos in education is particularly useful for low attaining students, as it has the ability to reduce their cognitive load and facilitate their understanding of abstract concepts (Han & Toh, 2019)

Reinforcing the concept of the lateral side faces of a pyramid.

Teacher Lucy-Marianne emphasized the sides and base of a pyramid from different angles and by decomposing a three-dimensional pyramid into two-dimensional parts. Teacher Lucy-Marianne next used a worksheet (Figure 1) to reinforce the identification of the sides. Here, she unravelled the next part of the "truth" that the base of a pyramid is not necessarily a square or rectangle. She introduced pyramids with various polygonal bases. This was also the first time she insisted on the precise mathematics terminologies (lateral sides and base of a pyramid) illustrated in the dialogue below Figure 1.

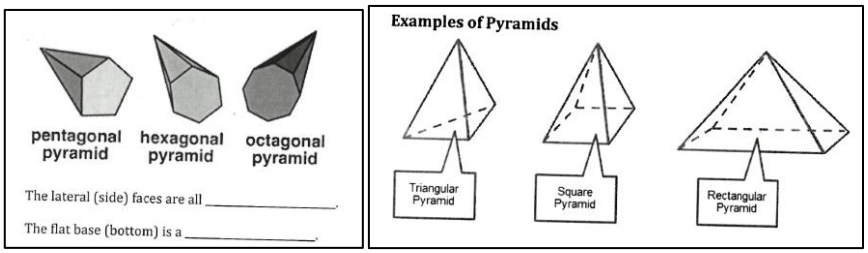

Figure 1. A portion of the worksheet used by Lucy-Marianne in introducing the faces of a pyramid

T: I want you to look at the word, the lateral side faces are? The word 'lateral' means side. Side means lateral. So the side faces are what kind of shape? … I will like to introduce a word, the flat base water missile, I call it 'polygon'. Polygon means it can be 3 sides, 4 sides, 5 sides, 6 sides, 7, 8, etc.

Deriving the procedure for calculating the total surface area of a pyramid.

The video clip and the identification the various parts of the pyramid led to the calculation of the surface area of a pyramid by considering the nets of a pyramid. She engaged her students in deriving the formulae using a hands-on approach by engaging them

to cut up a pyramid into its nets to identify the total surface area of a pyramid as the sum of the areas of the polygons in its corresponding net. This "experimental derivation" was observed in her lessons throughout this subtopic. In determining the volume of a pyramid in the succeeding subtopic, Lucy-Marianne conducted a "laboratory lesson" to demonstrate the relation between the volume of a pyramid and its related prism. The topic mensuration at the secondary level can be taught either in a very procedural manner, or one that engages the students with hands-on activities as proposed by Lim-Teo and Ng (2008). Teacher Lucy-Marianne had chosen the latter to better match the needs of her students.

Ample opportunity to practice. As in other observation of the Singapore classrooms, teacher Lucy-Marianne designed her worksheets to give sufficient structured and guided practice for her students. This will not be elaborated in this paper.

Conclusion

This is an episode of teaching mathematics to Normal Technical students by an experienced and competent teacher. While the teacher was cognizant of the importance of maintaining the rigor of the mathematics curriculum even for the low attaining students, the teacher was also skilful in engaging her students in activating their prior knowledge, exciting them with the mathematics in the real-world, and chunking up big group of mathematical content into manageable bites for her students. The teacher strove to develop in her students a relational understanding of the mathematical concepts through appropriate student engagement, while using video clip and storytelling to excite her students in the mathematical concepts. The lesson was evidence of her attempt at striking a balance between developing her students' cognitive and affective aspects of learning.

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Imbuement of desired attitudes by experienced and competent Singapore secondary mathematics teachers

Joseph Boon Wooi Yeo
National Institute of Education, Nanyang Technological University
<josephbw.yeo@nie.edu.sg>

This paper reports how 30 experienced and competent Singapore secondary mathematics teachers attempted to imbue desired attitudes in their students and some possible factors that might have influenced the teachers' choice of instructional approaches. It was found from the analysis of lesson observations of these teachers that most of those teaching lower-ability students tended to build their students' confidence and perseverance, while those teaching higher-ability students were more inclined to help their students appreciate the relevance of mathematics. Only a minority of the teachers tried to make lessons fun by using mathematics-related resources or telling non-mathematics-related jokes. It was also discovered from the teacher interviews that two factors appeared to influence the teachers' choice of the types of positive attitudes to develop in their students: the abilities of their students and the beliefs of the teachers on what mathematics is.

Most research on the affective domain in mathematics education focuses on finding out students' existing attitudes and their effect on other variables such as test performance (Aiken, 1970; Leder & Forgasz, 2006; McLeod, 1992), and students' and teachers' beliefs (Leder et al., 2002; Maaβ & Schlöglmann, 2009; Pepin & Roesken-Winter, 2015). In Singapore, research studies on affective variables also follow the international trend (e.g. Kay, 2003; Ng-Gan, 1987; Tan, 2011) and there are few intervention studies on changing students' attitudes (Yeo, 2018; Yeo et al., 2019).

This paper reports how some mathematics teachers attempted to imbue desired attitudes among their students as part of a programmatic research study on how 30 experienced and competent Singapore teachers enacted the secondary school mathematics curriculum. In the Mathematics Framework for the Singapore school curriculum (Ministry of Education, 1990; 2012), attitudes is one of the main components, consisting of beliefs, interest, appreciation, confidence and perseverance. It is beyond the scope of the research to study whether or how the teachers tried to affirm or change their students' beliefs about mathematics. Instead, this paper will report how most of these 30 teachers attempted to instil confidence in their students, encourage them to persevere, help them to appreciate mathematics and make lessons fun to interest them.

Methodology

In the programmatic research, 30 experienced and competent teachers were videoed teaching a topic for two to three weeks to find out how they implemented the curriculum. For the purpose of this project, an experienced and competent teacher was one who had taught the same course of study for a minimum of five years, and was recognized by the school or school cluster as a competent teacher who had developed an effective approach of teaching mathematics. There are four courses of study in Singapore secondary schools: Integrated Programme (IP), Express, Normal (Academic) (NA) and Normal (Technical) (NT). In general, the abilities of the students decrease from IP to Express to NA and then to NT. For each lesson, two different focus students were also videoed to observe how they responded during the lesson and how they did the mathematics tasks.

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Each teacher was also interviewed four times: once before the first lesson, twice at appropriate junctures during the series of lessons and the last time after the last lesson. The purpose of the teacher interviews was to find out more about how and why the teachers had chosen to enact the curriculum in the ways observed during their lessons. At the end of each lesson, the two focus students were also interviewed separately to find out their reactions to the lesson and how much they had learnt. For more details on the data collection, the reader can refer to Toh et al. (2019).

This paper only reports on one aspect of the curriculum enactment: the imbuement of desired attitudes in the students. To analyse the data, the 211 lessons of the 30 teachers were examined to pick up episodes of the teachers trying to cultivate positive attitudes in the classroom. These episodes were then classified according to the sub-components of attitudes in the Mathematics Framework described earlier. The transcripts of the teacher and student interviews were also analysed to triangulate the data obtained from the lesson observations.

Findings and Discussion

Table 1 on the following page shows the number (and percentage) of the 30 teachers in the four courses of study who attempted to imbue desired attitudes in their students using the respective instructional strategies. For each of the first three sub-categories of confidence, perseverance and appreciation, the teachers mainly utilised one instructional approach as shown in the table; while for the last sub-category of interest, the teachers generally employed two pedagogical strategies: using mathematics-related resources and/or telling non-mathematics-related stories or jokes. Some teachers also tried to develop more than one type of desired attitude.

From Table 1, we observed that most of the teachers (26 out of 30, or 86.7%) had tried to imbue desired attitudes in their students. Their foci were mainly in the areas of building students' confidence in doing mathematics by starting with tasks that students could do before progressing to more difficult tasks (20 out of 30, or 66.7%), followed by encouraging the class to persevere and to do well in mathematics (15 out of 30, or 50%). Of lower priorities were helping students appreciate the relevance of mathematics by showing real-life examples and/or applications (11 out of 30, or 36.7%) and making lessons fun to arouse the interest of their students (6 out of 30, or 20%). What was not shown in the table was that slightly more teachers (4 teachers) made lessons interesting by telling non-mathematics-related stories or jokes than those (3 teachers) who did this by using mathematics-related resources, including a teacher who did both.

On closer inspection, across the four courses of study, it is observed that all the teachers teaching the NT and NA courses (which are for lower-ability students) and 8 out of the 10 Express teachers (i.e. 80%) had attempted to develop desired attitudes in their students, but only two of the four IP teachers (i.e. 50%) had done the same. For the NT, NA and Express classes, most of the teachers focused on building students' confidence and encouraging the class to persevere, followed by helping students appreciate the relevance of mathematics and making lessons interesting. But for the IP course of study (which is for higher-ability students), the focus of the teachers was more on helping students appreciate the relevance of mathematics. In fact, only one of the four IP teachers had tried to encourage her class to persevere on only one occasion in all her seven one-hour lessons that were observed over more than two weeks, i.e. encouraging their students did not seem to be a high priority among IP teachers.

Table 1
Instructional Strategies for Imbuing Desired Attitudes in Students

Instructional Approach	Number (and Percentage) of Teachers				
	IP (n = 4)	EX (n = 10)	NA (n = 8)	NT (n = 8)	Total (N = 30)
Building students' confidence in doing mathematics by starting with tasks that students can do before progressing to more difficult tasks	0 (0%)	6 (60%)	8 (100%)	6 (75%)	20 (66.7%)
Encouraging the class to persevere and to do well in mathematics.	1 (25%)	5 (50%)	5 (62.5%)	4 (50%)	15 (50%)
Helping students appreciate the relevance of mathematics by showing real-life examples and/or applications	2 (50%)	2 (20%)	4 (50%)	3 (37.5%)	11 (36.7%)
Making lessons interesting by using mathematics-related resources and/or telling non-mathematics-related stories	0 (0%)	2 (20%)	2 (25%)	2 (25%)	6 (20%)
Attempting to imbue any desired attitudes in students	2 (50%)	8 (80%)	8 (100%)	8 (100%)	26 (86.7%)

From the above analysis, it seems that one factor that might have influenced the teachers' instructional strategies in imbuing what sub-category of desired attitudes is the abilities of the students whom they were teaching in their respective course of study: for lower-ability students, their teachers focused on building their confidence and encouraging them to persevere, but for higher-ability students, their teachers were more inclined to help them appreciate the relevance of mathematics. This is further confirmed by interviews with the teachers. For example, a teacher said that her type of students needed motivation to solve more difficult mathematical problems and so she used an amusing video to provide the link to real life and to entice her class to solve the problems. The following shows part of a transcript of an interview with the teacher.

Interviewer: So what is your purpose for showing them this video?

Teacher: It's actually to entice them to be interested in doing mathematics because … when you keep on practising and they don't see how it can be linked, it is very difficult. So we want to see, eh, ancient times people are already using Pythagoras' theorem ... Because, my class, I think they need this kind of motivation, because some of them will fall into a world of their own very easily. So we wanted them to … entice them to this kind of thing … so after this, what they will do is, the king [from the video] has a series of problems, so they will try to solve the king's problems by Pythagoras' theorem.

Another factor that might have influenced the teachers' instructional approaches in cultivating which kind of positive attitudes is the beliefs of the teachers. For example, a teacher encouraged his students to try to score at least a few marks for a difficult exam-type question because he revealed during an interview that he believed that mathematics was

about resilience and so he was attempting to convince his class not to give up on such examination questions.

Conclusion

The study has shown how some experienced and competent teachers in Singapore attempted to imbue desired attitudes in their students. They focused mainly on building their lower-ability students' confidence and perseverance, while helping higher-ability students appreciate the relevance of mathematics. The least priority among the teachers was making lessons interesting. An implication for local teachers is maybe they should emulate the examples of the experienced and competent teachers in developing confidence, perseverance and appreciation in their students (if they are not already doing so), but at the same time, they could perhaps pay more attention to arousing in their students interest in mathematics. A possible area for future research is to study whether the students had developed the desired attitudes under the instructional strategies adopted by the teachers.

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Singapore mathematics teachers' design of instructional materials

Yew Hoong Leong
National Institute of Education, Nanyang Technological University
<yewhoong.leong@nie.edu.sg>

Lu Pien Cheng
National Institute of Education, Nanyang Technological University
<lupien.cheng@nie.edu.sg>

This paper reports on one aspect of a bigger project: teachers' design of instructional materials. We found a number of design moves used by the teachers in our study. In this paper, we report three of them: Making things explicit, making connections, and re-sequencing practice examples.

This paper focuses on one major component of the project which examined the enactment of the Singapore mathematics curriculum in the Secondary Schools: the design and use of instructional materials by the teachers. We define instructional materials to be classroom-ready materials that teachers incorporate into their lessons for students' direct access for their learning. We make a distinction between *instructional materials (IM)* and *reference materials (RM)*. The latter are resources (including textbooks) which teachers refer to while planning for lessons; the former are the actual materials that are brought into their classrooms for use in their mathematics instruction. For most teachers which were the subjects of our study, their instructional materials differ substantially from their reference materials – it is this 'transformational space' that is an area of interest to us. For the rest of this paper, we will briefly describe a few such transformational moves as illustrated by some teachers in our study and their underlying intentions.

Transform Move 1: Making things explicit

The fuller version in the examination of this move is in Leong et al. (2019). We provide a brief description here. This move is illustrated by Teacher Teck Kim. Repeatedly, in the interviews with him, he mentioned "making explicit" as a major goal in the design of instructional materials. That is, in selecting and modifying from RM (mainly the textbook subscribed by the school), he considered some of the contents as displayed in the textbook not sufficiently clear to the students; in crafting the IM, he was thus consciously governed by the principle of making the mathematical content more explicit to the students.

Figure 1 shows an example of such an explication deliberated by Teacher Teck Kim. He made the following adaptations (among others): (i) ln the RM, the textual explanation of column vectors was located at a section that was separate from the vector diagram. in Teck Kim's IM, he merged the textual mode into the visual representation of column vectors. Not only was the label of $\binom{-3}{4}$ placed beside the drawn vector, the explanation of translation of "-3" and "4" was also summarily fused into the diagram. This merging of representational modes was the way in which Teck Kim made explicit—in this case the links among the drawn vector, the column vector notation, and the translational significance. (ii) The two examples in the RM were $\binom{2}{3}$ and $\binom{-1}{-4}$. The two examples in Teck Kim's notes were $\binom{-3}{4}$ and $\binom{-3}{-4}$[the latter is not shown in Figure 1 due to space constraints]. Apart from the fact that the magnitudes of these vectors yielded an integer value, not a surd, and thus potentially reduce computational complexity so that the focus was on the definition and method of obtaining the magnitude, the choice of $\binom{-3}{4}$ and $\binom{-3}{-4}$ shows a one-component variation only in the translation in the y-direction, allowing the teacher to focus students' attention on the

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translational significance when "4" is replaced with "-4", thus highlighting the need to attend carefully to signs. In other words, Teck Kim re-worked the examples to make explicit critical ideas (perhaps, even potential student mistakes) which may have otherwise been unnoticed by the students. (iii) [Not shown in Figure 1] The task implicit in the RM required students' to "write" the given drawn vector in column vector notation; the task in Teck Kim's IM [not shown in Figure 1] required students to do the reverse: to "draw" vector given its column vector notation. He made explicit by filling a gap in the textbook. In this case, the gap was the skill of drawing vectors.

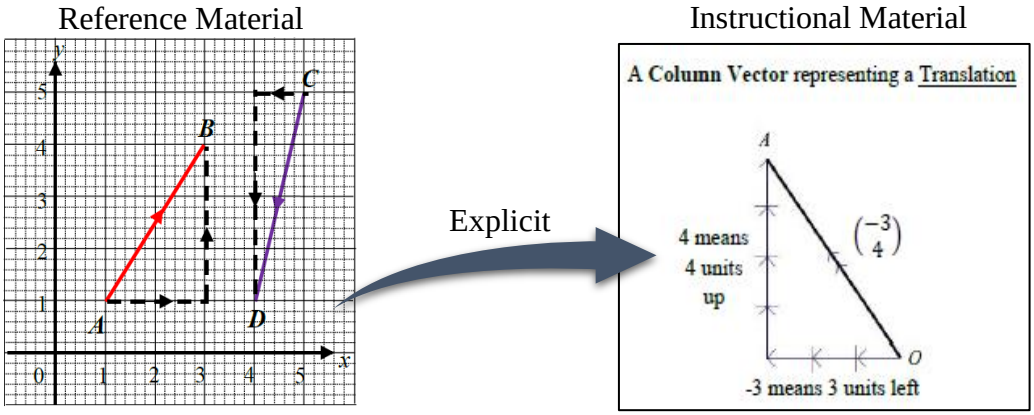

Figure 1. Making explicit from reference materials to instructional materials

Transform Move 2: Making connections

We illustrate this move by drawing upon the IM of Teacher Siew Ong. The phrase "making connections" – and similar phrases – occur frequently in her talk during our interview sessions with her. This move is particularly significant as connection-making in instructional work is highlighted as desirable in Singapore's official documents: "connections refer to the ability to see and make linkages among mathematical ideas …" (Ministry of Education, 2012, p. 15, emphasis added).

The context was the method of "completing the square". The RM presents an "investigative task" consisting of a table with four columns entitled (from left to right): "Quadratic Expression", "Number that must be added to complete the square", "Half the coefficient of x", "Quadratic expression in the form $(x+a)^2 - b$". An example as a row entry was then given for "$x^2 + 2x$" in the first column, "$1^2 = 1$" in the second column, "$\frac{2}{2} = 1$" in the third column, and the algebraic working to obtain $(x+1)^2 - 1$ in the last column. Other blank rows were given in the table below this first entry to provide working space for other samples of algebraic expressions of the form $x^2 + px$.

The IM designed by Teacher Siew Ong was an adaptation of the RM. She retained the four columns and kept largely to the titles of the first and the fourth columns (the 'beginning form' and the 'targeted complete square form'). She renamed the middle two columns as "Geometric representation" (second column) and "Term to be added" (third column). Figure 2 shows how the entry in the second column looks like for the same example of $x^2 + 2x$.

Different from the RM, she intended to help students connect "square" in "completing the square" to a "geometric square". There is thus a deliberate design decision to draw students' attention to intermodal links – between the algebraic mode and the geometric mode of representation. The geometric square provided a more natural motivation and hint as to

what value need "to be added" (language of Column 3) within the perforated small square to "complete the (geometric) square". This shift of focus rendered the step in Column 3 of the RM ("half the coefficient …") unnecessary as it would have become more intuitive from the geometric mode of representation within the context of forming a geometric square. [As an aside, the algebraic working in Column 4 now takes on a different function: it is not merely an algebraic procedure to complete; it is a static record (algebraically) of what happens dynamically over the entries in the last three columns. This further strengthens the algebraic-geometric connection].

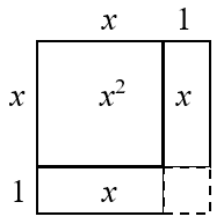

Figure 2. Geometric representation of $x^2 + 2x$ to set up for completing the square

In addition, to set up this way of thinking by students, that is, to view a quadratic expression as 'almost a square', she designed a prior page (not found in RM) where numbers (more accessible to students initially than algebraic expressions) were also represented geometrically as almost a square. As an example, 120 where written as $121 - 1 = 11^2 - 1$. This was also represented geometrically as a square of side 11 with a tiny square of 1^2 at the corner snipped off. This additional preamble that she designed revealed her deliberate effort at connection in at least these ways: (i) intermodal connections not only between algebraic and geometric representations, but also numerical to algebraic and geometric; (ii) conceptual connections – she recognised that students had prior familiarity with numerical perfect squares such as $121 = 11^2$. She drew from this prior conception to connect it to their other prior familiar imagery of geometric squares. These were then linked and further developed into 'almost square' in anticipation of connecting to the method of completing the square. In other words, she connected concepts by developing tightly from earlier concepts.

Transform Move 3: Re-sequencing practice examples

The details for this move can be found in Leong et al. (In press). As in the first move, we provide here a brief description. The teacher we studied for this move was Teacher Beng Choon. She designed the IM for the purpose of helping students gain proficiency with some 'rules' within the topic of differentiation. For the purpose of this paper, we restrict our consideration to the 'formula' of $\frac{d}{dx}(x^n) = nx^{n-1}$.

In her case, we were unsure as to the specific RM she relied upon most. Being an experienced teacher for many years, she could not specify a particular textbook she adapted from as her IM had evolved throughout the years over many rounds. For the purpose of this discussion, we referred to one common textbook to serve as a comparison to the examples she sequenced for this same section immediately after the introduction of the formula. The textbook provided three examples for application of this formula in this order: $\frac{1}{x^2}$, $\sqrt{x}$, and 1. The examples that appeared in Beng Choon's IM were: x^3, 5, $\frac{1}{x}$, and $\sqrt{x}$. Figure 3 provides a summary of what she wrote on the board for each item and how she explained the

procedure to obtain the final answer. Her main goal was to help students recognise the form x^n so that they can apply the formula correctly. As such, she needed to *vary the form* – so that they can 'see' how surface forms that do not initially look like x^n can be re-written in such a form for correct use of the formula. At the same time, she was cognizant that students did not get discouraged by difficulties and so she *proceeded gradually from simpler cases* of the form. A brief chronology: She started with x^3 as it is most recognisable as x^n. The switch to "5" was deliberate as she wanted to draw students away from fixation of formula-application; rather, they can think graphically and connect to differentiation as "finding gradient". The third and the fourth items show progressive complexity in recognising and rewriting into the form.

(a) $x^3 \xrightarrow[\text{formula}]{\text{apply}} 3x^2$

(b) $5 \xrightarrow{\text{gesture}}$ hor. line $\xrightarrow[\text{gradient}]{\text{find}} 0$

(c) $\frac{1}{x} \xrightarrow{\text{rewrite}} x^{-1} \xrightarrow[\text{formula}]{\text{apply}} -x^{-2} \xrightarrow{\text{rewrite}} -\frac{1}{x^2}$

(d) $\sqrt{x} \xrightarrow{\text{rewrite}} x^{\frac{1}{2}} \xrightarrow[\text{formula}]{\text{apply}} \frac{1}{2}x^{-\frac{1}{2}} \xrightarrow{\text{rewrite}} \frac{1}{2\sqrt{x}}$

Figure 3. Summary of the procedures explained for each item by Teacher Beng Choon

Discussion

Clearly, these moves as described are not exhaustive nor are they unrelated. A cursory reflection would reveal that a teacher who wishes to adopt such moves may do so in an integrated way for the same activity – that is, making things explicit, making connections, and re-sequencing of practice examples can be applied concurrently. The purpose, however, of this article is to illustrate examples of each of these moves as they were adopted by the teachers in our study. This paper highlights that Singapore secondary mathematics teachers do not merely 'teach from the textbook'; rather, they make intentional moves to adapt the reference materials in ways that fit their instructional purposes which are largely 'sound' both from a theoretical perspective and in terms of concurrence to policy mandates. Often, these moves are elusive to a casual observer. The results of this study reminds us as researchers that we should avoid the simple route of pigeonholing pedagogical enactments based on cursory observations.

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Strengths approaches in early childhood mathematics education

Amy MacDonald
Charles Sturt University
<amacdonald@csu.edu.au>

Fiona Collins
Charles Sturt University
<fcollins@csu.edu.au>

Angela Fenton
Charles Sturt University
<afenton@csu.edu.au>

Steve Murphy
Charles Sturt University
<smurphy@csu.edu.au>

Matt Sexton
Australian Catholic University
<matthew.sexton@acu.edu.au>

Joce Nuttall
Australian Catholic University
<joce.nuttall@acu.edu.au>

James Russo
Monash University
<james.russo@monash.edu>

Toby Russo
Bell Primary School
<toby@bellps.vic.edu.au>

This symposium discusses the use of strengths approaches in early childhood mathematics education. *Strengths approaches* can be conceptualised as educational practices that recognise, and utilise, children's strengths. Strengths approaches originate in the social work sector, but are growing in recognition in early childhood education. This symposium considers how strengths approaches might be adopted in early childhood mathematics education, specifically, encouraging pedagogical approaches that recognise, and build upon, young children's strengths in mathematics. This symposium presents theorisation and a case illustration of how strengths approaches can be meaningfully utilised in early childhood settings in order to enhance mathematical learning opportunities for young children. The symposium addresses three aspects: (1) Overview of strengths approaches; (2) Application of strengths approaches; and (3) Leadership to promote strengths approaches; illustrated within the context of early childhood mathematics education.

The symposium format is as follows:

Chair: Amy MacDonald

Paper 1: Fiona Collins & Angela Fenton *An introduction to the strengths approach*

Paper 2: Amy MacDonald & Steve Murphy *A strengths approach to birth-to-3 mathematics education: The case of Banjo Childcare Centre*

Paper 3: Matt Sexton & Joce Nuttall *Leadership of strengths-based approaches for early years mathematics education: Using CHAT as a framework for educational leaders' professional learning leadership*

Discussants: James Russo & Toby Russo

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An introduction to the strengths approach

Fiona Collins
Charles Sturt University
<fcollins@csu.edu.au>

Angela Fenton
Charles Sturt University
<afenton@csu.edu.au>

This paper provides a foundation for the Research Symposium, "*Strengths Approaches in Early Childhood Mathematics Education"* by providing an overview of the development of strengths-based approaches in social work and education. A framework, adapted from the Strengths Approach (McCashen, 2017), for applying a strengths-based approach in early childhood mathematics education is introduced.

An Overview of Strengths-Based Approaches

Strengths-based approaches, originally developed in social work practice and psychology (Glicken, 2004; Saleebey, 1996; Seligman, 1990), are gaining momentum as practitioners see applications in other human service fields such as education and health care (Pulla, 2017). Globally, there is a growing expectation that professionals working with children in their early years will adopt strengths-based approaches "to support the access and participation of all children and families, especially those with complex needs" (Fenton et al., 2015, p. 29). Furthermore, the Belonging, Being & Becoming: Early Years Learning Framework for Australia (EYLF) states that "in order to engage children actively in learning, educators identify children's strengths and interests" (DEEWR, 2009, p. 9) and extends this by explaining that "early childhood educators who are committed to equity believe in all children's capacities to succeed, regardless of diverse circumstances and abilities" (DEEWR, 2009, p.13). This paper provides an overview of strength-based approaches and then suggests a specific framework, adapted from the Strengths Approach (McCashen, 2017), for applying a strengths-based approach to support children in the early years in their learning of mathematics.

The development of strengths-based approaches in the 1980s and 1990s, alongside narrative therapies and solution-focused therapies, involved an entirely different approach to be adopted by professionals in human service practice (McCashen, 2017). Previously, therapy was pathology focused, where people and their problems were categorised according to diagnoses, behaviours and/or problems (McCashen, 2017); the focus was very much on what was wrong and as such has since been referred to as a deficit model. Later models shifted focus towards the specific circumstances of the client and the organisations around them available for support; the therapist was viewed as the "expert" and tasked with "fixing" the client in order to allow them to overcome their problem and return to a "normal" life (McCashen, 2017). However, these models raised concerns of imparting "power over" clients (McCashen, 2017, p. 54). In contrast, strength-based approaches are centred on the belief that all human beings are individuals who possess strengths, are experts of their own circumstances, and have the capacity for change if they are provided with opportunities and access to appropriate resources (Glicken, 2004; McCashen, 2017; Saleebey, 2009). Saleebey (2009, p. 97) states that "almost anything can be considered a strength under certain conditions," whilst McCashen (2017) goes further and defines strengths as

> anything people have that helps them to achieve, to overcome problems, to build on things that are already positive, to learn, grow, and be fulfilled. Strengths can be understood in terms of personal qualities – positive characteristics and things that people are good at. Strengths include people's skills and capacities, their aspirations and values and the resources in their environment. (p. 33)

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In education contexts, strengths-based approaches can also present an alternate point of view (Fenton, 2013) that is in contrast to a deficits view of learning, where emphasis is placed on 'gaps' in a child's knowledge and/or skills, or identified learning problems, such as a focus on children with learning disabilities (see Harry & Klingner, 2007). For example, educators working from a deficit model design learning experiences to help children remediate "gaps" in knowledge and/or model skills which are not evident. MacDonald (2018) warns that adopting a deficit view of a child's mathematical capacity can lead to a perpetual cycle of negative expectations, which can lead to opportunities for mathematical learning being blocked, which can contribute to negative mathematical learning experiences, ultimately resulting in disempowerment.

Instead, strengths-based approaches require practitioners to look at "individuals, families, and communities … in light of their capacities, talents, competencies, possibilities, visions, values, and hopes" (Saleebey, 1996, p. 297). In essence, strengths approaches within education are student-centred, and focussed on measuring children's strengths, catering for individual children's needs, collaboration, and the deliberate application and intentional development of children's strengths (Lopez & Louis, 2009). Mathematics educators working with a strengths approach will focus on what mathematics children can do, as well as the opportunities and resources available to assist in the development of their strengths and capacities to meet identified learning goals. MacDonald (2018) described this process as a competency cycle, "a process of creating positive expectations and opening the way for the development of new competencies" (p. 144).

Whilst strengths approaches are being encouraged in early childhood education, a number of critiques of this philosophy have also been expressed, including: that it is simply another way of describing being positive, and/or a way of reframing deficits through ignoring or denying real problems (Saleebey, 1996). The strengths approach has also been criticised for being "overly simplistic and superficial" (Glicken, 2004, p. 14) and for being an ideological theology (Epstein, 2012). Glicken (2004) cautions strengths practitioners about the complexity of discovering and applying strengths and warns that it can be a time consuming process. Furthermore, there is the potential for educators and children to adopt fixed mindsets if practice is limited merely to the identification and affirmation of strengths, without the nurturing and development of new talents (Lopez & Louis, 2009).

An Introduction to the Strengths Approach

Building on the foundations of strengths perspectives' origins in the United States, the Strengths Approach, was developed further in Australia by St. Luke's, a social services organisation based in Bendigo, Victoria, as a philosophy for collaborating with others in an effort to achieve a positive transformation (McCashen, 2017). St. Luke's sought to develop practice-based principles to guide their practical work with children and families. The approach "encourages the identification of resources and the use of challenges, as they occur, to create resilience and aptitude when working with issues" (Fenton et al., 2016, p. 46). A number of principles guided the development of the Strengths Approach, including: the dignity and capabilities of each person as their own change agent; the ability of each person to enact their own strengths and capabilities; the identification and mobilisation of resources to support development; and a collaborative sharing of power between all stakeholders (McCashen, 2017).

The Strengths Approach is a framework for practice that encompasses reflection, learning, planning, action and review. It is important to emphasise that the Strengths Approach not only looks at the positives. In fact, the approach generally starts from clearly

exploring a challenge, complex issue or need. The Column Approach (McCashen, 2017) is provided as a scaffold for applying a Strengths Approach in five steps. Practitioners are encouraged to consider with all stakeholders: (i) What is the challenge here? (ii) What is the ultimate goal/vision? (iii) What existing strengths and capacities can we utilise? (iv) What extra resources are available? (v) With the previous steps in mind – what is our plan of action? A table version of the Column Approach (Table 1) can be used by educators, to assist children in their early years to develop their mathematical knowledge, skills and understanding.

Table 1
*The Column Approach**

Stories and issues	The picture of the future	Strengths and capacities	Other resources	Plans and steps
Ask questions that invite children to share their mathematical stories and enable them to clarify the challenges, such as: • What's the mathematical challenge or problem? • What's happening here? • What are you trying to do? • What have you discovered? • Have you solved a problem, or overcome a challenge like this before? If so, can you tell me about it?	Ask questions that help children explore their mathematical aspirations, dreams, interests and goals, such as: • What do you want to know/be able to do? • What do you want to discover? • Why do you want to overcome this mathematical challenge/solve this mathematical problem? • What do you need to know? • What will solving this allow you to do? • What are you interested in?	Ask children questions that help them explore their strengths, as well as their mathematical capacities such as: • What are you good at? • What do you like doing? • What do the special people in your life think you are good at? • What were you thinking about when this happened? • What do you know that might be helpful here? • What have you done in the past when you have experienced similar mathematical challenges / problems?	Ask questions that help children to identify resources that might help them reach their mathematical goals, such as: • Who else might be able to help? • What other skills or resources might be helpful? • What have people done already that has helped? • Who or what has been helpful in the past when you have had mathematical challenges / problems like this?	Ask questions that help children to specify steps towards the achievement of their mathematical goals, such as: • What are you going to do next? • What information will you use? • What skills and strengths will you use? • Who will help? How will they help? • What resources will you use? • When will it be done?

* Adapted from McCashen (2017) and MacDonald (2018).

Implications

The Column Approach provides a "mind map" (McCashen, 2017, p. 97) for working with children to help them: develop a narrative of their opportunities for learning in mathematics; identify their mathematical hopes and dreams; consider their strengths and mathematical capacities; identify resources that are available to them; and map out a way for them to move forward. It is also recommended that a proactive first step for educators is to

identify what they do well (for example pedagogical approaches, resource development, leadership etc.) and ensure that they continually model and refine these strengths as they work with children to help them recognise and utilise their own strengths in the learning process and environment (Lopez & Louis, 2009). In this way, drawing on its social service and psychological origins, and particularly guided with a Column Approach, the Strengths Approach can be a practical collaborative framework for acknowledging children's mathematical curiosity and challenges, honouring their existing mathematical knowledge, and importantly drawing on their strengths and mathematical capacities as their learning develops.

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A strengths approach to birth to 3 mathematics education: The case of Banjo Childcare Centre

Amy MacDonald
Charles Sturt University
<amacdonald@csu.edu.au>

Steve Murphy
Charles Sturt University
<smurphy@csu.edu.au>

This paper contributes to the Research Symposium, *"Strengths approaches in early childhood mathematics education"* by providing an illustration of how an early childhood centre adopts a strengths approach to mathematics education for birth to three-year-old children. A case illustration is drawn from a current Australian Research Council-funded study focusing on mathematics education for children under three years of age. The case is analysed and described using a five-step strengths-based framework.

Introduction

It is well-established that young children, prior to starting school, are capable of engaging with a range of mathematical ideas (Gervasoni & Perry, 2015; MacDonald & Carmichael, 2018). However, most of this research has focused on children aged four years and older, with birth to three mathematics education receiving very little attention (MacDonald & Murphy, 2019). However, a current Australian Research Council-funded study being conducted by the lead author of this paper is addressing this dearth of research through a national study of mathematics education for children aged under three years. As part of the larger study, case studies of birth to three education settings are being conducted in order to examine mathematics education opportunities afforded to very young children, and the beliefs and practices of their educators which influence these opportunities. Drawing from the larger study, this paper presents a case illustration of the birth to three learning environment at Banjo Childcare Centre (pseudonym), a long day care service located in regional New South Wales (NSW), Australia. Six early childhood educators and 17 children participated in the case study, and the children ranged from 13 to 40 months in age. The authors of this paper spent two days in the site, gathering data in the forms of continuous video recordings; video and photographic observations; documents such as learning stories and daily reflections; and anecdotes from educators. This case has been selected as it illustrates how a strengths approach to mathematics education can help an early childhood service overcome a range of challenges and barriers, and utilise their unique strengths and resources in order to provide high-quality mathematics education for very young children. In the case illustration that follows, we apply the Column Approach as described by Collins and Fenton (under review) (Paper 1 in this Symposium) in order to analyse how Banjo Childcare Centre are taking a strengths-based approach to mathematics education for the birth-to-three-year-olds in their centre. The case is structured according to the five-step framework, namely: (i) Stories and issues; (ii) The picture of the future; (iii) Strengths and capacities; (iv) Other resources; and (v) Plans and steps.

The Case of Banjo Childcare Centre

Stories and Issues

As noted in Paper 1 in this symposium (Collins & Fenton), a strengths approach does not only focus on the positives; rather, the use of the approach generally starts from clearly

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exploring a challenge, complex issue, or need. Banjo Childcare Centre, and the community it serves, experience a range of challenges and complex circumstances. The service has a maximum of 50 approved places; however, at the time of this study, only 42 of these places were filled. The community receives a relatively low score on the *Socio-Economic Indexes for Areas* (SIEFA) - 869 compared to the NSW average of 1001. This score indicated a disadvantaged socio-economic position characterised by attributes such as low income, low educational attainment, and high unemployment (Australian Bureau of Statistics, 2011). According to the 2018 Australian Early Development Census (AEDC; Commonwealth of Australia, 2019), 38.1% of children in this community are developmentally-vulnerable on one or more AEDC domains; a figure nearly double the NSW average (19.9%). Moreover, 23.8% of children are developmentally-vulnerable on two or more domains, compared to the NSW average of 9.6%. The centre itself experiences challenges in the current early childhood reform climate, receiving a 2018 *National Quality Standard* (NQS; Australian Children's Education and Care Quality Authority, 2019) rating of "Meeting" the NQS, a decline from their 2013 rating of "Exceeding" the NQS.

The Picture of the Future

The data presented above paint a deficits-focused picture of Banjo Childcare Centre and their community. However, these data are not how they see themselves nor the future they see for their children. The centre's handbook states that educators "maintain a high level of professionalism through working together, supporting each other and continuously expanding [their] knowledge base", that educators are "confident in children's ability to learn" and that they "encourage the children to develop a positive attitude towards learning". This positive picture of the future extends to mathematics learning at the centre. While not explicitly articulated, a strengths-based picture of the future is communicated in various ways. The importance of mathematics is highlighted through displays and explicit weekly reporting focussed on mathematics learning. There is an expectation that children at the centre, including very young children, can engage in sophisticated mathematical activities. Records showed in one week children three years old and younger were engaged in various activities that involved measuring height and volume, additive thinking, and counting using Wiradjuri words (the local Indigenous language). Analogue clocks were displayed alongside daily events in the toddler's room (see Figure 1). Collectively, this evidence suggests the Centre pictures a future where their children are capable and confident users of mathematics.

Strengths and Capacities

The centre's handbook makes explicit that educators respond to the strengths and capacities of the children to guide learning and teaching. The handbook states that educators use their observations of children "to develop an educational play based program". Further, "children are given the chance to make decisions, experiment, and explore with a wide range of activities." This philosophy was evident in the way educators responded to children's strengths and capacities through their play in order to engage them in mathematical activities. Counting was regularly introduced to children's activities; for example, ball bounces being tallied, and the time before a jump counted. Measuring concepts were incorporated into play, such as big and small when kicking a football, fast and slow when bike riding, tall and taller when measuring each other's heights, and volumes when cooking. Locating language was built into children's play; for example, when children were playing on a pretend horse (see Figure 2) an educator led a discussion of who was in front, on, and

under. Educators helped children develop plans and procedures associated with their games. In one instance, two children were endeavouring to untangle a ball on a rope, with one up the tree and one underneath, and another child playing nearby accidentally impeding the task. An educator supported the children in a complex series of actions to safely and successfully free the ball.

Figure 1. Clock display in toddler's room.

Figure 2. Pretend horse using saddles and pipe.

Not only did educators notice and capitalise on children's strengths and interests as they presented during play, but they deliberately shaped the learning environment so that these mathematical learning opportunities regularly arose. The physical environment was spatially challenging, with winding and intersecting paths, objects of various heights, and spaces of irregular form (see Figure 3). These spaces encouraged children to problem pose and engage in mathematical activity. Further, the learning culture supported children to fully exploit these spaces to exercise their strengths and capacities. Educators did little to structure play, allowing children to structure their own play opportunities. For example, the play space included a rope and pulley system attached to a tree. It was only once a small group of children were engaged in play that involved getting buckets of bark high into the branches did an educator join to discuss alternate ways of using the ropes to move the buckets higher. A culture of permitting risk also supported children to fully engage in this complex learning environment. Rather than discouraging tree climbing, objects were deliberately placed to facilitate it. Similarly, when a group of children were jumping from objects in the yard, the educator nearby did not restrict the activity, but rather supervised and engaged in discussion about the height of objects and the size of the jump.

Figure 3. Spatially complex learning environment.

Other Resources

Banjo Childcare Centre, and its community, does not have significant financial resources. Despite this, they have been able to create a rich environment to facilitate mathematical learning through resourceful behaviours that are both strategic and opportunistic. Reclaimed, recycled and repurposed objects make up the play spaces, including tyres of various sizes, wooden pallets, restored old play equipment, and items such as the pipe and saddle described earlier (see Figures 2 and 3). The centre also makes excellent use of the resources of its local community to enhance children's engagement and learning. In particular, Wiradjuri culture—the culture of the traditional owners of the land where the centre is located—is strongly represented in the displays and practices of the centre, and, as previously mentioned, the Birth to Three program includes the use of Wiradjuri language in mathematical activities.

Plans and Steps

As noted, Banjo Childcare Centre works with a community facing complex issues, and has limited financial resources with which to do this work. The centre adopts a strengths orientation in their aims and planning for the future, including their approaches to mathematics learning experiences for their birth-to-three-year-olds. Children are empowered mathematically through a "secure environment with opportunities for risk-taking and self-regulation" (Centre Handbook). Educators are trusted to constantly develop mathematics education programs "through reflective practice and our commitment to training" (Centre Handbook). Mathematics learning is deliberately and explicitly included in documentation such as programs, learning stories, and classroom displays, thus highlighting the value placed on mathematics education within the Birth to Three program.

Summary

This brief case illustration has highlighted how an early childhood service experiencing challenging circumstances uses a strengths approach to provide a quality mathematics education program for children aged birth to three years. Educators draw on community strengths and their own resourcefulness in order to create a learning environment that encourages birth-to-three-year-olds to pose and solve mathematical problems, engage with complex spatial environments, utilise number and measurement concepts in meaningful ways, and use mathematical language and representations to add meaning to everyday routines and activities.

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Leadership of strengths-based approaches for early years mathematics education: Using CHAT as a framework for educational leaders' professional learning leadership

Matt Sexton
Australian Catholic University
<matthew.sexton@acu.edu.au>

Joce Nuttall
Australian Catholic University
<joce.nuttall@acu.edu.au>

We present a model that employs *cultural-historical activity theory* (CHAT) concepts to inform research with designated Educational Leaders in early years settings. We theorise practice change in early years mathematics education in terms of motive objects of activity and mediation by cultural tools. We show how CHAT can be used to lead development of a strengths-based approach to support young children's early mathematics education through systematic professional learning activity. Our overarching aim is to understand how educational leadership in early learning spaces can be reimagined, drawing on CHAT to theorise this under-researched area of mathematics leadership in early learning settings.

In this MERGA symposium paper, we present a model that employs concepts from *cultural-historical activity theory* (CHAT) to inform research and learning opportunities with designated Educational Leaders in early years settings. We show how the model can be used to lead a strengths-based approach (e.g., Fenton et al., 2016) to support young children's mathematics education. Our overarching aim is to understand how educational leadership in early learning spaces can be reimagined, drawing on CHAT to theorise this under-researched area of mathematics leadership in early learning settings. This reimagining and expansion of work sees Education Leaders lead enactment of strength-based approaches for early years mathematics education through on-site professional learning.

Culturally and historically, there have been limited expectations for mathematics education in early childhood programs (for children aged from birth to five years), relative to the focus on mathematics in the early years of schooling (for children aged from five to eight years). The work of Piaget has long influenced thinking about children's learning in early years education, with a focus on discovery learning of mathematical thinking (Stipek, 2013). This situation has been compounded by early years educators' underestimation of young children's capacity to think mathematically and misunderstandings about how young children come to understand mathematical ideas. Many educators hold negative affective responses to mathematics in general (Knaus, 2017; Moss et al., 2016; Stipek, 2013), and they also tend to have limited understanding of mathematical content knowledge (MCK), particularly understanding mathematical concepts and terms (Knaus, 2017).

The position of Educational Leader has been mandatory in all early childhood services in Australia since 2012. This policy move aims to improve program quality through the leadership of suitably qualified staff who foster changes in pedagogical practice. In Aotearoa New Zealand, there is no such mandatory position, possibly because the proportion of degree-qualified staff in the sector is higher than in Australia. In this paper, we position Educational Leaders as *mathematics professional learning leaders* who direct their leadership activity towards developing colleagues' mathematics teaching practice using strength-based approaches. We show how this leadership-of-learning process can be researched through CHAT concepts.

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Theorising Leadership as a Research and Learning Opportunity

We argue for the explanatory power of CHAT for researching and transforming long-standing workplace practices, such as the historical neglect of mathematics in early years education. Professional learning can enhance educators' knowledge and practices for mathematics education, including their dispositions and expectations for young children's mathematical learning (Perry & MacDonald, 2015). We are concerned specifically with professional learning focused on strengths-based approaches for mathematics education with young children, including the use of documentation associated with those approaches (Fenton et al., 2016). We suggest that that documentation, including the concepts and practices of strength-based approaches, offer new cultural tools to inform professional learning in early years settings. These offer opportunities for educators to work on new motive objects focused on young children's mathematics learning. In this sense, we believe that research and learning opportunities lie in expanding the work of designated Educational Leaders to identify as mathematics professional learning leaders in their work sites.

We draw on three core concepts of CHAT: *motive object of activity*, *cultural tools*, and *mediation.* CHAT understands all human activity as *object-oriented* (Kaptelinin, 2005); that is, psychological and practical activity are simultaneously drawn forward by attention to collaborative tasks (*motive objects of activity*) that result in desired outcomes (Engeström, 2015). This differs from dominant understandings of motivation, which see it as an individual and internal force of will. We use the well-known triangular representation of collaborative activity (Figure 1) to show how subjects of the activity system (designated Educational Leaders) are motivated to enhance teaching practices of their colleagues. The Educational Leaders' motive object of activity is the development of mathematics teaching practices. The desired outcome is quality mathematics education for young children.

This relationship between Subject and Object is *mediated* by valued *cultural tools*. The mediating function of cultural tools is due to culturally-specific meanings that inhere in those tools. Buttons, for example, are mostly associated with clothing, but in early years education, another contextually-specific meaning inheres in a box of buttons: the pedagogical opportunity they offer to teach higher-order concepts (e.g., classification & subitising).

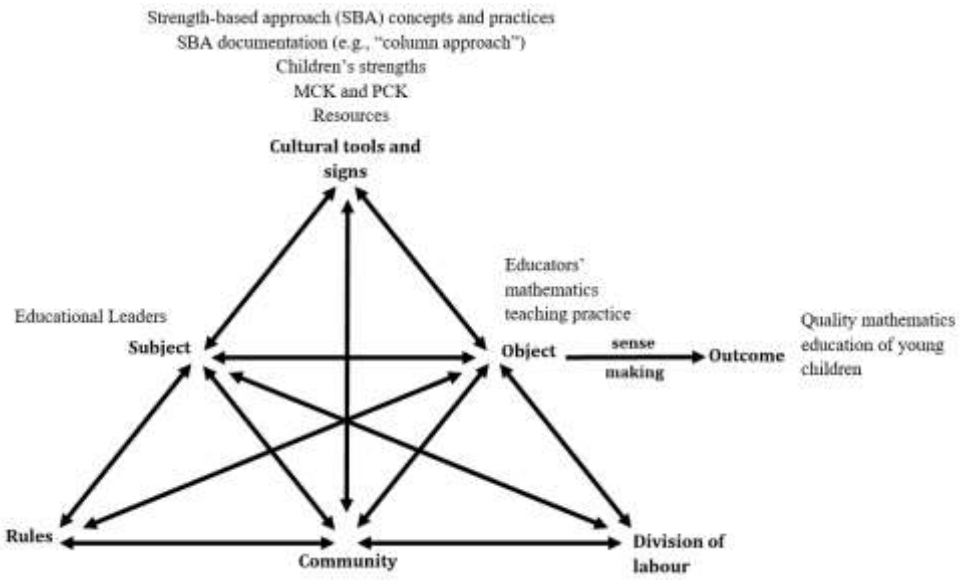

Figure 1. Representation of the Educational Leaders' mathematics professional learning activity system.

Children's Strengths as a Temporary Motive Object of Activity for Educational Leaders as Mathematics Professional Learning Facilitators

A way for researchers to use these CHAT concepts and to understand changes in the professional work of Educational Leaders is to address and transform long-standing practices that have impeded mathematics education in early years learning spaces (Knaus, 2017; Moss et al., 2016; Stipek, 2013). This could be achieved by using the example of 'children's strengths' as a cultural tool that Educational Leaders can deliberately reposition as a temporary motive object of activity. Cultural tools do not become effective components of practical and psychological activity without deliberate efforts to understand and expand the meanings that inhere within them. A key "move" for Educational Leaders in early years education therefore is to make the definition, identification, and valuing of *children's strengths* a temporary focus in their work with colleagues (i.e., a temporary motive object of the collaborative professional learning activity they are leading).

Without this critical first stage of meaning-making in professional learning, the capacity to mobilise any new concept in the context of teaching practice, including strengths-based pedagogical activity, will be severely limited. Once children's strengths takes on a stabilised meaning across early years educators' conceptualisation of young children's learning, pedagogical strategies for applying strengths-based approaches can become the *next* temporary object of activity in an *ongoing sequence* of professional development focused on a series of *related motive objects*. Educational Leaders therefore have a critical role in progressively introducing new and more complex cultural tools to support educators' professional learning of strength-based approaches. For example, in Figure 1, we included mathematical content knowledge (MCK), pedagogical content knowledge (PCK), and resources (both in the classroom and for professional learning) as further cultural tools (and therefore potential temporary motive objects for professional learning) in the mathematics professional learning leadership activity of Educational Leaders. As noted earlier, early years educators may not feel adequately knowledgeable or disposed toward mathematics pedagogy due to their own limited mathematical knowledge (Knaus, 2017). Their own internalisation of specific mathematics concepts may therefore be a critical temporary motive object of professional learning leadership activity to support educators' confidence in teaching mathematics to young children.

In the context of this symposium, the "column approach" described by Collins and Fenton (Paper 1 in this symposium) offers a key cultural tool to enhance the PCK of early years educators. A temporary focus on the use of this tool has been shown to effectively foster the uptake of strengths-based approaches (Fenton et al., 2016). According to our conceptualisation, we suggest this success is due to the new meanings the column approach makes available to mediate early years mathematics pedagogical practice.

Educational Leaders can employ a variety of approaches in directing colleagues' psychological and practical activity toward new cultural tools as temporary motive objects. These strategies include providing reading materials, practice development through action research, collaborative design-based research activities, or through the practice methodology developed within CHAT, known as Developmental Work Research (DWR) (Virkkunen & Newnham, 2013). Strengths of DWR include its incorporation of simultaneous research and learning activities, allowing researchers to track shifts in meanings and practices at close hand, and its emphasis on the volitional action of the research participants to solve practical problems found in their work (Sannino, 2015). This would prove to be helpful in expanding the work of Educational Leaders to include mathematics professional learning leadership.

Conclusion

Given the insights from Fenton et al. (2016) and MacDonald and Murphy (Paper 2 in this symposium) regarding early childhood educators' use of strengths-based approaches for mathematics education in early years settings, a research focus on the role of the Educational Leader in these settings is timely. Strategies to expand their work activity as mathematics professional learning leaders who can mobilise concepts, practices, and documentation of strengths-based approaches as motive objects of activity is one way of fostering mathematics education in early years settings. CHAT and DWR methodology have been shown to transform sedimented practices in early education (e.g., Nuttall, 2013) and is effective in expanding Educational Leaders' work (Nuttall et al., 2016).

However, this work has not hitherto focused on young children's mathematics development or educators' mathematics education knowledge, practices, and dispositions. We suggest that interventions informed by CHAT and DWR offer researchers and Educational Leaders the opportunity to conceptualise new, expanded work activity together for the professional learning leadership of strengths-based approaches for early mathematics education. Such a conceptualisation draws on the role of motive objects, specifically the adoption of new cultural tools that support the development of educators' understanding and use of strengths-based approaches for mathematics education. This would be a significant shift in the cultural and historical norms of early years mathematics education, but one that appears necessary if sedimented practices related to mathematics education in the early years are to be transformed. This research and learning opportunity, concerning the professional learning leadership of strength-based approaches, might be the investment that Stipek (2013) called for in evolving mathematics education practice in early years settings.

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Let's Count: Success and expansion

Bob Perry
Charles Sturt University
<bperry@csu.edu.au>

Ann Gervasoni
Monash University
<ann.gervasoni@monash.edu>

Amy MacDonald
Charles Sturt University
<amacdonald@csu.edu.au>

Sue Dockett
Charles Sturt University
<sdockett@csu.edu.au>

Anne Roche
Monash University
<anne.roche@monash.edu>

Paige Lee
Charles Sturt University
<palee@csu.edu.au>

This symposium reports on recent developments for *Let's Count*, the preschool mathematics program implemented across Australia since 2010 by The Smith Family, a national, independent children's charity helping disadvantaged Australians to get the most out of their education, so they can create better futures for themselves. *Let's Count* is an early mathematics program that has been designed to assist educators in early childhood contexts to work in partnership with parents and other family members to promote positive mathematical experiences for young children (3-5 years). The program aims to foster opportunities for children to engage with the mathematics encountered as part of their everyday lives, talk about it, document it, and explore it in ways that are fun and relevant to them. The success of *Let's Count* has been reported many times at MERGA conferences, including the Beth Southwell Practical Implications Award paper in 2016.

The papers presented in the symposium will build on the success of *Let's Count* by considering a number of recent initiatives in delivery and scaling up of the project in order to make it available to a more extensive set of participants across Australia and internationally. Based on a series of program evaluations, the three papers in the symposium will consider delivery methods beyond the usual face-to-face workshop presentations to early childhood educators and will anticipate future developments as *Let's Count* undergoes a program revision during 2020-2021.

The proposed symposium program is as follows.

Introduction to Let's Count (Bob Perry) – 5 minutes

Paper 1: Ann Gervasoni & Anne Roche *Let's Count* in an online environment

Paper 2: Amy MacDonald & Paige Lee *Let's Count* in early childhood teacher education

Paper 3: Sue Dockett & Bob Perry *Let's Count* and community professionals

Discussant – Wendy Field, Head, Programs and Policy, The Smith Family - 10 minutes

Questions and Discussion

The symposium will be chaired by Bob Perry and there will be ample time for discussion and questions.

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Let's Count in an online learning environment

Ann Gervasoni
Monash University
<ann.gervasoni@monash.edu>

Anne Roche
Monash University
<anne.roche@monash.edu>

Let's Count Online is a new e-learning approach to delivering *Let's Count* professional learning. It was evaluated in 2018. The findings suggest that the e-learning platform was successful, and that the outcomes for educators were similar to those achieved by participants using the face-to-face workshop professional learning model. Several key differences in outcomes were noted, and these inform recommendations for refining *Let's Count Online*.

Introduction

Let's Count (Gervasoni & Perry, 2017) is an early mathematics program that assists educators, in early childhood contexts, to work in partnership with parents and other family members to promote positive mathematical experiences for young children. Professional learning associated with *Let's Count* was first offered for educators in 2010 using a face-to-face workshop learning environment and between session activities and investigations. Following the positive evaluation of *Let's Count*, (Gervasoni & Perry, 2015a, 2015b; Perry et al., 2016), The Smith Family received Federal Government support to make *Let's Count* available to more communities across Australia. It was then decided to develop and pilot a complementary e-learning professional learning approach, *Let's Count Online,* with the capacity to reach more educators across Australia.

An important goal when developing *Let's Count Online* was maintaining the successful outcomes achieved through the original *face-to face* professional learning model. For this reason, *Let's Count Online* was evaluated in 2018 to determine the extent to which the outcomes achieved by educators who participated in the *Let's Count Online* course were similar to or varied from the outcomes achieved by educators who participated in the face-to-face model during the *Let's Count* longitudinal evaluation (Gervasoni & Perry, 2015a, 2015b; Perry et al., 2016). It was anticipated that the evaluation findings would assist The Smith Family to determine the effectiveness of the *Let's Count Online* platform for delivering the professional learning underpinning the *Let's Count* initiative for families. The evaluation also sought to gain insight about participants' experiences of the e-learning platform, and its effectiveness, so as to recommend any improvements for the *Let's Count Online* Course. The evaluation method and findings are presented in this paper, along with recommendations for further developing *Let's Count Online*.

Evaluation Method

The *Let's Count Online* evaluation used a mixed methods approach, drawing on both quantitative and qualitative approaches. Data were collected through online surveys, and telephone interviews with participants. The design of the surveys and interview schedules were informed by the instruments used in the *Let's Count* Longitudinal Evaluation (Gervasoni & Perry, 2015a) to enable valid comparisons to be made between the participant outcomes for the two program delivery formats.

All those who registered for *Let's Count Online* during the 2018 evaluation period (n=814) were invited to participate in the evaluation and complete two online surveys – one prior to commencement of the *Let's Count Online* course (Time 1) and two weeks after

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completion of the course (Time 2). The Time 1 (T1) survey was completed by 207 participants and the Time 2 (T2) survey by 60 participants. Thirty-three participants completed both surveys. Participants were drawn from every state and territory in Australia. Telephone interviews with seven case-study participants took place twice – two weeks after the commencement of the e-learning course and two weeks after its completion. The duration of the course was approximately 8 weeks and took place at a time of participants' choosing.

Qualitative and quantitative data from the surveys were used in conjunction with interview data to provide a picture of any changes in the respondents' reported attitudes to mathematics and mathematical pedagogies, and the effectiveness of the e-learning platform for professional learning. Data from the *Let's Count Online* Evaluation were compared with findings from the *Let's Count* Longitudinal Evaluation (Gervasoni & Perry, 2015a) to determine whether the outcomes for participants varied in respect to their mathematics dispositions, skills, and levels of confidence in developing children's mathematical knowledge. Data were also analysed to determine how *Let's Count Online* might be improved to deliver the *Let's Count* professional learning program more effectively.

Key Findings

A summary of the key evaluation findings is presented below. Of particular interest are comparisons between educators' dispositions, skills and confidence; their attitudes to a range of teaching strategies; and their engagement with the professional learning models.

Dispositions, Skills and Confidence of Educators

With respect to educators' attitudes to mathematics (either increasing or decreasing) between T1 and T2 surveys, the findings showed that these were similar for most statements for both the online and face-to-face cohorts. For example, for both programs at T2 there was an increase in the proportion of participants who believed *mathematics is something that I do every day*, and *their liking of maths*. Also, the *Let's Count Online* participants' confidence in developing children's mathematical knowledge increased more than for the face-to-face course participants, however, their confidence was lower overall.

Educators' Attitudes to a Range of Mathematical Teaching Strategies

At both T1 and T2, educators were presented with 24 statements about a range of mathematical teaching strategies and asked to indicate whether they agreed or disagreed on a five-point Likert scale. For 15 of the 24 statements, the initial and final percentages, as well as the change in percentage, are relatively similar between participants in the two programs. In contrast, for some statements there was a reduction in the proportion of educators in the face-to-face program who indicated that they agreed with the statement from T1 to T2, but this proportion increased for the online course participants. These statements suggest that the online course appeared to have promoted, for some participants, pedagogies that were more school like or traditional, than did the face-to-face course. These trends are reflected in the increased 'schoolification' of much of early childhood education (Moss, 2013), but are not well-aligned to approaches recommended for mathematics education in the early years. Illustrative statements were:

> It is important that children represent their mathematics through the use of conventional symbols.
>
> Workbooks and worksheets are essential in learning and teaching mathematics in early years settings.

It is important that the experience of *Let's Count Online* is strongly aligned with the theoretical underpinnings of *Let's Count*, early childhood approaches to learning and teaching, including those espoused by the *Early Years Learning Framework for Australia* (Department of Education, Employment and Workplace Relations [DEEWR], 2009), and reform approaches to mathematics education. The findings suggest that this is mostly, but not always true, of *Let's Count Online*.

A key focus of *Let's Count* is engagement between educators and family members centred on children's mathematics learning. In the T2 survey, *Let's Count Online* participants rated their engagement with a set of teacher practices before and after Let's Count Online. They reported lower levels of *'talking about children's mathematics learning with family members'* or *'building on the mathematics that family members tell them children are using at home'* prior to the program, (means of 4.4 and 4.1 out of 10 respectively). The mean rating for these practices after *Let's Count Online* was 7.0 and 6.9 respectively. This suggests that the course prompted an increase in both practices, but these activities were less common for some.

Comparison Between Let's Count Online and Face-to-Face

Interview data indicated that there was not as much accountability for participants' engagement and learning in the online course compared with the face-to-face model. This was possibly due to the different level of accountability for the between session tasks embedded in *Let's Count Online*, compared to the *Family Gatherings Report* required of the face-to-face participants. In the face-to-face model, participants presented the outcomes of family engagement strategies to other participants and received feedback and inspiration from the experiences of colleagues, and from the course facilitators. They also discussed their observations of children's mathematics learning during the period between workshops, and had the opportunity for this learning to be extended through the guidance of facilitators. This learning opportunity was not included in the *Let's Count Online* model.

The findings also suggest that there was a lesser understanding of the aims of *Let's Count* developed by *Let's Count Online* participants. Interview data suggested that the course was more likely to reinforce the pedagogical practices that the educators were already using, rather than stimulating new pedagogical practices. Also, the *Let's Count* mantra of *Notice, Explore, and Talk About Mathematics* was less a feature of *Let's Count Online* participants' reflections in the interviews and survey data than for face-to-face participants.

Low Level of Difficulty for Let's Count Online

The findings suggest that the same level of professional and academic rigour may not be afforded by the *Let's Count Online* learning environment compared with the face-to-face workshop environment. This view was reinforced by one participant stating that *Let's Count Online* did not reach the level of challenge he was seeking for his staff, and another who explained that *Let's Count Online* was the sort of course she could complete while watching TV with her family. Perhaps the *online* course is more characterised by passive engagement with the intended learning opportunities than active engagement. Possible strategies to increase the level of difficulty and active engagement for participants may include providing a *Let's Count Online* facilitator who can provide online or real-time feedback, or the opportunity to complete the course in workplace groups to promote discussion and feedback.

Conclusion and Recommendations

Overall, the findings from the *Let's Count Online* evaluation suggest that the e-learning platform was successful for delivering professional learning for educators associated with the *Let's Count* program. The participants in the evaluation were very positive about *Let's Count Online*, and many appreciated the chance to access the professional learning when opportunities for the face-to-face workshops were not available in their region. However, some educators endured technical issues and a lack of online support for rectifying these. There were some important differences noted when comparing the *Let's Count Online* professional learning model with the face-to-face model. For example, the reported low level of difficulty, passive engagement and lack of accountability for learning reported by some *Let's Count Online* participants suggests that the *Let's Count Online* course may benefit from some further development.

The following recommendations provide direction for how *Let's Count Online* may be refined and strengthened to better assist educators meet the aims of *Let's Count*.

1. Develop opportunities for feedback associated with the learning activities embedded in *Let's Count Online*. This may include a facilitator to provide online or real-time feedback, or the opportunity for participants to complete the course in groups within a workplace or early years setting, with a leader in each setting to facilitate discussion about the professional learning, and monitor and support engagements with parents, and observations about children's mathematics use, language and learning.
2. Review the *Let's Count Online* content and materials to identify and alleviate any dissonance with the theoretical underpinnings of *Let's Count*.
3. Ensure that any refinement of the *Let's Count Online* course includes:
    a. Sustained emphasis on the *Let's Count* mantra – notice, explore and talk about mathematics in everyday contexts.
    b. Strategies to sustain educator/parent communication across an entire year of implementation.
    c. A prominent, actively monitored help-line, including email and phone support.

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Let's Count in early childhood teacher education

Amy MacDonald
Charles Sturt University
<amacdonald@csu.edu.au>

Paige Lee
Charles Sturt University
<palee@csu.edu.au>

In 2011, the *Let's Count* professional learning program was developed into an elective distance education subject offered at Charles Sturt University. The resulting subject, EMC101: Let's Count, has been offered every year since 2012, and has to date been completed by 796 students. This paper details the subject design and provides enrolment and evaluation data that attest to the success of the subject.

History and Development

In 2011, the first author was contracted by The Smith Family to develop the *Let's Count* program into a distance education subject at Charles Sturt University, as a means of sustaining the *Let's Count* initiative and achieving a wider impact on the early childhood field (MacDonald, 2015). The subject EMC101: Let's Count has been offered at Charles Sturt University since mid-2012, and is primarily offered as an elective in the Bachelor of Education (Birth to Five Years) degree program. It is also available as an elective in a number of other degree programs across the University, and is available for single subject study, independent of a degree program. The authors of this paper have both been Subject Coordinators of EMC101, and have been responsible for teaching, developing, and evaluating the subject.

Subject Design

EMC101: Let's Count is designed to be an elective subject that brings together pedagogy and practice. The subject provides a link between the workplace or community of the student and their professional practice. The subject is designed so that a series of six modules deliver the content, which is supported by current literature, anecdotes, reflective discussion questions, and practical examples. The modules provide various ways for students to engage with the content and critically reflect on their pedagogy and practice in relation to young children noticing, talking about and exploring mathematics in everyday situations. Key examples are provided, and students can use discussion forums and text-based chat sessions to engage with the modules and associated activities as well as their peers and tutors. After the modules have been delivered, the *Let's Count* program ideas are put into practice through two assessment items: (1) Family Gatherings; and (2) Learning Stories.

Family gatherings

For assessment item 1, students are required to plan, implement and reflect on a Family Gathering, and present this using Microsoft PowerPoint©. This assignment is a workplace or community-based assessment item, where students actively engage with families in their setting to support them to notice, talk about and explore maths in everyday situations with their children. The Family Gathering can be organised and run in any way that suits students and the families with whom they collaborate. Family Gatherings have taken many forms, and each session new and inventive ways are explored by students. Examples include: using private social media groups, email, early years communications apps; individual face-to-face meetings; larger group information sessions; casual conversations during pick up and drop

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off times; home visits, park play sessions, excursions; and often, a mixture of some of the above. Students are encouraged to consider the context of their families as well as their own context during the planning of their Family Gathering, and also to be flexible and responsive to the needs of the families they work with, as well as their own circumstances. There is no one 'right' way to complete their gathering; the aim is simply to support families to notice, talk about and explore maths with their children.

At the end of the session, after assessment item (2) has been submitted, students are invited to share their Family Gathering presentation with their peers. Students who consent to this, have unmarked and de-identified versions of their presentations uploaded by the Subject Coordinator to a showcase location in the learning management system, and all students are able to access and view these presentations. On average, between five and ten students per session opt to share their work with their peers; however, many more view the presentations. Once some are uploaded, it is not uncommon for other students to email with permission to share theirs, after seeing the value in the showcase. Interestingly, students who received all variation of grades opt to share their work.

Learning stories

For assessment item (2), students are required to write three short learning stories as well as present a 1,000-word statement on the role of learning stories in early childhood mathematics education, including assessment and communication with families. The learning stories can be taken from the Family Gathering or from additional observations of children that were involved in the Family Gathering. Students are required to include information on the context, an analysis of the mathematical learning that occurred, as well as provide meaningful feedback and suggestions to the child and family, and suggest ways they plan to support the child as the educator. The statement requires students to critically consider the role of learning stories in early childhood mathematics education. Students are asked to specifically consider learning stories as a form of communication with families, as well as a method of mathematics assessment.

Enrolment Data

EMC101 has to date been completed by 796 students. Charles Sturt University offers three sessions of study per year: Session 30 (for example, titled 201630), which runs March-June; Session 60, which runs July-October; and Session 90, which runs November-February, including the Christmas-New Year period. The subject was first offered in 201260, and was offered in all three sessions of study until 2018, at which point a change in the BEd (Birth to Five) course structure reduced the subject offerings to the 30 and 90 sessions only. Figure 1 displays the enrolment patterns for EMC101 across the nine years for which it has been offered. The student numbers displayed represent the number of students who *completed* the subject in each session. As can be seen in Figure 1, enrolments have consistently trended upwards across the years of offering the subject. Dips are evident in the summer session offerings, as one might expect. Unsurprisingly, the majority of enrolments are drawn from the BEd (Birth to Five) program. The subject also consistently attracts enrolments from the Bachelor of Educational Studies degree program; a program servicing students who are pursuing careers in, for example, community education or classroom support. However, it is interesting to note the participation from a range of other degree programs including Bachelor of Arts, Bachelor of Accounting, and Bachelor of Science. Anecdotal evidence indicates that students from these diverse degrees are attracted to the subject because it

develops their skills in working with children and families, as well as communicating mathematical ideas.

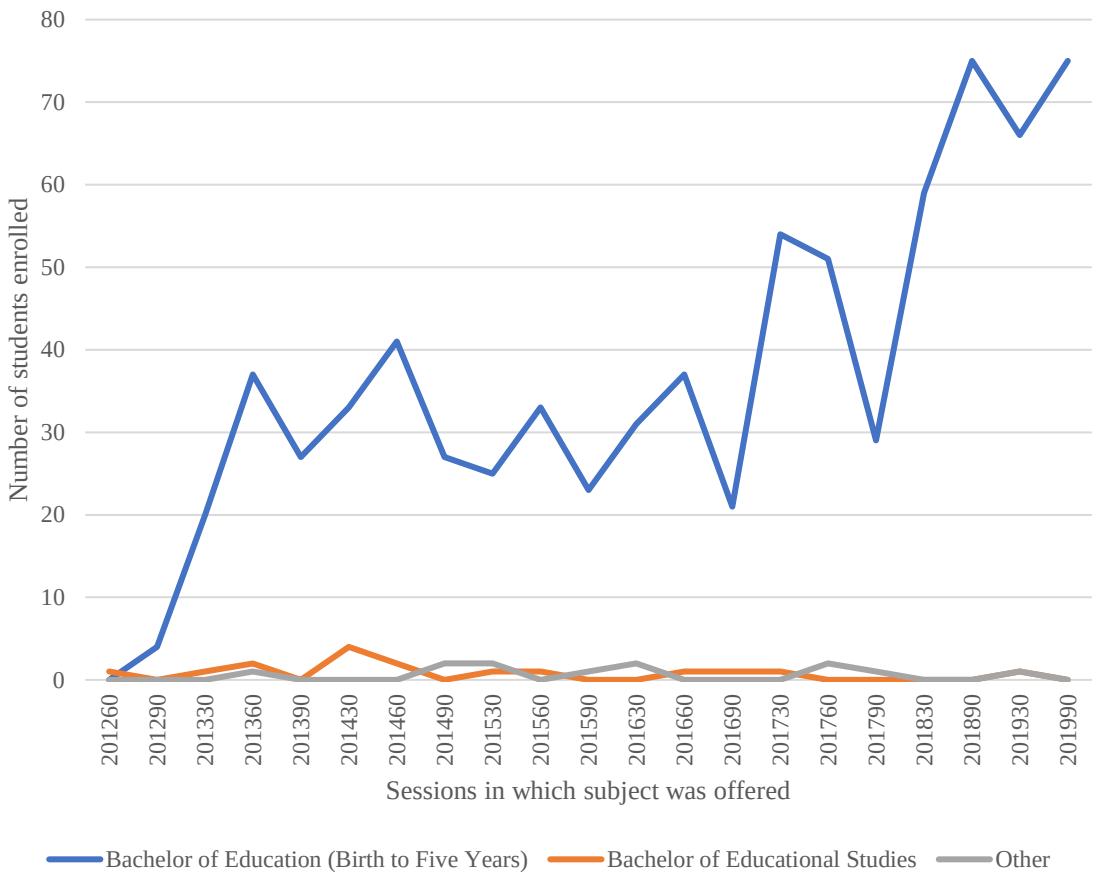

Figure 1. Enrolment pattern for EMC101 2012 - 2019

Evaluation Data

Subjects at Charles Sturt University are formally evaluated through a Subject Experience Survey (SES), which is completed by students in all subjects across the university. The survey consists of 21 compulsory core items (18 Likert scale items and three short response items) as well as a number of optional items at the Subject Coordinator's discretion (Charles Sturt University, 2020). EMC101 consistently achieves SES scores which are both very high (>4 on a 5-point scale) and higher than the School mean. Example SES data from three recent offerings is presented in Table 1.

Table 1
Example Student Evaluation Survey (SES) data

	201830		201890		201930	
Item	Subject Mean	School Mean	Subject Mean	School Mean	Subject Mean	School Mean
The learning activities in this subject helped me to learn effectively.	4.4	3.9	4.3	3.8	4.4	4.0

The learning activities in this subject created opportunities for me to learn from my peers.	3.8	3.7	4.0	3.5	4.1	3.7
This subject incorporated study of current content.	4.3	4.1	4.3	4.1	4.4	4.2
The assessment tasks in this subject helped me to learn effectively.	4.4	3.9	4.3	4.0	4.4	4.0
I could see a clear connection between the learning outcomes, learning activities and the assessment tasks in this subject.	4.3	4.1	4.3	4.1	4.5	4.1
The learning activities enabled me to judge the quality of my own work.	4.3	3.7	4.3	3.7	4.2	4.1
The learning activities in this subject extended my knowledge.	4.4	4.0	4.3	4.0	4.4	4.1

In addition to the SES data, the subject has been evaluated through a small-scale research evaluation. Past EMC101 students were invited to participate in an email interview about their experiences in the subject (MacDonald, 2015). Eighteen educators participated in the evaluation and all reported positive experiences in the subject, evident through comments such as the following:

> I'm not confident with maths but after undertaking the course I felt I benefitted as well as the children. It gave me the confidence to implement more 'maths' type activities and to talk confidently about maths [Stephanie, VIC].

> I've learned so much from this subject and it deepened my knowledge in maths. I can understand maths better through children's play and I discovered that I can 'see' mathematics all around me every day [Apple, Brunei Darussalem].

> I enjoyed doing the learning stories, in particular giving advice to the parents on how they can extend on mathematics learning at home. I encourage parents to be more hands on in their child's learning and recognise that they are the number one teachers of their child [Carissa, NSW].

> Through working on such projects with children and families as equal partners we are enabled to share and celebrate children's learning. The family I worked with were clearly proud of the child's numeracy understanding and thinking. The child was seen as competent by all and her family expressed an intention to further extend on her numeracy learning in their everyday lives [Sarah, NSW].

Conclusion

It appears that the translation of the *Let's Count* program to a university subject has been a successful endeavour. The elective subject consistently has a high participation rate, with 796 students completing the subject to date. The subject consistently performs well on formal subject evaluation surveys. Moreover, it can be seen from the research evaluation that students find the subject valuable for developing their confidence in mathematics, their ability to identify mathematics in children's everyday lives, and their skills in communicating with families around their children's mathematics learning.

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Let's Count and community professionals

Sue Dockett
Charles Sturt University
<sdockett@csu.edu.au>

Bob Perry
Charles Sturt University
<bperry@csu.edu.au>

The *Let's Count* Community Professional Pilot 2019 took place in six sites across three states (New South Wales, Queensland and South Australia). The aim of the pilot program was to implement the *Let's Count* face-to-face program for the first time with a group of people who work with young children and their families but who are not trained early years educators working within early childhood education and care centres. This paper reports on the evaluation of the pilot program with specific emphasis on expanding the reach of *Let's Count* whilst maintaining its integrity and outcomes.

The authors were commissioned by The Smith Family to undertake an evaluation of the *Let's Count* Community Professional Pilot 2019. The aim of the evaluation was to ascertain the effectiveness of face-to-face implementation of the *Let's Count* program in mixed groups of early years trained, centre-based educators and other community professionals. Data were generated using surveys before and after the training sessions and telephone conversations after each of these sessions. Seventy-nine participants and six facilitators or program coordinators were involved in at least one aspect of the evaluation.

Background

Since 2010, the *Let's Count* program in mathematics has supported centre-based early childhood educators using a face-to-face professional learning model in geographical sites across Australia consisting of two workshop days with approximately 4-6 weeks between the workshops. In 2019, The Smith Family specifically targeted these community professionals when mixed groups of early childhood educators and such community professionals undertook face-to-face *Let's Count* program sessions together and engaged in the between-sessions requirements of the program in their own workplaces. The *Let's Count* program and its impact on early childhood educators, young children and their families has been well documented (Gervasoni & Perry, 2017; Gervasoni et al., 2016; Perry et al., 2016; Perry & MacDonald, 2015). This paper reports on the evaluation of the *Let's Count* Community Professionals Pilot 2019. The research questions for the evaluation are listed in the Results section of the paper.

Methodology

The Community Professionals Pilot 2019 was undertaken in six sites across three states (two sites in each of NSW, Queensland and South Australia). The evaluation used multiple methods involving both qualitative and quantitative approaches.

Both authors were present for the first session of each group in order to meet participants and undertake preliminary surveys and background discussion with all participants, *Let's Count* facilitators and Program Coordinators willing to be involved in the evaluation. As well, participants were asked if they would undertake the follow-up activities in the evaluation – two telephone conversations – one between the two program sessions and one approximately three weeks after the second session - and post-Session 2 online surveys. No child data were generated in this evaluation.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.*), Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia),* pp. 105-108. Singapore: MERGA.

The numbers of participants in the Community Professionals Pilot 2019 and the evaluation are provided in Table 1. The community professionals came from many different backgrounds and endeavours including education (other than early childhood); social work; library and information science; business administration; aged care; sports coaching; sociolinguistics; music therapy; and law. There were paid and volunteer workers from libraries, playgroups, HIPPY (Hippy Australia, n.d.) and other community support groups.

Table 1
Participation in data generation

Participant Type		Data Generation Approach			
Early Childhood Educator (E)	Community Professional (CP)	Survey 1	Survey 2	Conversation 1	Conversation 2
44	35	44 E, 33 CP	12 E, 13 CP	14 E, 11 CP	7 E, 7 CP

Results and Discussion

Only a summary of the results can be provided here. This is done by answering each of the research questions, with a particular emphasis on the responses of the community professionals.

What were the community professionals' expectations of the program?

Many of the community professionals who participated in the *Let's Count* Community Professionals Pilot 2019 knew little about what to expect from the program before Session 1. All of the community professionals anticipated that the 'mixed' model would be of benefit to them as they would be learning alongside experienced early childhood educators. Some wondered whether they would be able to 'keep up' with the early childhood educators and some brought long-held reticence about their own abilities both to do mathematics themselves and to facilitate young children's learning of mathematics. There was no indication from the early childhood educators that they experienced any difficulties arising from the presence of the community professionals.

> Great networking. Great experience. A big thing was that ideas bounced off each other. (CP)

> There were no disadvantages [with the mixed group]. It was great to have different ideas, read about some, and get some ideas not out of long day care such as ways to give different ideas at home. Opportunity to think outside the box and give us new ideas. No problems, only advantages with community professionals group. It opened up my eyes. (E)

> It was great to see the different perspectives of the community professionals, especially perspectives on what parents are doing and thinking when the community professionals go to family homes. We can't do that. It was great to see what they're doing – they often don't have a lot of resources, so must use basic things at home. (E)

What did the community professionals see as the benefits of engaging with Let's Count to themselves and their organisations?

Many of the community professionals have not only learned a great deal about facilitating young children's learning of mathematics from their experiences in *Let's Count* but have also used this knowledge in their own contexts. Many of them have different links with the families of the children with whom they interact than early childhood educators typically have, and these strong links have encouraged their use of *Let's Count.* Contexts

such as HIPPY, playgroups, library-based experiences, music therapy and several volunteering opportunities with children and families who have complex support needs have facilitated interactions around mathematics learning for children and families. Many of the community professionals now see that they can be leaders in their organisations around the establishment of effective practices in mathematics education.

> It went really well and was an opportunity for us to grow and expand on what we learnt. It was a great starting point for young people's programs in the library.

> *Let's Count* provided opportunities to think about what we could do and what is possible in our environment. It provided space and opportunity to brainstorm and hear about what other places are doing re talking with families about numeracy concepts and to reflect on what we are doing and what we can do as a team.

> I will add *Let's Count* to the programs I am already involved in, including neighbourhood networks and refugee and migrant hubs.

What do community professionals see as the benefits of engaging with Let's Count to the children and families of their communities?

Being able to provide children and families who do not access centre-based early childhood education with appropriate, interesting and play-based mathematical experiences was seen as a major benefit of the community professionals' engagement with *Let's Count.* Many of the community professionals who participated in the *Let's Count* Community Professionals Pilot 2019 also enjoyed the opportunity to be involved in group professional development and in the recognition that the group gave them for their own work in the early childhood space.

> This is valuable work because the focus is on parent engagement. It is important to influence a number of areas as not all children attend early childhood education centres. *Let's Count* has a place targeting and promoting needs of working with children and families in whatever context.

> I liked the diversity of the group, across different learning environments. I enjoyed meeting people and seeing how *Let's Count* really helped across the programs, from very young children to Kindergarten aged 3-5. Learning about how people integrate maths with very young children as well was interesting. It made you think outside the square, more than about your own little environment. You can learn so much from each other. It is important to be aware of other groups and programs in your community.

In what ways did the early years trained educators experience the Let's Count program sessions?

As for the community professionals, early childhood educators participating in the *Let's Count* Community Professionals Pilot 2019 were very satisfied with the 'mixed group' model. They were particularly grateful for the diversity of perspectives which the community professionals brought to the training sessions and for the variety of approaches they adopted in using *Let's Count* in their contexts. Many of the early childhood educators praised the ways in which some community professionals were able to interact with both children and families and wished for the same flexibility in their own settings. Many early childhood educators recognised that the *Let's Count* program was not 'rocket science' and, in some cases, reinforced and extended current practice while others were grateful for the 'reminder' about what was possible.

> Different perspectives were an advantage. We are supporting all children, not all of them are at early childhood education centres. A lot of children are at home not attending early childhood education

> centres but may go to library, so we can reach more children and families. We all learn from each other and there were some really good ideas. We are here for all children and the whole community.
>
> Did the pilot work? Really well. Some non-educators apologised when presenting, but we thought they brought different perspectives that were very helpful. They made us think about different ways and about how they engage with different contexts, it added a new dimension. It was really good. I would encourage everybody to take the opportunity to do *Let's Count* training.
>
> Librarians do it differently. They have parents there, can share parent information and have games out for parents to try. All groups should be mixed. It is much more beneficial with community professionals than just early childhood educators. All [participants] took something different away from the training.
>
> *Let's Count* is applicable to all working with children and families.

Conclusion

The 'mixed group' model of the *Let's Count* training program where early childhood educators and community professionals undertake the program together has worked well for all involved. There have been real benefits to early childhood educator participants in that they have seen different ways for interacting with children and families and different ways of facilitating the mathematics development of young children than they would have been exposed to in a more homogenous group of participants. Community professionals have not only learned that mathematics learning can be incorporated into their core work but also that they can do this with minimal disruption to their programs. All participants have indicated that they really valued the opportunities to network with other professionals from across their communities who are also committed to the education and wellbeing of children and families. A number of participants have indicated that they would like to see the community professionals model as the norm in terms of face-to-face *Let's Count* training and this recommendation has been accepted by The Smith Family.

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Issues and affordances in studying children's drawings with a mathematical eye

Jennifer Way
The University of Sydney
<Jennifer.way@sydney.edu.au>

In this third consecutive MERGA symposium focused on young children's drawings, three separate groups of researchers discuss the benefits and issues of using drawings as a source of data in their studies. Although drawings are ubiquitous in early years classrooms and in studies of children's learning, there is no comprehensive framework for analysing children's drawings in mathematical contexts. The overarching purpose of these symposiums has been to explore the qualitative methods that researchers have developed in their distinct projects and advance our critical perspectives on interpreting drawings and understanding the role they can play in children's learning of mathematics.

Broadly, the researchers view drawings as an external representation of mathematical concepts, mathematical thinking, or perceptions of mathematical contexts. Typically, researchers trust that children's drawings express to some extent the developing internal systems of the child, including the affective domain. In studying the interplay between children's internal and external representations, researchers must grapple with the ambiguities of interpreting representational drawing, as explained in quotation below.

> "Internal systems, … include students' personal symbolization constructs and assignments of meaning to mathematical notations, as well as their natural language, their visual imagery and spatial representation, their problem-solving strategies and heuristics, and (very important) their affect in relation to mathematics. The *interaction* between internal and external representation is fundamental to effective teaching and learning. Whatever meanings and interpretations the teacher may bring to an external representation, it is the nature of the student's developing internal representation that must remain of primary interest." (Goldin & Shteingold, 2001, p.2).

In this symposium, as well as sharing results from recent research, the authors reflect on some of the issues and affordances in studying children's drawings with a mathematical eye.

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Chair & Discussant: Jennifer Way

Paper 1: Jill Cheeseman, Ann Downton, Anne Roche & Sarah Ferguson *Drawings reveal young students' multiplicative visualisation*

Paper 2: Katherin Cartwright, Janette Bobis & Jennifer Way *Investigating students' drawings as communication and representation modes of mathematical fluency.*

Paper 3: Kate Quane, Mohan Chinnappan & Sven Trenholm *Children's drawings as a source of data to examine attitudes towards mathematics: Methodological affordances and issues*

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Drawings reveal young students' multiplicative visualisation

Jill Cheeseman
Monash University
<jill.cheeseman@monash.edu>

Ann Downton
Monash University
<ann.downton@monash.edu>

Anne Roche
Monash University
<anne.roche@monash.edu>

Sarah Ferguson
Clairvaux Primary School
<sferguson@ccsbelmont.com.au>

In the context of a multiplicative problem, our study investigated young children's ability to visualise and draw equal groups. This paper reports the results obtained from 18 Australian children in their first year of school (age 5-6 years). The task *12 Little Ducks*, taught by their classroom teacher, provoked children to visualise and to draw different solutions. Fifteen children (83%) could identify and create equal groups via drawings; eight of these children (44%) could also quantify the number of groups that were formed. These findings show that some young children can visualise multiplicative situations and can communicate their reasoning of equal group situations through drawing.

The accepted wisdom of earlier research was that the intuitive pathway for children to multiplication is through repeated addition (Anghileri, 1989). Research reported by Sullivan et al. (2001) showed a relatively large cognitive step for children to move from using models with counting to abstract multiplication. These authors recommended that the teaching of multiplication require children of 5-8 years of age to imagine objects as well as model with objects.

The theoretical framework of this research is a social constructivist theory of learning which holds that meaning is created between individuals through their interactions (Ernest, 1991). The mathematical content was framed by the research literature related to problem solving with children, early multiplication and division, and children's drawings. The ability to solve problems is a fundamental life skill and develops naturally through experiences, conversations and imagination (Cheeseman, 2018). The perceived importance of problem solving stimulates educators to look for authentic problem-solving situations in which children behave as mathematicians (Baroody, 2000). The task reported in this paper is one such non-routine mathematical problem.

Multiplicative thinking involves making two kinds of relations: the many-to-one correspondence between the three units of one and the one unit of three (Clark & Kamii 1996). Doing so requires an ability to form visual images of composite unit structures and is fundamental to multiplicative thinking (Sullivan et al., 2001). Young children are only able to abstract this notion of a composite unit when they have constructed meaning in their own minds (Bobis, 2008). In order to determine children's meaning of groups, this study used children's drawings as a research tool, and to potentially be a "window into the mind of a child" (Woleck, 2001, p. 215). Children were asked to draw a picture of what they were visualising and to describe their thinking as they solved the problem. Materials and modelling were used only when a child was unable to solve the problem (Sullivan et al., 2001). We conjectured that many children make mental images and visualise quantities when situations provoke them to do so. Our challenge was to create a context that would elicit children's thinking, and to interpret and understand what children imagine. The research question we set out to answer was: *How do children's drawings, explanations and actions reveal the ways they visualise group structures?*

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 110-113. Singapore: MERGA.

Method

A teaching experiment methodology was used to explore and explain students' mathematical actions and thoughts about recognising and making equal groups. As researchers we wanted to experience, first-hand, students' mathematical learning and reasoning (Steffe & Thompson, 2000). The study included the four basic elements of teaching experiment methodology. The "teaching episode" in this case, a sequence of five consecutive days of mathematics lessons in one school with a class of 5-6 year-olds in their first year of school. Three researchers witnessed the teaching and video-recorded each lesson.

The exploratory teaching was undertaken by Sarah (fourth author). While not privy to the team's design of learning contexts, she contributed to the theoretical framing of the study, and was conversant with the purpose of the research. Sarah was familiar with the Launch, Explore, and Summarise lesson structure (Lappan, & Phillips, 2009), and she believed that children should not be shown possible solution strategies before they attempt a task. The research team noted that the lesson content was beyond the intended curriculum and would present conceptual challenges for 5-6-year-olds, as would the exploratory teaching. Analysis of the children's mathematical thinking was based on their drawings, mathematical language and actions, and on the researchers' theoretical interpretation of events in accordance with a teaching experiment methodology. We closely observed children's interactions to infer their thinking about multiplication as seeing "groups of groups".

Participants were 21 children (13 girls and 8 boys) from a primary school in a large rural city of Victoria, Australia. The mean age was 5 years and 6 months. Sarah's class provided a convenience sample for investigating our research question. The results are from the 18 children who were present on the day. We devised lessons as contexts in which 5-6 year-old children could be stimulated to recognise and create equal groups and to quantify those groups. One lesson, Twelve Little Ducks, is the setting for the results presented here. Sarah was given a lesson outline and encouraged to implement the ideas in any way that she felt suited her children. The problem was originally written as: *Can you make 12 little ducks into equal groups? Can you do it a different way? Draw or write what you did.* To introduce the task to her children, Sarah told a story:

> In order not to lose any of her ducklings the mother duck put them into some groups that were the same. She put them into equal groups, because it was easy for her to see that she still had her 12 baby ducks. Can you make a picture in your head of those 12 little ducklings? The mother duck put them into groups with the same number of ducks in each group. I wonder what groups she put them into … I would like you to draw a picture of what is in your head (video transcript).

Sarah chose not to show a picture of ducks or to model the problem with materials, she explained that it might interfere with children's thinking. She was keen to learn what her children could imagine without objects - in a context her children would understand. Sarah was conscious of the challenge of the task's mathematical vocabulary as her diary showed:

> These children have not heard the term "equal groups" from me at school at all until today. I did say "the same number in each group" but I didn't go into great detail about what I meant by equal groups.

These pedagogical decisions deliberately created a challenging for 5-6 year-olds. Blocks were not provided initially but a child was offered blocks when it was apparent that s/he could not begin to solve the problem.

Data were collected from two fixed video cameras, three tablet cameras operated by the observer-researchers recording children working or in conversation with an adult. Subsequently, photographs of work in progress, children's finished work samples, classroom

observations, and the video and photographic data were closely examined and interrogated. Data analysis began with each university researcher describing in detail what they observed soon after the lesson. In this way, we built a shared understanding of the events in the classroom. Each child's work sample was examined. Tentative categories of responses were proposed and iteratively tested to refine category definitions.

Findings

Analysis of the work samples together with our observations, conversations with the children, and video evidence revealed that three distinct categories of thinking could be described in terms of demonstrated multiplicative thinking.

Evident - could simultaneously quantify objects in groups and enumerate the groups as new units

Eight children (44%) produced 12 ducks by drawing and simultaneously creating equal groups. The ducks in their drawing were located in identifiable groups, indicating that they had perceived or imagined such groups before drawing the ducks. Elise drew two groups of six, circled each group and labelled her drawing, "2 groop 6" (*sic*) (Figure 1). She could make equal groups and quantify the groups. It appears Elise had determined the group size prior to drawing her solution because the ducks are drawn in equal rows.

Figure 1. Elise's first solution

Partial - having some awareness of the quantity of each group but not the number of groups shown

Figure 2. Georgie's first solution.

Six children (33%) were categorised as having "partial" understanding because they made equal groups but were not able to quantify the number of groups. Georgie drew three groups of four ducks (Fig 2), and when asked about her groups she said:

Georgie:	There are four here, and four there and four there. (Pointing to each group.)
Teacher:	How many groups of four have you got?
Georgie:	Twelve.
Teacher:	Twelve altogether. How many groups of four?
Georgie:	Um, I'm not sure yet.

Emergent - unable to find a solution – even with 12 cubes to model the problem

The four children (22%) who we described as emergent thinkers had several observed misunderstandings. For example, Conrad was unable to make six groups of two, from his drawing. It appears that Conrad did not have a solution in mind when drawing the 12 ducks as they were not drawn in identifiable clusters or rows. The random arrangement may have contributed to the difficulty of circling groups of two. Other emergent thinkers were unable to make equal groups in their drawings or when provided blocks to do so.

Figure 3. Conrad's drawing

To Conclude

We investigated whether children could visualise and construct equal groups and recognise the composite units they formed. Our research question was answered. Some children can imagine and draw equal group structures and in doing so recognize composite units. Some children can also enumerate the composite units. More than 80% of the children in the present study exhibited early multiplicative thinking. Children seemed to have intuitive understandings of equal group structures based on their experiences because they came to the problem we posed without any prior formal instruction about equal groups. This finding is novel - we have found no studies that have reported similar results with 5-6 year-old children.

Children communicated their visualisation of equal group situations through their drawings and elaborated their meaning with verbal descriptions and gestures. Such drawings of visualisations represent abstract thinking and call into question the accepted view of the way early multiplication typically develops via direct modelling to partial modelling, then to thinking abstractly (e.g., Anghileri, 1989).

We argue that it is productive to require young children to abstract problems earlier. Requiring visualisation together with drawings is an alternative approach to direct modelling. We acknowledge this is a small study and the results are only indicative of the ability of young children to visualise multiplicative situations. Further research might investigate other provocations that elicit children's thinking about multiplication. Children's drawings of their mathematical reasoning are fascinating and the intuitive understandings that young children develop about aspects of multiplication are worthy of detailed examination.

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Investigating students' drawings as a representational mode of mathematical fluency

Katherin Cartwright
The University of Sydney
<Katherin.cartwright@sydney.edu.au>

Janette Bobis
The University of Sydney
<Janette.bobis@sydney.edu.au>

Jennifer Way
The University of Sydney
<Jennifer.way@sydney.edu.au>

In sharing solutions of mathematical tasks, students may use various modes of representation such as: language (oral/written), numerical and symbolic, or drawings (pictures, diagrams or markings). In this paper we explore the potential of student drawings to provide evidence of mathematical fluency. Examples of young students' (5-8 years old) solutions to mathematical tasks are examined through the lens of drawing representations. The investigation suggested that students' drawings are valuable data when analysing work samples for evidence of mathematical fluency alongside other representations.

Drawn representations are a window into students' thinking and are worthwhile to explore in a mathematical context. Cai and Lester (2005) assert that representations not only help students make sense of mathematical problems but allow for communication of thinking to others. Bakar et al. (2016) agree that students use drawings to share solutions and suggest that "drawing was a *translation* from other types of representations, used [by students] to confirm and explain their answers" (p. 92). Within Way's (2018) research she utilised drawing to "reveal the variety in … drawings, and to explore similarities and differences across the age range" (p. 98). There exists an important transitional point during the early years of schooling for students between drawing (personal expression) and mathematical representation (function and purpose) (Bakar et al., 2016; Way, 2018). These representations require further analysis in observing students' mathematical fluency.

Data reported on in this paper is part of a larger research project (Cartwright, 2019) investigating students' characteristics of mathematical fluency and teachers' noticing of fluency. Within the study, many students produced drawings in their written work to convey their mathematical understanding in solving tasks. The drawings, as a mode of representation, became a vital aspect of analysis when observing a students' mathematical fluency. The purpose of this paper is to build on the drawing representational analysis conducted by Way (2018). In-depth analysis of the drawing work samples addresses the following research question: *How can students' drawing representations provide evidence of their mathematical fluency?*

Method

For the analysis, 39 Kindergarten to Grade 3 work samples were selected from schools involved in the research study. All students responded to the same problem: *The farmer saw 16 legs in the field. How many animals might he have seen?*

To analyse the drawings, previously researched drawing categories (Bakar et al., 2016; Way, 2018) pertaining to students' development of drawings within a mathematical context were employed. The drawing types *pictographic* and *iconic* (Bakar et al., 2016) were used to initially sort the data. Bakar et al. (2016) define drawing as pictographic "if it has realistic

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depictions of the objects stated in the problem" and iconic drawing as containing "only simple lines and shapes to embody the intended objects" (p. 89). Cartwright's (2019) mathematical fluency characteristics were then used as an additional lens through which to view the drawings. Four fluency characteristics were used as deductive analysis categories: use of other representations (numerical or symbolic), correct process or solution, multiple solutions, and efficient strategy. Following the characteristics analysis, data were ordered into a developmental sequence based on Way's (2018) drawing categories: picture, partial story, partition and solution.

Findings

Overall, 17 students (44%) used pictographic representations, 14 used iconic (36%), and 8 used no drawn representations (20%). Interestingly, a few students used both pictographic and iconic representations. During analysis it was necessary to split the iconic category further as a distinct difference between the way students used shapes and lines emerged. Instead of using shapes and lines to represent the animal or its legs, students used lines and circles to cordon off solutions. Some students also used lines, arrows, or circling to connect numerical solutions to symbolic or language representations (see Figure 1). The new category was named *iconic (as organisers)* to distinguish between the two uses of iconic drawings: in place of a picture, or as part of explaining the mathematical process.

The second level of analysis took the sorted work samples (pictorial features) and analysed the data using Cartwright's (2019) mathematical fluency characteristics. All Kindergarten students (N=6) used pictographic representations. Most students also included a numerical representation. One sample included multiple solutions and the majority of students were able to use an efficient strategy to count the legs (see Figure 2). Most students obtained the correct number of legs (16) but did not mention the number of animals.

Figure 1. Example of using lines and circling to organise solution

Figure 2. Kindergarten example of counting by ones

The Grade 1 samples have not been reported on in this paper as there were only three work samples, not enough to make significant statements. For Grade 2 (N=22) twenty of the students included a numerical representation to support their process or solution. Students used pictographic and iconic drawing types, however, there were significant differences in the mathematical features across the samples. One significant difference was the use of symbolic representation. Almost all students who used no drawings included symbols. Whereas only a few students who drew pictographic or iconic representations used symbols. Another significant difference was with solutions and types of efficient strategies. Most students who used pictographic representations did not produce multiple solutions and

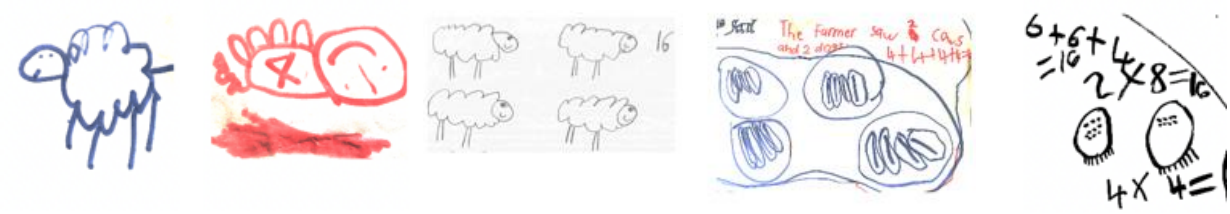
showed no strategy or an additive strategy. Compared with students who drew iconic representations or no drawings where multiple solutions and higher strategies (multiplicative) were observed. All but one Grade 3 sample (N=8) used numerical representations and six included symbolic representations as well. Most students recorded a correct process and solution and the majority of students used multiplicative strategies. Of the students who drew pictographic representations, none produced multiple solutions. Students who drew iconic representations or used no drawings were able to produce multiple solutions, often using their knowledge of number patterns to find different combinations.

Way's (2018) developmental sequence was used in analysing both pictorial and mathematical features of the work samples. Levels (described in Table 2 and illustrated in Figure 3) were adapted as the analysis progressed.

Table 2.
Developmental Sequence of Mathematical Drawings (Adapted from Way, 2018)

Level	No.	Level description
1. Scribble	0	Incoherent, no representation of the mathematical story
2. Picture	2	Shows pictures from the story problem (i.e. animal, farm) but no numerical labels or symbolic representations attached
3. Emergent Story - incorrect process/ solution	2	Shows pictures or iconic representations of the story and includes numerical values. No correct mathematical process or solution are visible.
4. Partial Story - errors with process or solution	7	Uses pictures or iconic representations and numerical values to show process of solving the problem. Correct process but incorrect/incomplete solution. Or correct solution with incomplete/incorrect process.
5. Partition and Solution	7	Uses pictures or iconic representations and numerical values during the process. Shows a correct solution.
6. Advanced Partition and Solution	13	Uses pictures or iconic representations and numerical values during the process. May include multiple solutions or patterns to find solutions.

N=31 (students who did not use drawings have not been included within this analysis)



Level 2 Level 3 Level 4 Level 5 Level 6

Figure 3. Illustrations of mathematical drawing levels

The use of a developmental sequence was beneficial when analysing the mathematical fluency features. For example, both Ellen and Daniel (Figures 4 and 5) used pictographic representations and in the initial analysis were grouped together. However, once these student samples were analysed using the developmental sequence of drawing levels, differences in their use of the representations appeared. Ellen used pictographic and iconic representations in an advanced way compared to Daniel. She labelled her pictures numerically which aligned to her cumulative count by fours. Ellen also drew lines to explain her partitioning of 16 into eights, then fours to describe her process. Although Daniel used a correct process and found a correct solution, his pictographic and numerical representations were separate. It is unclear if Daniel made a connection between the animals' legs and his

count of four. Both samples show characteristics at *Level 5: Partition and solution.* However, if we see the drawings along a continuum of development, Ellen's would be placed higher.

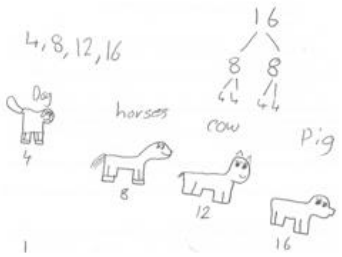


Figure 4. Ellen's work sample

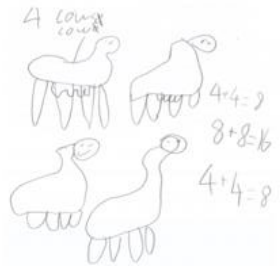


Figure 5. Daniel's work sample

Discussion and Conclusion

It was clear that drawing ability by itself did not always correspond to a student's mathematical understanding. However, students who made direct links between drawings, numerical, and symbolic representations, showed a higher level of mathematical fluency. The findings suggest that there are both affordances and issues with utilising students' drawings to analyse their mathematical fluency. One benefit was that drawings were a visual depiction of students' mathematical strategies. The way students grouped animal legs or drew arrays assisted in deciding if students were applying additive or multiplicative thinking, especially when the symbolic representations were not present. Some impacting factors emerged. Drawing ability *was* an issue for students unable to draw animals appropriately, i.e. incorrect number of legs. For students who drew pictographic representations time was a factor. The time it took to draw the animals resulted in only one solution being found, whereas students who used iconic representations generally found multiple solutions. Future research could explore iconic drawing further, specifically when students created array structures, and could be aligned to Mulligan and Mitchelmore's (2013) levels of Awareness of Mathematical Pattern and Structure (AMPS). Iconic representations revealed students' knowledge of number structure and provided scaffolding to efficiently solve the task.

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Children's drawings as a source of data to examine attitudes towards mathematics: Methodological affordances and issues

Katherine Quane
University of South Australia
<Kate.Quane@unisa.edu.au>

Mohan Chinnappan
University of South Australia
<Mohan.Chinnappan@unisa.edu.au>

Sven Trenholm
University of South Australia
<Sven.Trenholm@unisa.edu.au>

Ascertaining young children's attitudes towards mathematics has its challenges. Methodologically, limitations exist regarding the type of research techniques that can be employed. The use of children's drawings as a data source has both methodological affordances and issues. The study was conducted with 106 children in Years 2 and 3 from three South Australian primary schools. This paper identifies some of the methodological affordances and issues of using children's drawings to ascertain and describe their attitudes towards mathematics.

For Vygotsky, a "young child's creative forces are concentrated on drawing not by chance, but because it is precisely drawing that provides the child with the opportunity to most easily express what concerns him at this stage" (Vygotsky, 2004, p. 43). Children's drawings act as a list or "graphical narration" about what a child is portraying (Vygotsky, 2004, p. 77). Numerous researchers have used children's drawings in the mathematics domain. However, few researchers have used children's drawings to ascertain and describe young children's attitudes towards mathematics. Bobis and Way (2018) state that "representations are an integral part of learning mathematics" (p. 56) and while these authors refer to representations primarily from a conceptual and working mathematically perspective, children representations of themselves are ubiquitous in their drawings. This research connects the ubiquitous nature of children's drawings of themselves with mathematics education by asking children to draw themselves "doing mathematics" as a means of ascertaining their attitudes towards mathematics.

The use of children's drawings is an innovative approach to ascertain an individual's attitude which moves away from traditional research methods such as attitudinal questionnaires. The use of children's drawings provides several affordances that traditional research methods do not allow, including providing a method to children to voice their attitudes which can then describe the nature of their attitudes in depth. Conversely, the innovative nature of this research raises several issues related to the interpretation and analyses of children's drawings. This paper examines some of the affordances and issues of using children's drawings to ascertain young children's attitudes towards mathematics.

The purpose of this study was to investigate the attitudes of young Australian children in Years 2 and 3 have towards mathematics. This investigation answered the broad question: *What are the range and nature of attitudes young children exhibit towards mathematics, in both lesson and non-lesson contexts?* It is essential to distinguish between the range and nature of young children's attitudes towards mathematics. In this paper, a distinction has been made to ensure clarity around the two words. Additionally, the words 'nature' and 'range' have often used interchangeably, but both describe specific aspects of this research. The range refers to the scope or extent of young children's attitudes towards mathematics, providing a broad view of the issue. The nature of attitude is descriptive, providing the basic

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qualities, structure, and the essence of individual attributes of children's attitudes towards mathematics. In other words, the nuances or fine-grain view of attitudes.

Method

This paper discusses findings from the non-lesson context where children drew a picture of themselves doing mathematics, provided a written description of their drawing and participated in a semi-structured interview. One hundred and six children, aged between 7 and 9 years of age, participated in a mixed-method research design where children's drawings started a conversation about their attitudes towards mathematics.

Utilising the work of Bachman et al. (2016) the prompt "Draw yourself doing mathematics" was given to participants on an A3 piece of paper. The researcher read a prompt (see Quane et al., 2019) to children with no time limit given to children to produce their drawing. Children provided a written description of their drawing and then participated in a semi-structured interview. Using the three research techniques is viewed as "complementary methods" to "understand children's lived experiences" (Macdonald, 2009, p. 48). The generated data from the three research techniques was analysed using a modified version Three Dimensional Model of Attitude (TMA) (Zan & Di Martino, 2007). The original TMA framework comprised of three aspects of attitude: an emotional dimension; a vision of mathematics; and perceived competence. In the discussion below we take up the methodological affordances of using children's drawings in terms of TMA, in the course of our research.

Findings and Discussion

The use of children's drawings was effective in identifying the range and describing the nature of young children's attitudes towards mathematics. However, while the use of children's drawing as a research technique has benefits, it raises some issues. In this discussion, the affordances and issues pertaining to the use of children's drawings is reviewed.

Attitude is a multi-dimensional construct (Zan & Di Martino, 2007) that can be complex to unpack. Any research method employed to ascertain attitudes towards mathematics needs to disentangle the different strands of this complexity. That is, the use of children's drawings as a research tool needs to be sensitive to the multi-faceted nature of the construct in question, namely attitude. Additionally, data about attitudes towards mathematics has to capture the dynamic interplay between the dimensions of attitudes.

Drawings constitute an accessible vehicle for communication, expressing what is important for the child. Unlike surveys, drawings are open-ended, expressive and are child-centred tasks (Stiles et al., 2008). Stiles and colleagues (2008), found that "attitudes towards mathematics expressed in drawings significantly correlated with attitudes expressed in the TIMSS [The International Mathematics and Science Study] statements about mathematics" (p. 1) and are "superior to the TIMSS statements" (p.13).

Drawing affords children to express what is important to them in a medium that they feel comfortable. Further, children could express a variety of emotions, as shown in Figures 1 – 3. Children articulated connections between the emotions that they expressed to specific mathematical topics and their perceived competence in mathematics.

The second dimension of attitude is children's vision of mathematics (Di Martino & Zan, 2011). For this research, children's vision of mathematics was characterised by the topics, tasks, and processes that they depicted and described as well as their value and appreciation.

The use of children's drawings provided insights into children's vision of mathematics in terms of how children depicted the mathematics that they were doing. The drawings show the interconnectedness of the three dimensions of attitude with children indicating their emotion and self-concept. Figures 4 – 6 show the mathematical topics and the children's representations of these topics. Further data from the non-lesson context provided insight into children's perceived competence, particularly their mathematical mindset and self-concept. For example, C16 (Figure 1) indicated that she hated mathematics, finds it hard but wants to try "make friends" indicating she has a low perceived competence in mathematics.

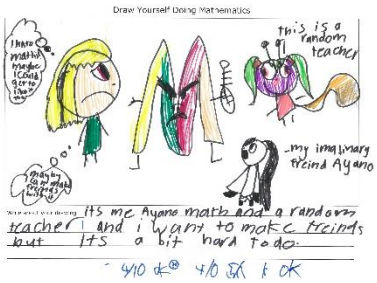

Figure 1. C16; female, negative attitude

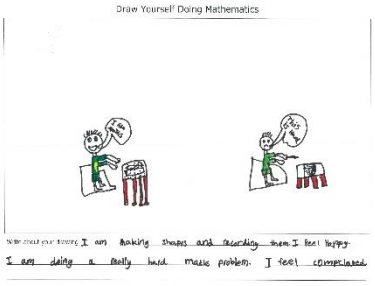

Figure 2. B8; male, neutral attitude

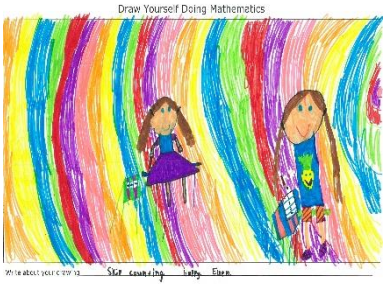

Figure 3. A25; female, positive attitude

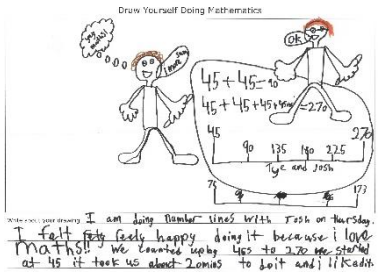

Figure 4. A13; male, extremely positive attitude

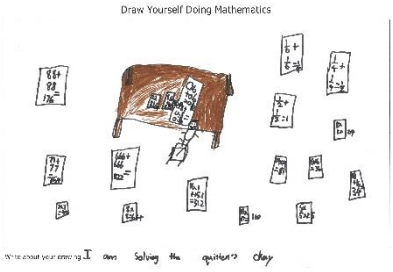

Figure 5. B45; male, positive attitude

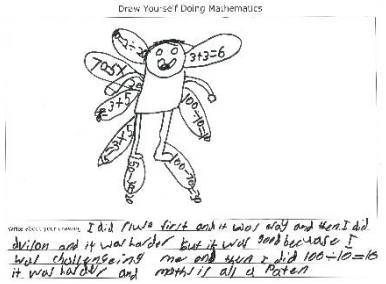

Figure 6. C6; male, positive attitude

Lowenfeld and Brittam (1964) were instrumental in describing the developmental nature of children's drawings. In so doing, these authors drew attention to the principle of 'deviation' as a means for children to emphasise, exaggerate or omit pictorial elements. It is important to note how an observer views these three principles. Lowenfeld and Brittam (1964) cautioned the observer of a drawing regarding making incorrect judgments about a child's intention of using disproportional elements within a drawing. Correct judgements and interpretations can only be made by asking the child about their drawing to understand the reasons for using disproportionally or drew a particular object. When children have used the three types of deviations, the child has drawn what is real, significant, and relevant to them (Lowenfeld & Brittam, 1964).

The principle of deviation is seen in A25's drawing (Figure 3), where she has emphasised the background of her drawing. The child explained that she loved patterns. The emphasis that the child placed on her rainbow background would not have been realised without asking the child open-ended questions about her drawing. The background in A25's drawing consumed A25's attention and focus including her responses to the interview questions. Understanding the importance A25 has placed on the background was required to minimise the potential for the generation data that may have been unreliable. Asking the child about the other elements within her drawing and other open-ended questions such as "what is maths?" provided indicators for all three dimensions of her attitude.

A second emerging issue with using children's drawings as a research technique is the interpretation. The following example illustrates the potential for misinterpretation. Two boys have used the same colour for their face, but the reasons for their colour choice is very different. B17 (Figure 7) has chosen the colour as he believes it reflects his skin colour. B42 (Figure 8) has chosen the colour to show that he is feeling frustrated. Examining the drawings in isolation from the other data sources may produce very different conclusions. It is only when the child is asked about what they have drawn and why they have chosen to draw it in the way that they have, do we truly understand the meaning in their drawings.

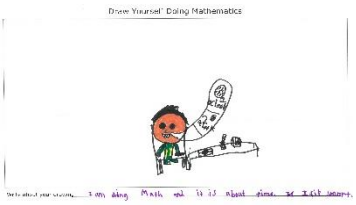

Figure 7: B17; male, extremely positive attitude

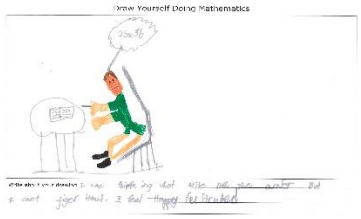

Figure 8: B42; male, neutral attitude

Conclusion

The use of 'Draw yourself doing mathematics' elicits children's drawings that were personal stories about their complex relationship with mathematics revealing their attitude towards mathematics. The process of drawing was a means for children to feel comfortable sharing their thoughts in a familiar manner (Macdonald, 2013). Children were given the time to "comprehensively explain the intended meanings of their drawings through extended conversations and further questioning" (Macdonald, 2013, p. 72). An affordance not offered in quantitative measures. Children's written responses complemented the visual and verbal accounts adding further insights into what was important to them. By providing children multiple opportunities to share their thoughts about mathematics, rich narratives were told about individual attitudes towards mathematics.

In conclusion, our experiences thus far showed that there are challenges in using drawings particularly in unpacking the developmental aspects of attitude. On balance, however, the affordances outweigh the hindrances in deploying the technique. The affordances of using children's drawings can be summarised as giving children the freedom to choose what they depict and how they portray themselves. For children's drawings to be understood by adults, Anning and Ring (2004) offer the following: "We need a society that can listen to children and recognise that perhaps their drawings may tell us much more about childhood than we ever imagined" (p 125).

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Understanding secondary school students' motivations for mathematics subject choice: First steps in construct validation and correlational analysis

Jacky Tianmi Pei Bell
The University of Sydney
<tianmi.pei@sydney.edu.au>

Jennifer Way
The University of Sydney
<jennifer.way@sydney.edu.au>

Paul Ginns
The University of Sydney
<paul.ginns@sydney.edu.au>

With the increased workplace demand for STEM specialists, and the trend in capable students opting out of higher levels of secondary mathematics, the psychological influences on mathematics subject choice are important issues to explore. Expectancy-value theory is used to examine the factors influencing such achievement choices. In the present study, as part of a larger programme of research on mathematics subject choice, we sought to validate self-report measures of students' expectancies for success, values, and perceived costs associated with participation in mathematics. Confirmatory factor analysis supported the hypothesised factor structure, with the measures displaying acceptable levels of internal consistency.

The growing demand for specialist STEM practitioners is undercut by a decline in participation in sciences and advanced mathematics in school and university (Australian Government Department of Education and Training, 2016), and an associated labour shortage in these fields. In the final two years of secondary education, a trend exists in which talented and capable students are turning away from the more rigorous calculus-based mathematics courses. In the state of New South Wales, Australia, there has been a decline from 34 to 22 percent of students opting for these higher-level courses over the past two decades (Jaremus et al., 2018). The calculus-based courses of study available to Australian students are Advanced Mathematics (previously known as Mathematics), Extension 1 and Extension 2 mathematics, and these courses lay the groundwork for meeting the challenges of many tertiary STEM pathways. Not completing adequate levels of high school mathematics preparation is associated with attrition from undergraduate STEM majors, with students being almost twice as likely to fail certain first-year science units if they did not complete a calculus-based mathematics course in secondary school (Nicholas et al., 2015).

It is important, therefore, to explore the antecedents of students making choices either for or away from participation in higher levels of upper secondary mathematics. Why are students, especially girls, increasingly dissuaded from choosing calculus level mathematics? What are the psychological influences on this choice? If we had a better understanding of these factors, might we be able to increase participation in these courses through levers in the middle school experience and the curriculum?

The current research investigates motivations for pursuing mathematics subjects in senior secondary school, with a focus on examining gender differences in motivational influences. The *Expectancy-Value Theory* (EVT) is drawn on as the guiding theoretical framework (Eccles et al., 1983; Wigfield & Eccles, 2000), being one of the most comprehensive frameworks for studying the psychological and contextual factors influencing individual and gender differences in achievement choices (Wigfield & Eccles, 2000). This theory has been used extensively to examine the short-term and long-term motivations and achievement outcomes in a variety of achievement domains.

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Research into student motivations using the EVT has predominantly been quantitative longitudinal variable-centred studies tracking changes in positive motivations throughout school, and their contributions to achievement-behaviour. These positive aspects of motivations are broken into the student's expectancy for success (perceived competence or self-concept) and their valuing of mathematics in the forms of utility value (perceived usefulness), attainment value (importance of doing well) and intrinsic value (inherent interest, a similar concept to intrinsic motivation). Historically, few studies have incorporated the negative "cost" component of the theory into empirical analyses. Cost refers to the perceived drawbacks of engaging in an activity and has been defined as the negative consequences derived from participating in an activity, such as perceived difficulties, fear of failure and loss of valued alternative activities (Wigfield, 1994). In more recent years, a fast-growing literature has begun to focus on measuring cost as a multidimensional construct (e.g., Barron & Hulleman, 2015; Battle & Wigfield, 2003; Chen & Liu, 2009; Chiang et al., 2011; Conley, 2012; Flake et al., 2015; Perez, et al., 2014; Watkinson et al., 2005).

The ongoing study extends this work by focusing on the influence of this negative cost factor in relation to the motivation of high school mathematics students, to further explore its associated effects on mathematics-related academic choices. The results of 500 survey responses collected from Year 10 students in New South Wales are analysed to derive motivational profiles of students with similar beliefs in their levels of expectancy for success, the values they hold for mathematics, and the costs they associate with this subject. A latent profile analysis will be conducted to identify and classify clusters of individuals with similar beliefs based on patterns of categorical responses, followed by interviews of students with high-cost profiles in an attempt to capture a further understanding of their experiences, the complexity of the interrelated influences on their motivation, and their influences on their choice in mathematics studies.

This paper presents initial results of the quantitative analyses assessing the construct validity and reliability of hypothesised constructs. It provides the groundwork for subsequent intended research exploring gendered relationships between and among the various motivational profiles, as well as their relationship to achievement background, language background, dependency on selective schools, coeducational/single-sex learning environments, amongst other educational contexts. These further factors will be explored through both quantitative and qualitative research components to follow. Results from this study can help provide information for school-teachers in understanding factors affecting student motivation and the types of classroom experiences and programs that may help shift students into more favourable motivational profiles, so students may be more likely to persist with a level of mathematics commensurate with their ability.

Method

Participants

Survey data were collected from 521 Year 10 students from 10 high schools in the Sydney metropolitan area. Data were gathered from participants at a critical decision point in relation to subject choice: students completed the surveys after having submitted their subject selection forms, and so were able to report their chosen level of mathematics for Year 11. All students in the Year 10 cohort were invited to participate and were given paper consent forms to be signed by parents or guardians and themselves. Their teachers were asked to remind students to return forms to maximise returns from each school.

New South Wales secondary school contexts vary significantly in demographic characteristics, numeracy performance, and level of participation in senior mathematics. The current study's sample includes a range of coeducational and single sex schools, comprehensive and selective schools. Some of this complexity was reduced by only including government schools (none from the Catholic or private sector), and by employing strategic sampling of schools. Participating schools were matched for socio-economic status to minimise its influence as a confounding variable, measured by the Index of Community Socio-Educational Advantage (ICSEA). This index is calculated based on student-level data on a raft of factors including family background, parental level of education, and remoteness of the school (Australian Curriculum and Assessment and Reporting Authority, 2018). The participating schools' mean ICSEA was 1082 (*SD* = 78.33), above the sector-wide mean ICSEA value of 1000 (*SD* = 100). Three schools were academically selective, seven were coeducational, two were girls-only and one was boys-only.

Instrument

The survey instrument first collected information on school attributes, including subject preferences and academic aspirations. The questions that followed gathered information on the level of mathematics the students had chosen for their final two years of high school and how they believed that it matched with their ability level ("Was this level of mathematics higher than/the same as/lower than what you believe you're capable of?"). There were also three open-ended short-answer questions eliciting students' reasons for their choice in level of mathematics. Follow-up interviews in the second qualitative part of this study will further clarify student responses to these questions.

This section was followed by 31 items gathering students' perceived expectancy and value (utility, attainment, intrinsic) beliefs, which were sourced from Eccles' Expectancy Value measures (Eccles, 2005; Eccles & Wigfield, 1995), with grammatical and contextualising modifications for the Australian sample developed and psychometrically validated in Australia (see Watt, 2004). Examples of some items are: "How well do you expect to do in your next maths task?" to measure expectancy for success or self-efficacy, "How useful do you think maths is in the everyday world?" to tap on utility value, "Being someone who is good at maths is important to me" to tap on attainment value, and "How enjoyable do you find maths?" to tap on intrinsic value.

The items measuring the dimensions of cost, including effort cost, outside effort cost, loss of valued alternatives cost, and emotional cost, were based on Flake et al.'s (2015) comprehensive scale validation study, with "this class" replaced by "mathematics". Examples of items were "I worry too much about mathematics" to tap on emotional cost, "Mathematics requires me to give up too many other activities I value" to tap on loss of valued alternatives, "Because of the all the other demands on my time, I don't have enough time for mathematics" to tap on outside effort cost, and "Mathematics demands too much of my time" to tap on task effort cost. Each expectancy, value, and cost item was rated on a 7-point Likert scale from 1 (not at all) to 7 (extremely). A question at the end of the questionnaire elicited student interest in participating in a short, individual, semi-structured interview early in the following year to further explore quantitative results and how subject choice are shaped by the various interrelated and interacting facets of motivation. For a copy of the full survey please contact the first author via email.

Procedure

Surveys were conducted in class, online via the Qualtrics survey platform, and were led by the students' normal classroom teacher. Respondents (N = 21) who provided insincere responses (e.g. pattern drawing, string responses) were excluded from the analyses. Missing data were rare as the online format of the survey ensured that important questions could not be skipped; however, respondents who exited the survey without completing it were also excluded from the analyses. The final sample consisted of 500 students (239 boys, 250 girls, 11 other, mean age = 15.69, SD = 0.77).

Confirmatory factor analysis (CFA) was used to assess the dimensionality of latent constructs using M*plus* 6.12 (Muthén and Muthén 2004). Multivariate normality is a key assumption of a range of multivariate statistical methods, including CFA (Kline, 2016). Mardia's (1970) test indicated the data were multivariate non-normal. To account for this, robust maximum likelihood estimation of covariance matrices was used, as this procedure is less sensitive than other estimation methods to violations of the normality assumption (Boomsma & Hoogland, 2001). Each of the latent motivation constructs of expectancies, values and costs were analysed for fit, with their corresponding 3 to 6 items as indicators for their assigned latent constructs.

To assess the reliability of survey measures, McDonald's (1999) omega was used as an estimate of internal consistency. There has been increasing criticism of the use of Cronbach's alpha in behavioural science research due to some of its untenable assumptions. Some of these assumptions include the requirement that each indicator variable contributes equally to the factor (tau-equivalence), and that error variances must be uncorrelated (Dunn et al., 2013). McDonald's omega takes into account the strength of association between items, as Cronbach's alpha's failure to do so may overestimate the reliability of results (Dunn et al., 2013). These initial procedures will ensure the consistency, validity and reliability of the latent constructs measured for the purposes of this study.

Results and Discussion

Confirmatory factor analysis confirmed that the eight-factor model of motivation to be a good fit to the data. Model fit was evaluated using recommendations by Kline (2016) and Marsh et al. (2004), focusing on the Comparative Fit Index (CFI), Tucker-Lewis Index (TLI), the root mean squared error of approximation (RMSEA), and the standardised root mean squared residual (SRMR). By these recommendations, RMSEA values at less than 0.08 are considered acceptable fit and values less than 0.05 are considered excellent fit (Marsh et al., 1996). For the CFI, values at or greater than 0.95 are taken to reflect excellent fit to the data (McDonald & Marsh, 1990). Cut-off values close to 0.95 for TLI; close to 0.08 for SRMR (Hu & Bentler, 1999) are considered acceptable fit.

The present eight-factor model showed an acceptable fit for each of the constructs (χ^2 = 788.76, df = 437, CFI = 0.964, TLI = 0.959, RMSEA = 0.040, SRMR = 0.042). Table 1 presents factor loading ranges of items against the hypothesised latent constructs, as well as descriptive statistics, reliability (using estimates of McDonald's omega). Where single item indicators (e.g. Gender, NESB, Co-educational/Single-sex school, Comprehensive/selective school, NAPLAN achievement) were used, the variance of these indicators was fixed at one, and standard deviation fixed at zero. For each sub-dimension, estimates of reliability using McDonald's omega ranged from 0.87 to 0.93, which indicates a high degree of internal consistency for all scales; factor loadings were strong (> .60),

indicating items were suitably measuring hypothesised constructs. Descriptive statistics, McDonald's omega measures of reliability are also provided in Table 1.

Table 1
Descriptive statistics, reliability using McDonald's omega, confirmatory factor analysis factor loadings and measurement errors for each subconstruct

	Mean	SD	Skewness	Kurtosis	McDonald's ω	CFA Factor Loading Range	Mean Residuals
Expectancies	4.91	1.32	– 0.77	0.57	0.93	0.88 – 0.92	0.01
Intrinsic value	4.39	1.63	– 0.56	– 0.19	0.93	0.89 – 0.94	0.01
Attainment value	4.62	1.55	– 0.42	– 0.66	0.87	0.83 – 0.84	0.02
Utility value	5.02	1.37	– 0.45	– 0.53	0.89	0.84 – 0.89	0.02
Task effort cost	4.08	1.36	– 0.01	– 0.40	0.89	0.63 – 0.82	0.02
Outside effort cost	3.83	1.43	0.20	– 0.47	0.92	0.81 – 0.88	0.02
Loss of valued alternatives	3.48	1.41	0.22	– 0.23	0.88	0.74 – 0.85	0.02
Emotional cost	4.17	1.52	– 0.06	– 0.70	0.92	0.60 – 0.88	0.02

A correlational analysis showed that each measure of cost was negatively correlated to each of the positive subconstructs of motivation, which was to be expected. Some cost subscales were found to be highly correlated with one another, for instance, the correlation between task effort cost and emotional cost being 0.87. This level of correlation is not ideal, as a correlation of 1 means that the constructs are indistinguishable. The scale development study from which the current survey is based (Flake et al., 2015) found similar correlations in their initial analyses into measuring and operationalising the "cost" component for motivation. Their confirmatory factor analyses provided the strongest support for the four-factor solution of cost.

However, Flake et al. (2015) also found that the higher order factor model, which included a general unidimensional cost factor, also provided a good fit to the data. They argued that although the four-factor solution provided four highly-correlated dimensions, it showed adequate reliability and model fit, and a further correlational study revealed that the four cost factors had different relationships to the other positive motivation factors. This particular scale development study was conducted in a tertiary calculus setting with a smaller cohort (N = 228), which may explain the discrepancy between those results and the ones produced in the current secondary setting. Flake et al. suggested that future research should investigate the empirical structure of cost within different groups of students in different contexts to see how their findings might replicate across educational settings. The current study provided one such further context and showed that the cost factors also displayed high levels of multi-collinearity. Table 2 presents a latent correlation matrix for the constructs under analysis.

Table 2
Latent factor correlations for Expectancy, Values and Costs perceptions

	EXP	IV	AV	UV	TEC	OEC	LOVA	EMC
EXP	–							
IV	0.66	–						
AV	0.65	0.80	–					
UV	0.37	0.57	0.68	–				
TEC	– 0.39	– 0.55	– 0.42	– 0.35	–			
OEC	– 0.41	– 0.46	– 0.37	– 0.28	0.80	–		
LOVA	– 0.35	– 0.43	– 0.32	– 0.29	0.86	0.82	–	
EMC	– 0.51	– 0.63	– 0.46	– 0.31	0.87	0.72	0.73	–

Note. All correlation coefficients are statistically significant at the $p < 0.001$ level. EXP = expectancy for success, IV = intrinsic value, AV = attainment value, UV = utility value, TEC = task effort cost, OEC = outside effort cost, LOVA = loss of valued alternatives, EMC = emotional cost.

Correlations between the expectancies and values dimensions echoed that of comparable previous studies of secondary students' mathematics motivations using EVT (e.g. Watt, 2004). The highest correlation between these positive factors were between intrinsic value and attainment value (r = 0.80), followed by correlations between utility value and attainment value (r = 0.68) and between expectancy for success and intrinsic value (r = 0.66). In the current sample, although the correlations between the cost constructs were found to be high, the model also had a good level of fit and the cost sub-constructs were found to be differentially related to expectancies for success and values. Interestingly, emotional cost was related to intrinsic value more than any other cost component, which brings up the question of how emotional cost and the psychological cost of failure impacts on high school students' intrinsic valuing for mathematics. This question, along with others, will be further explored in the subsequent interview study with a subset of the survey participants.

Conclusion

The factorial structure of the underlying constructs was validated using CFA, with the measurement model confirmed to be valid and ready to be used for further analyses on relationships between the latent variables. High degrees of internal consistency showed that the items were reliable in measuring the constructs they were designed to measure. The fit indices were adequate, which was expected because the expectancy and value scales have undergone rigorous scale validation through multiple studies, across many year groups and in a variety of subject contexts. However, the cost scales displayed a level of multi-collinearity, and were problematic in some pairs of sub-constructs having a higher level of correlation. As the particular scale development study was conducted in a tertiary calculus setting with a smaller cohort, further work needs to be done examining the construct and dimensionality of the cost factor in the secondary context.

The present study provides a foundation for subsequent intended research linking students' mathematics motivational profiles with their school contexts and choice of

mathematics course. The next steps in analyses include conducting a latent profile analysis to explore how students hold multiple motivational beliefs simultaneously to make decisions on persisting with difficult mathematics subjects, rather than examining the isolated effects of single variables. Previous work on motivational profiles have shown different profiles to be differentially related to persistence outcomes (Perez et al., 2014; Watt et al., 2019). Studies of cost have repeatedly shown that the theorised dimensions of cost contribute differentially to student motivations and have suggested that future research should seek to understand the sources of cost.

Without an understanding of how costs interact with the other expectancy and value components, and by excluding it from the EVT framework, research findings about motivational influences may be compromised. An imbalanced value-cost relationship may hinder motivation, so the planned interviews will seek to understand the experiences of students to gather the reasons and sources for the costs they perceive. "What could teachers do to optimise student motivation if they knew students were experiencing high cost?" was a question that Flake et al. (2015) posed in their study, and highlighted that it is a question that remains unanswered. The ongoing study aims to contribute to the literature on how the components of expectancy, value, and cost influence student motivation in the context of high school mathematics.

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Exploring the ‘high’ and ‘low’ points in primary preservice teachers’ mathematics-related identity development

Janette Bobis
The University of Sydney
<janette.bobis@sydney.edu.au>

Janet Nguyen
The University of Sydney
< jngu6830@uni.sydney.edu.au>

Heather McMaster
The University of Sydney
<heather.mcmaster@sydney.edu.au>

We report on the use of a data-gathering task requiring preservice primary teachers to ‘graph’ their emerging relationships with mathematics. A cultural-historical activity approach was used to analyse data from nine final year preservice primary teachers to reveal what and how key events in their lives helped shape their current mathematical identities. Oscillations between “high” and “low” points in their relationships with mathematics was a feature of participants’ graphs regardless of their current mathematical identities. Combined with semi-structured interviews, the graphing task is posited as a valuable method for researchers and practitioners to explore mathematics-related identity.

The development of a positive mathematical identity is considered critical to student learning because of its potential to influence career and higher education aspirations (Black et al., 2010). Selecting a mathematics-related career is not just about being academically successful in mathematics, it is also determined by how a person identifies with mathematics as a discipline (Sfard & Prusak, 2005). It is therefore unsurprising that the development of a healthy student identity with mathematics is considered of major importance to achieving current goals for the Australian government’s mathematics and science related education agenda (Australia Government, 2015). Research indicates that teachers’ personal identities with a particular discipline can profoundly influence how they teach that discipline and position their students to learn it (Leatham & Hill, 2010; Reay & Wiliam, 1999). Unfortunately, it is well established that many primary teachers have not experienced healthy relationships with mathematics as students (Maasepp & Bobis, 2014), making it difficult to nurture positive identities in their own students. Such a situation can be detrimental to primary students’ long-term decisions to undertake further study in mathematics areas as early negative experiences can have enduring negative influences on students’ achievements and aspirations in those disciplines (Black et al., 2010).

Numerous researchers have studied preservice primary teachers’ mathematics-related identities, often with the intention of better understanding the personal experiences that shape certain identities (Darragh, 2016). Studying identity is problematic due to its complexity—commonly conceptualized as dynamic, multidimensional and formed through a blend of personal characteristics and long-term socio-cultural experiences. Such complexity has raised questions about the capacity of researchers to provide an adequate measure of mathematical identity (Kaspersen et al., 2017). With this challenge in mind, we sought to explore the mathematical experiences of primary preservice teachers (PSTs) that helped shape their current mathematics-related identities. Additionally, the merits of a relatively novel task that required participants to graph the high and low points in their relationship with mathematics was explored. We conclude the paper by advocating the graphing task as a valuable qualitative strategy for researchers and teacher educators to understand the experiences and conditions under which mathematical identities develop.

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Defining Identity

Definitions of identity vary from those who consider it to be how individuals are perceived by themselves and others (Grootenboer et al., 2006) to those who emphasise the socio-cultural context in which individuals act (Kaspersen et al., 2017). However, Sfard and Prusak (2005) conceptualized identity as discourse comprising endorsable stories or narratives about 'who one is' independent from one's actions (Kaasila et al., 2005). No matter how it is defined, researchers generally conceptualize identity as a multidimensional construct, combining elements such as knowledge, beliefs, attitudes, emotions, confidence and dispositions that influence how individuals view themselves and are viewed by others (Beauchamp & Thomas, 2009; Kaasila et al., 2012). In essence, we see identity as relational by nature, incorporating both cognitive and affective aspects (Kaasila et al., 2012; Leatham & Hill, 2010) and is dynamic in nature in that an individual's identity is considered to be constantly shifting as a result of social interactions. More specifically, in the current study, we use Lutovac and Kaasila's (2019) term 'mathematics-related' identity because it encompasses all aspects of a preservice teacher's identity related to mathematics.

Preservice teachers' mathematical-related identities can be influenced by their socio-cultural backgrounds. A study by Watkins and Noble (2008) involving 35 Year 3 students from different ethnic backgrounds revealed that Chinese parents had higher expectations for their children's achievements than their Anglo and Pasifika peers. Such cultural background influences could impact the developing identities of young children in either positive or negative ways. Socio-cultural factors that can potentially influence identity go beyond ethnicity to include a range of family, religious, educational and socio-economic background elements.

Theoretical Perspective

Studies that are framed in cultural-historical activity theory (CHAT) view identity as essentially a social experience, whereby the context must be considered when interpreting an individual's activity or responses (Engeström, 2001). In this study, we were interested in primary PST's shifting relationship with mathematics (the context) over time (historical) and how they responded (the activity) to salient events (socio-cultural) in their lives.

The mathematics-related identities that PSTs develop as students via various socio-cultural contexts can not only influence the actions that they take regarding their own relationship with mathematics but those of their future students (Maasepp & Bobis, 2014). Thus, it is of utmost importance that primary PSTs not only develop healthy mathematics-related identities, but that mathematics educators can easily assess information about their PST's identities to ensure adequate interventions might take place.

Research Design

While most investigations adopt a qualitative tradition to explore mathematics-related identity (e.g. Black et al., 2010; Darragh, 2016), some quantitative studies exist (Kaspersen et al., 2017). Given an aim of this study was to gain a deep understanding of the socio-cultural experiences of prospective primary teachers, we adopted qualitative methods including a reflective task to elicit the historical information we needed. Hence, a second aim of our study was to explore the merits of a qualitative identity task that encourages individuals to graphically represent the high and low points in their relationship with mathematics over their life experiences. The identity graphing task, accompanied by an individual semi-structured interview, is appropriate for studies adopting a cultural-historical

approach given its capacity to capture reflective insights into the impact of past events that may not have been obvious to PSTs when they occurred. The research questions addressed were:

1. *What experiences in the lives of prospective primary teachers do they report as influencing their emerging identities with mathematics?*
2. *To what extent can an identity 'graphing' task provide information about the socio-cultural and historical contexts in which mathematical identities are shaped?*

Setting and Participants

All prospective primary teachers (N = 96) enrolled in the final year of a four-year Bachelor of Education program (B.Ed. Primary) at a large university located in an Australian state capital were invited to participate in the study. Nine PSTs (six female, three male) aged 20-24 agreed to participate. Background data were collected at the start of the interview for all participants and are summarized in Table 1. All participants completed mathematics in their final year of secondary school and were born and schooled in Australia. Four PSTs had Asian born parent(s).

Table 1
Background Details of the Nine Participants

Participant Pseudonym	Parents' Birth Country	Level of Mathematics completed in Year 12 (final year of secondary school)
Abigail	Both Australian	Intermediate
Arthur	Mother UK Father Hong Kong	Intermediate
Angela	Both Sri Lankan	Advanced
Brenda	Both South Korean	Advanced
Benjamin	Both Australian	Intermediate
Blake	Both Australian	Lowest Level
Charlotte	Both Australian	Lowest Level
Celeste	Both Australian	Lowest Level
Caitlyn	Both Chinese	Intermediate

In this paper we report detailed findings for two of the PSTs to illustrate the capacity of the identity graphing task. However, we draw upon data from all nine participants when referring to common themes. Brenda and Caitlyn were selected for closer focus due to the clarity of annotations on their identity graphs and because the end-point on their graphs (their relationship with mathematics as they perceived it at the time of this study) were very different, despite sharing some similar socio-cultural experiences.

Data Collection Tasks, Procedure and Analysis

The introductory phase of the interview involved questions intended to gather information on PST's family backgrounds, schooling and involvement in mathematics study. The second phase comprised an identity graphing task. The 'Me and Mathematics' instrument developed by Lewis (2013) was adapted to gain a visual representation of each PST's relationship with mathematics. This instrument was modified to specifically capture how PSTs' mathematical identities had been shaped by their past experiences. Participants were asked to reflect upon their memories (as far back as they could recall) and involvement with mathematics that they felt helped shape their current relationship with the discipline. They were then provided with a A4 sheet of paper containing a pre-drawn horizontal and vertical axis. The horizontal axis was labelled "key events that have shaped my identity with mathematics" and the vertical axis was labelled "degree of enjoyment/dislike/confidence/ anxiety". Participants used a black pen to construct a line graph representing the high and low points in their 'relationship' with mathematics and then annotated it with a different coloured pen to describe the nature of each experience.

In the final phase of the interview and immediately after drawing their identity graphs, participants were questioned to clarify reasons for turning points in their graphs. Our focus here, is on those turning points. In particular, we wanted to gain a better understanding of the socio-cultural aspects underpinning such key events and of PSTs' behavioural, cognitive and affective responses to them. The open-ended questioning offered in-built flexibility during data collection as it encouraged PSTs to comfortably share their stories, eliciting 'how' and 'why' they possessed certain mathematics-related identities (Neuman, 2013). Blending the strengths of identity graphing and semi-structured interviews improved data validity as the focus for the data collected was specified, and the participant role in the data generation process was increased (Tashakkori & Teddlie, 2010).

Individual interviews were audio recorded and transcribed to assist with analysis. Data from the interviews and the identity graphs were combined for analysis to provide a wholistic picture of each PST's data. An across case thematic analysis was conducted involving Braun and Clarke's (2006) six phases—data familiarisation, generation of initial codes, search for themes, review and defining themes, and report production. Given our interest in primary PSTs' shifting relationships with mathematics from both a cultural and historical perspective, initial coding adopted the major theme of *culture* which was soon divided to the subthemes of *ethnicity-culture*, *family-culture*, and *school/classroom-culture* as analysis of data proceeded.

Results and Discussion

Thematic analysis involving all nine participants revealed that cultural expectations, parental and teacher influences were among the key factors that shaped PSTs' mathematics-related identities. A notable feature common to all identity graphs, was the oscillations between "high" and "low" points in PSTs' relationships with mathematics throughout their lives. This characteristic oscillation occurred regardless of whether they perceived their current mathematics-related identity in a positive or negative light. These graphic representations affirm conceptualisations of identity as a dynamic construct that is constantly shifting. Moreover, just one event has the potential to instigate a downward or upward trajectory in mathematics-related identity formation. Of interest was the nature of events that could influence trajectory changes and why some PSTs could experience similar events to others but develop very different mathematical identities. It is reassuring to note that a

downward trending relationship with mathematics can be reversed with the right combination of socio-cultural experiences. We now restrict our presentation and discussion of data to Brenda (Figure 1) and Caitlyn (Figure 2).

Brenda and Caitlyn both expressed the view that their Asian heritage greatly influenced their mathematics-related identity formation as they were growing up. Brenda, who went to a selective high school, reflected: "If I told someone who was not Asian that I did 3-Unit maths, they wouldn't be surprised … . Non-Asian students would be surprised if they did perform better than I did." While the stereotype assumption that "Asian students are good at mathematics and non-Asian students are not" in both PSTs' schooling experiences was prevalent, they responded differently. On inspection of her graph, it is clear that Brenda considered her earliest relationship with mathematics as quite positive (the first high point in Figure 1). It was recalled in terms of her academic performance relative to her peers. She used a different coloured pen (blue) to record each memory referencing her parents and family – successive comments pertaining to family appear at the two lowest points on her graph. Brenda eventually opted to take advanced mathematics for her final years of secondary school and despite some low points associated with poor test scores in Year 11 (as represented in Figure 1 at the fifth turning point), managed "through effort" to improve.

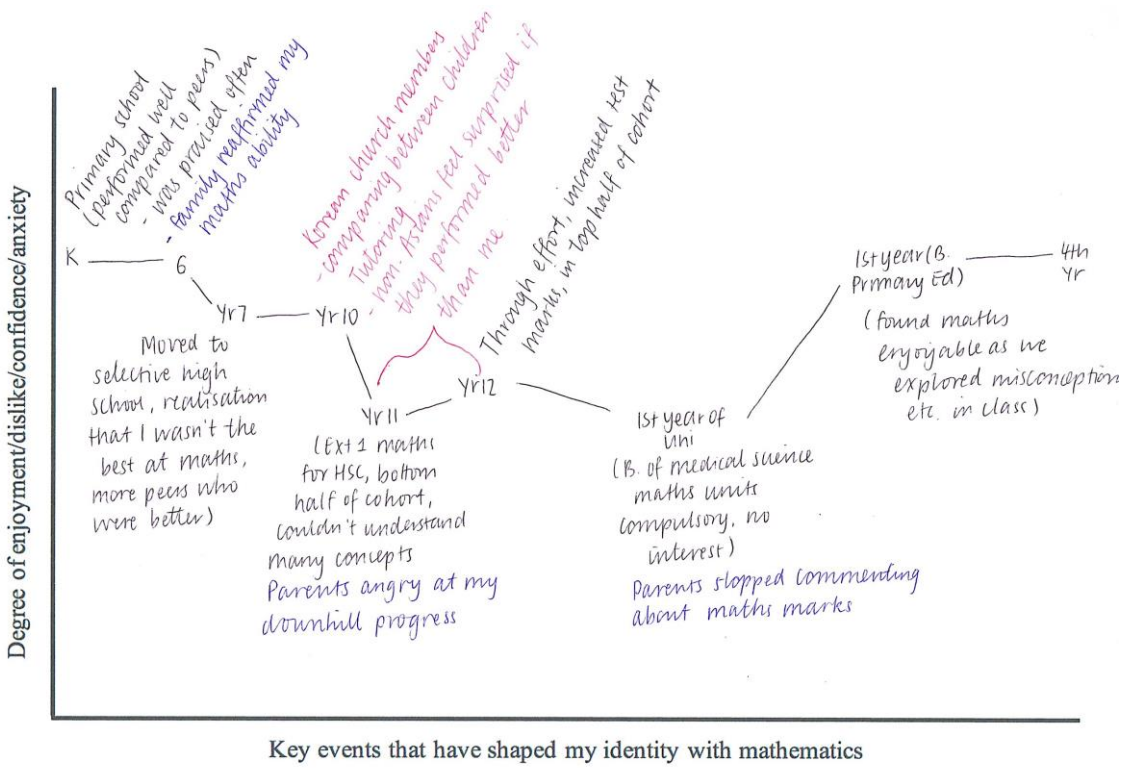

Figure 1. Brenda's identity graph

The same stereotype had a negative impact on Caitlyn. In early high school, she wanted to maintain the expectation that Asians are good at mathematics but when she achieved poorly in Years 9 and 10 (as represented in Figure 2 by the dip between Years 7 and 9/10), she actually felt "proud not fitting into that Asian stereotype". She also stopped caring or "trying" to do well in mathematics, believing she had already failed the Asian expectation.

Resistance to the Asian stereotype image by Caitlyn had a lasting impact on her relationship with mathematics, continuing into her B.Ed. program and contributed to her resultant "indifferent" attitude toward mathematics.

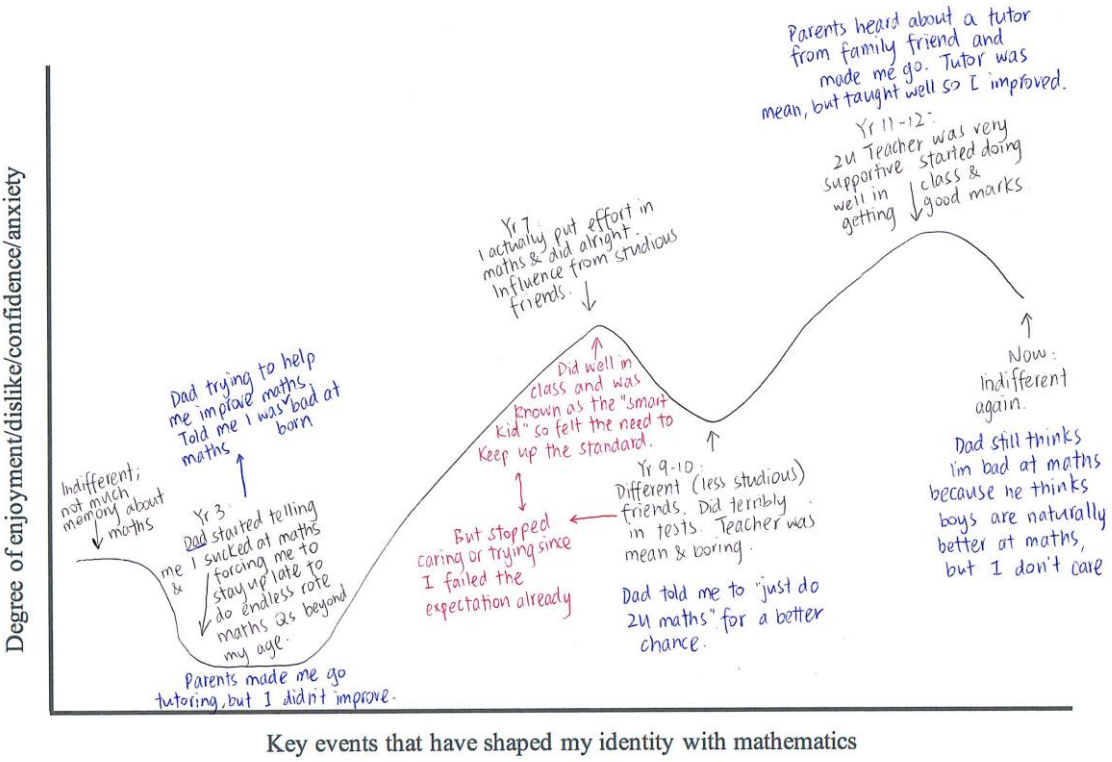

Figure 2. Caitlyn's identity graph

Brenda's Korean parents placed significant pressure on her to perform well in mathematics. She shared that her parents "put a lot of emphasis on maths" and that "other subjects weren't considered" as important. Brenda experienced reduced self-confidence in mathematics, as illustrated by the dip between Year 10 and 12 (Figure 1). A relaxing of parental pressure to achieve in university mathematics was one of the factors that Brenda attributed to regaining enjoyment in mathematics during her preservice program. Similarly, Caitlyn's parents, particularly her father, applied a great deal of pressure to perform well in mathematics and attributed her 'less than expected' performance to the belief that "boys are naturally better at maths". Caitlyn indicated that once her parents concluded that she was not going to achieve the level they expected of her, she stopped trying to improve and became content to be "indifferent" to mathematics (see final down-turn in Figure 2). Sadly, such indifference can be detrimental to her ability to nurture positive relationships with mathematics by her future primary students.

Mathematics teachers and friendship groups were also substantially involved in shaping PSTs' mathematical identities. For example, Caitlyn attributed her upward slopes and peaks on her identity graph to the "better", "kind and amazing" mathematics teachers she had late in high school and to the influence of "studious friends" in her first year of high school. Similarly, Brenda considered her positive experience of mathematics teaching in first year university as the reason for a spike in her interest and enjoyment of mathematics—a

relationship that steadily increased to her final year of the program (as represented by the final two turning points in Figure 1).

Conclusion

The identity graphs revealed that a range of socio-cultural experiences, including cultural stereotypes, parental expectations, peer pressure, school culture, teacher expertise and teacher empathy had the potential to shape PSTs' personal views of and attitudes towards mathematics, resulting in different mathematics-related identities. The semi-structured interviews were critical to the interpretation of the reasons underlying individual PST's responses to each critical experience.

In this paper we have shown how personal mathematics-related identities can be elicited from primary PSTs using a simple graphing task and interpreted via a cultural-historical activity perspective. We posit the graphing task as a valuable qualitative method for researchers and teacher educators to understand the experiences and conditions under which mathematics-related identities develop. Combined with a semi-structured interview, the task encourages participants to provide rich descriptions of past experiences and reasons as to how/why they were considered influential in the formation of their mathematics-related identities. Such information can assist mentoring processes to help prospective teachers reflect upon identity formation and the experiences that can positively shape the mathematics-related identities of their future primary students.

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Coding and learning mathematics: How did collaboration help the thinking?

Nigel Calder
University of Waikato
<nigel.calder@waikato.ac.nz>

Kate Rhodes
University of Waikato
<ktleg99@gmail.com>

This paper reports on two teachers' perceptions as part of a project examining the learning that took place when 9 and 10-year-old children used *ScratchMaths* in their programme. The project used design-based methodology, which incorporated video-recorded classroom excerpts, teacher interviews, and teacher analysis and review of their practice. The teachers identified the students' problem solving, collaborating using explicit mathematical and coding language, and being cognitively engaged. They also recognized that their own practice evolved into a more faciliatory role, while their understanding of coding processes grew through learning beside, and through, their students.

In 2020, the new Digital Technology Curriculum (DTC) became a mandatory part of the New Zealand (NZ) Curriculum but research indicates that NZ teachers and schools will find adopting and implementing DTC challenging. This is because it encompasses proficiencies such as coding that are outside the expertise and experience of many NZ primary teachers' current understanding of digital technologies (Crow et al., 2019; ERO, 2019). Crow et al. (2019) indicated a gap in the availability of resources that are specifically situated in curriculum contexts, which would practically assist engagement with coding. They also advocated that teachers and schools develop unique implementations. This paper reports on a small research project that examined teacher practice with coding through the use, evaluation and adaption of University College London's *ScratchMaths* resources, and the associated student learning. The project also aimed to enhance teachers' coding and computational thinking-based pedagogies and student learning while simultaneously addressing the limited resources available for teaching coding in NZ.

Some NZ research has evaluated similar curriculum implementation at high-school level (Johnson et al., 2017) and international research has examined some aspects of DTC (e.g., Falkner et al., 2014; Johnson et al., 2014). However, none of this research specifically examined the affordances and implementation of DTC in the NZ primary-school context. There has been very little research on the use and influence of coding in NZ schools, hence the implementation of the DTC would benefit from being analysed by a collaborative partnership of teachers and researchers, as teachers consider how, when and where it will best be integrated into existing classroom practice, and explore how to support student learning.

Scratch is a free-to-use graphical programming environment that provides opportunities for creative problem-solving. It is a media-rich digital environment that utilizes a building block command structure to manipulate graphic, audio, and video aspects (Peppler & Kafai, 2006). Studies have shown its potential for developing computational and mathematical thinking in an integrated way, particularly in geometry and algebraic thinking (Calder, 2018). *ScratchMaths* aims to integrate computing and mathematical thinking effectively. Mathematics is used as a context and gives purpose for developing computational thinking, while the process of coding, particularly with *ScratchMaths*, is identified as being influential on the development of mathematical thinking (Benton et al., 2018) and the understanding of mathematical ideas such as algorithms and the 360 degree turn (Benton et al., 2017).

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 139-146. Singapore: MERGA.

However, the *ScratchMaths* resources, while well-tested and effective resources, are structured, with small incremental steps to be undertaken by students individually, whereas in NZ learning is seen as a more collaborative, creative process (Ministry of Education, 2007). The project examined how the *ScratchMaths* resources might evolve to be more conducive for learning in the NZ context. For instance, the development of collaborative student-led projects in *Scratch* (e.g., Calder, 2018), which might also emerge with *ScratchMaths*, would be conducive to collaborative problem solving.

Collaborative Problem Solving

In the consideration of collaborative problem solving, collaborative learning is first discussed, together with its potential to improve learning and understanding. Ways that collaboration supports learning when digital technologies are used and the influence of both in facilitating problem solving are next briefly identified. The connection between collaborative problem solving, the use of digital technologies, thinking, and student engagement is then considered. Collaborative learning occurs when two or more students are engaged in an activity, interacting with each other and learning together (Dillenbourg, 1999). This perspective of learning in mathematics repositions learning more as participation in a social practice then as an acquisitional process (e.g., Cobb & Bowers, 1999; Sfard, 1998). Educational collaboration associated with problem solving has been connected to academic success. For example, Mercer and Sams (2006) showed how students collaborating while engaged in an online task produced enhanced learning outcomes in mathematics. Other studies have illustrated how the collaborative use of digital technologies can support students in developing more flexible approaches to problem solving (e.g., Mercier & Higgins, 2013).

Mercer and Littleton's (2007) definition of collaborative learning goes beyond the sharing of ideas and task coordination to "reciprocity, mutuality and the continual (re)negotiation of meaning" (p. 23). Collaborative learning in line with this definition involves the utilization of individual understandings and expertise, with the collaborative interaction influencing the thinking of at least one participant in the interaction, even if there is only a minor adaption, coupled with a repositioning of the learners' perspective and understanding. When students work collaboratively on a task there is frequently a coordinated approach to the sense making and the approach taken when engaging with the task. The joint coordination of a task enables students to communicate and negotiate in order to support decision-making (Zurita & Nussbaum, 2004), and, as such, they are involved in "a coordinated joint commitment to a shared goal" (Mercer & Littleton, 2007, p.23).

In general, digital technologies can enable opportunities to explore and organize data or mathematical phenomena in ways that might facilitate mathematical thinking, and to see patterns and trends more quickly in mathematical situations that might otherwise be too complex to do so. With coding, this offers potential to learn through the iterative process of engagement with the coding process, and reflection on the output that the coding generates. The coder can try something and instantaneously identify the effects of the new coding, enabling them to generalize coding attributes and refine their approach. With a visual environment such as *Scratch*, where the coding and output screen sit side by side, these relationships are even more easily identified (Calder, 2018).

Computational thinking can be considered a collection of problem-solving skills that relate to principles of computer science (Curzon et al., 2009). At times, computer science involves creating applications to solve real-life problems using computational thinking, an analytical, computing approach for problem solving, modeling situations and designing systems (Wing, 2006). Abstraction, allied with logical thinking, innovation, and creativity,

is considered central to the constitution of computational thinking (Wing, 2006). These elements also resonate with mathematical thinking and problem solving in mathematics. *ScratchMaths* appeared to be an engaging and relatively easy to use space for problem solving.

Research has indicated that students become more engaged when using digital technologies, with enhanced mathematical learning also evident (e.g., Attard & Curry, 2012; Bray & Tangney, 2015; Pierce & Ball, 2009). In educational settings, engagement is recognized as more than the student being interested or participating positively, but as a complex, eclectic relationship between the student and classroom work (Fredricks, et al., 2004). They perceived it as being multi-faceted and operating at cognitive, affective and behavioral levels. With regards to using mobile technologies in the process of learning mathematics, Attard (2018) concluded that they do improve student engagement at operative, cognitive, and affective levels.

Additionally, studies have indicated that *Scratch* was an effective medium for encouraging communication and collaboration (e.g., Calder, 2010, 2018). This paper considers teachers' observations and perspectives of the students' problem solving, collaboration and engagement as they undertook coding tasks using *ScratchMaths.*

Research Methodology and Design

Using a design-based research methodology, with the teachers as co-researchers, the project examined two teachers and their 9 and 10-year-old students' use of the *ScratchMaths* resources. This methodology, designed by and for educators, endeavours to enhance the impact and implementation of educational research into improved classroom practice (e.g., Anderson & Shattuck, 2012). It can illuminate the challenges of implementation, the processes involved, and the associated pedagogical and administrative elements (Anderson & Shattuck, 2012). Design research necessarily comprises multiple cycles, which involve a number of different design and research activities. Nieveen and Folmer (2013) divide these activities into three distinct phases: the preliminary research phase; the prototyping or development phase; and the summative evaluation phase. These three phases, involving the teachers and including videoing of their classes, were implemented through iterations of use, reflection and modification of the resources and the associated pedagogy.

The research design was also aligned with teacher and researcher co-inquiry whereby the university researchers and practicing teachers work as co-researchers and co-learners (Hennessy, 2014). Allied to this was an emphasis on collaborative knowledge building. The research design was based on a transformational partnership arrangement that aims to generate new professional knowledge for both academic researchers and teachers (Groundwater-Smith et al., 2013).

The *ScratchMaths* resources identified by the teachers to use initially were from module one and included: Moving, turning and stamping, and creating circular rose patterns. The *ScratchMaths* resources and existing projects were used as starting points for the lessons, with the "unplugged" activities also incorporated into the sessions. Some of these class sessions and individual groups working on the tasks were video recorded. There were two iterations of the review and design process with videoing of classes each time, followed by co-researcher meetings to examine the classroom practice. One element of these meetings was the analysis of classroom video recordings. Discussions in the meetings were recorded, as were the teacher interviews.

The research question related to this paper was: *In what ways might the use of coding embedded within a mathematics curriculum context, influence teacher practice and children's coding and mathematics engagement?*

Results and Discussion

The paper reports on teachers' perceptions of how using *ScratchMaths* facilitated the learning process in four key areas: problem solving, collaboration, mathematical thinking and the teachers' pedagogical approach. The teachers consistently commented on how using *ScratchMaths* fostered a problem-solving approach as the students found solutions to unfamiliar problems in mathematical contexts, through a variety of approaches. For example:

Annie: The children were problem solving, risk taking and learning from failure

Marama: It's massive (problem solving). For some activities there are no instructions for how to get them from there to there, they just had to work it out.

The students use of *ScratchMaths* within the problem-solving process at times led to enhanced engagement. The process of debugging code was a particular aspect that some students became immersed in. This is a part of computational thinking that involves reviewing the code through trialing and when it didn't produce the desired output, collaboratively problem-solving possible solutions. It might also involve the output unexpectedly stopping or going into continuous loops. While the aspect of debugging was highlighted by the teachers at times, usually students were self-motivated with this process through wanting the script to be consistent with their expectations of the output. Marama commented on the student engagement consequential of the debugging process:

There would not be many things that would have them that focused on what they're doing so intensely. They would be doing debugging the whole time.

The teachers identified that the students not only appeared more cognitively engaged but that the process facilitated enjoyment and a sense of fun.

Marama: They're having a laugh as well you know... it's not all serious... even though it's heavy duty problem solving. They're having fun, they're smiling and enjoying working with each other too.

Marama: Well, it's not quiet in our classroom but it's not off task noise, it is completely on task noise. It's talking about what they are doing and it's excited talk.

The students interacted with each other in a relatively natural, seamless manner as they explored potential solutions and then collaborated to make their codes more efficient. As they worked to design the scripts and subsequently make the codes more efficient, they shared ideas and potential solutions using language that used coding terminology, or was related to the mathematical or coding processes that they were discussing. The teachers noted this in the interviews. For instance, Annie indicated how the collaboration fostered their shared understanding of language, and hence from her perspective, their mathematical and computational thinking:

Annie: It supported students' learning through communicating with friends, problem solving, increasing their mathematical knowledge and mathematical and coding language, bringing that all into the norm of how we can talk about coding.

Annie: So, then we can look at different ways of how children create a script to get to an end product and look at just simplifying the script.

Marama identified instances when students found efficient ways to code that were valued by other students, enhancing their mana (respect) within the class. Sometimes this wasn't the students who were usually perceived as being more capable in mathematics so it readjusted those perceptions.

> Marama: There are kids that are capable but then someone quietly just comes up with this really simple code to do something that someone else has taken a long time to do and they think they're good so it's kind of just levelled everyone out

This also indicated how using *ScratchMaths* facilitated collaboration. Collaborative learning can be perceived as going beyond the sharing of ideas and task coordination to the ongoing negotiation of perspectives and meanings (Mercer & Littleton, 2007). Collaborative learning in line with this definition was identified:

> Annie: So, it gives a context for social interaction to happen where they're learning to code and learning maths.

> Marama: They're definitely getting extended in their maths but also that social side of it, working together collaboratively like that and not... someone not (always) taking a lead role, they're all in different roles all the time, sometimes they're teachers, sometimes they're learners.

While the ongoing negotiation and evolving perspectives are indicated here, this also indicates that the students' roles were flexible and contingent on their personal, and the group's understandings. Observational data also suggested that there was contestation of ideas during the collaborative work. Not only did the students interact through the ongoing dialogue as they problem solved to find solutions, students did at times became leaders of learning.

> Marama: One of the girls solved this thing that really no-one else was managing to do and she managed to crack it. Well the whole class was whoosh over there, so that's fantastic that she's having to explain it and off they go all excited.

Much of their work involved mathematical thinking. Further, the interview data revealed that at this later stage, for one teacher, the activity focused on the mathematics to begin the task. So, the coding in some instances was a way to enact the mathematical ideas. This was the perception of one of the teachers:

> Annie: It's the maths first and then the coding.

After several weeks they decided to make the work with *ScratchMaths* an integral part of their mathematics programme, so one of the classes usual mathematics sessions became the session using *ScratchMaths.* The teachers also found that the mathematical thinking related to both concepts and processes arose more naturally within the *ScratchMaths* activities. For instance:

> Annie: I think because maybe the opportunities with this program and what it's actually focused on with the angles and the measurement side and the negative numbers – we've been going through this for three terms so it's that continual weekly learning of that that's probably been more cemented than what it could have been if we had been teaching it in isolation.

While the teachers made the mathematical thinking explicit to the students by referring directly to the mathematics and using mathematical language, some of the mathematics emerged through attempting to solve and accomplish the tasks, and the collaboration on the coding aspects. In this way, some of the mathematical thinking and learning was more incidental as the need arose, and outside the usual curriculum level for that age group.

> Annie: It was just-in-time learning around the maths concepts. The use of angles was very in-depth. They used negative numbers, degree turns and always mathematical language.

For instance, negative numbers were not part of the curriculum for this particular age group. In a later discussion they identified some of the other mathematical thinking that occurred: Relationships, exploring variations, precision with language, methodical thinking, and strategies for problem solving. Their spatial awareness, understanding of angles, and positioning sense through the use of coordinates, were all engaged to varying degrees. There was also evidence of relational thinking as the students made links between their input, the actions that occurred on screen, and the effect of specific variations of size in coding procedures. They discussed how the students came to conclusions and gave explanations of what they had done.

The fourth aspect reported here is the teachers' pedagogical approach, which varied from their usual approach when teaching mathematics.

> Marama: I don't know that I need to know everything. Most of the time it's the kids that are the ones that solve things. They are learning off each other a lot more, they're going to each other a lot more, they're talking a lot more.
>
> Annie: The classroom approach is to explore, but the mathematics and coding objectives are explicit. At times (we) start with *ScratchMaths* for say, angles. There is a purposeful context for the learning.
>
> Marama: The teachers' role is facilitating learning – actively scaffolding processes and content.

The teachers were consistent in their belief that positive student learning had occurred and also regarding students' collaboration and engagement when problem solving. They articulated their personal learning regarding coding processes, while acknowledging that their role in the classroom had evolved.

Conclusions

Although findings are presented as four separate aspects, they were mutually-influential elements that the teachers perceived had contributed to student engagement and learning. The work with *ScratchMaths* simultaneously influenced teacher practice, moving them towards a more faciliatory approach and greater understanding of coding processes. The students' mathematical thinking and learning in coding were tied to their solving of both mathematical and coding problems, while the explicit language of both contributed to the communication of processes, concepts and solutions. Students at times became leaders of the learning.

Much of the conceptual understanding and thinking related to the Geometry and Measurement strand of the NZ curriculum, in particular, angles and spatial perception. However, the process the participants undertook more directly facilitated mathematical thinking through the creative problem-solving process it evoked, and the development of logic and reasoning as they responded to the various forms of feedback.

While the findings were limited by the size of the project and the particular context in which they were enacted, they nevertheless give insights into the ways learning in both mathematics and coding might be enhanced through the *ScratchMaths* resources. The research is ongoing, with more schools and a broader range of classes and teachers now involved, and there is still analysis of the data to be completed, but further research into a broader range of contexts and some assessment and analysis of students' mathematical and computational thinking is anticipated and will give clearer, more comprehensive insights.

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Adapting curriculum materials in secondary school mathematics: A case study of a Singapore teacher's lesson design

Sze Looi Chin
National Institute of Education, Nanyang Technological University
<nie19.csl@e.ntu.edu.sg>

Ban Heng Choy
National Institute of Education, Nanyang Technological University
<banheng.choy@nie.edu.sg>

Yew Hoong Leong
National Institute of Education, Nanyang Technological University
<yewhoong.leong@nie.edu.sg>

When mathematics teachers plan lessons, they interact with curriculum materials in various ways. In this paper, we draw on Brown's (2009) *Design Capacity for Enactment* framework to explore the practice of *adapting* curriculum materials in the case of a Singapore secondary mathematics teacher. Problems from the textbook used and the worksheets she crafted were compared to determine how she adapted the content. Video-recordings of the lessons and post-lesson interviews were used to clarify how her personal *teacher resources* contributed to her design decisions. The findings suggest that her seemingly casual use of problems from the textbook are in fact unique variations of adapting curriculum materials.

Singapore's success in large international studies (e.g., TIMSS, PISA, etc.) has left many nations curious to learn about its pedagogical practices. However, a common assumption is that Singapore teachers predominantly employ a "drill and practice" approach and are reluctant to deviate from curriculum materials (e.g., printed textbooks, workbooks) to meet the specific needs of their students (Toh et al., 2019). Despite this, Leong et al. (2018) demonstrated that such was not the case for Singapore secondary mathematics teacher, Teck Kim, who created worksheets by modifying content from a textbook for "making things explicit" (p. 47). His modifications included: (i) filling in gaps in the content he felt were fundamental; (ii) linking different representations to deepen students' understanding; and (iii) highlighting ideas he deemed critical. In light of this, we argue that a key feature to Singapore teachers' practices, which may generally go unnoticed, is their transformative use of curriculum materials in planning instruction tailored for their students. In this paper, we explore another case of a Singapore secondary mathematics teacher, Mrs Fung (pseudonym), who demonstrated another way to adapt curriculum materials that was different from Teck Kim when she crafted trigonometry worksheets using a textbook for her lessons. To do so, we utilise Brown's (2009) *Design Capacity for Enactment* (DCE) framework to explore the adapting process and to answer the question: *How does Mrs Fung, an experienced and competent mathematics teacher in Singapore, adapt curriculum materials to design worksheets?*

Theoretical Underpinnings

Teachers' use of curriculum materials has been conceptualised in many ways. For instance, Shulman (1987) broadly described teachers' interactions with textbooks as *pedagogical reasoning and actions*, which involves comprehension, transformation, and instruction, informed by their knowledge and understanding of the text. For Sherin and Drake (2009), these interactions were referred to as *reading*, *evaluating*, and *adapting*, which drew on teachers' curriculum strategies. For Amador et al. (2017), these interactions

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involved a set of skills, known as *curriculum noticing*, in which teachers attend to the materials, interpret what they attended to, and decide how to respond (e.g., to include or omit a problem). Likewise, Brown and Edelson (2003) described this as a teachers' *pedagogical design capacity* (PDC), their ability to "perceive and mobilize resources in order to craft instructional contexts" (p. 13). First, teachers perceive and interpret curriculum resources, then they evaluate their potential to achieve instructional goals, and finally these evaluations inform their decisions for teaching. To demonstrate the factors involved when teachers interact with curriculum materials, Brown (2009) proposed the *Design Capacity for Enactment* (DCE) framework. The framework is composed of two types of resources: curriculum resources and teacher resources. *Curriculum resources* are physical objects and their representations (e.g., manipulatives), the representation of tasks (e.g., instructions for teachers, structure of lesson), and representations of concepts (e.g., models, descriptions of concepts). *Teacher resources* include the teacher's goals and beliefs, their subject matter knowledge, and their pedagogical content knowledge.

Brown (2009) characterised teachers' interactions by considering the varying degrees of responsibility shared between the curriculum and teacher resources. On one end of the scale, teachers can *offload* their responsibility as designers of the lesson and instead choose to rely primarily on the curriculum resources (e.g., teaching in direct alignment with the textbook). On the other end of the scale, teachers can *improvise* by predominantly relying on their own resources. According to Brown, improvisations are typically spontaneous and occur due to unexpected events, such as realising students held fundamental misconceptions about a related concept. As a result, a conscientious teacher may deviate from the textbook to address these misconceptions by generating their own content. Lastly, an intermediate of the two processes is when teachers *adapt* the curriculum materials. By sharing the responsibility to design between the curriculum and teacher resources, teachers can use content in a textbook as inspiration for instruction. For example, instead of directly using an example given in the textbook, the teacher could generate a similar example by changing the context and figures, thereby applying their own subject matter and pedagogical content knowledge to ensure the lesson goals are still achieved.

The DCE framework has also been used by Amador (2016) to describe teachers' approaches to lesson planning in relation to their consideration for students' thinking. Three *planning themes* emerged from the study: (i) *adapting* in response to students' understanding (e.g., editing exercises to highlight features that students had neglected in the previous lesson); (ii) *producing* competence in students' procedural fluency (e.g., frequently including in-class quizzes to demonstrate ability to solve); and (iii) *regulating* content to ensure students keep up with the curricular pace, regardless of students' progress (e.g., strictly following the school syllabus, maintaining the same lesson structure).

In the context of Singapore, the teaching practices and supposed curriculum are often perceived by those outside of Singapore as overwhelmingly aimed at producing and regulating. Thus, students would rarely have opportunities to engage in "genuine" problem solving experiences that would be more conducive to their knowledge growth, such as experiencing productive struggle (Schoenfeld, 2017; Henningsen & Stein, 1997). Instead of adapting or improvising materials to accommodate students' needs (e.g., to stretch their thinking), Singapore teachers are believed to be offloading responsibility to the curriculum resources which aligns with more traditional teacher-centred practices (Toh et al., 2019). In the context of the aforementioned teacher, Teck Kim, Leong et al. (2018) reported that he purposely adapted content from the textbook by changing the representations and improvised his own self-created tasks. This brought us to wonder, how does Mrs Fung, an experienced

and competent teacher similar to Teck Kim, negotiate curriculum resources and her teacher resources to inform her decisions in adapting curriculum materials? To what extent are her goals achieved through her decisions?

Methods

The data presented was taken from a larger project, which explored the distinctive instructional practices enacted by Singapore mathematics teachers. Mrs Fung had taught secondary mathematics for over 10 years and had been recognised by the local professional community as being an effective mathematics teacher. The class that Mrs Fung taught was a Year 9 class, which comprised students who scored between the 25th to 60th percentiles in the nationwide Primary School Leaving Examinations (PSLE) at Year 6. Mrs Fung was selected as the subject of the study after the first author, a non-native to Singapore, observed her unique implementation of personally authored worksheets to teach introductory trigonometry in place of the textbook, Discovering Mathematics 3B Normal Academic (Chow et al., 2015a). The trigonometry unit consisted of seven lessons between 30-60 min in duration. In this paper, we discuss Lesson 6 of the trigonometry unit.

Three sources of data are presented in this paper. The first are the physical materials that Mrs Fung used and created. This includes one worksheet (Worksheet 6.4) crafted by Mrs Fung, and the curriculum materials she drew on for the design of her worksheet – Section 6.4 from the textbook (Chow et al., 2015a) and the teachers' guide (Chow et al., 2015b). The second source of data is a video-recording of the post-lesson interview conducted with Mrs Fung after Lesson 6, which discussed her goals and the events of the lesson. Some prompts that were used in the interview were:

- What were your goals for the lesson?
- Do you think you have achieved your goals that you have set out to achieve? How were the goals achieved?
- What is the most ambitious or challenging thing you did in the lesson?

The third source of data is a video-recording of Lesson 6 when Mrs Fung implemented Worksheet 6.4, where a researcher took a non-participant observer approach.

Data analysis was conducted over three phases. In the first phase, the problems from Section 6.4 (Chow et al., 2015a) were categorised according to the mathematical processes required to solve them (e.g., insert an auxiliary line, two-step calculations). The model examples from the teachers' guide (Chow et al., 2015b) were also consulted to confirm these were the expected solving methods.

In the second phase, the categories found from Section 6.4 were applied to the questions in Worksheet 6.4 to determine if Mrs Fung had offloaded, adapted, or improvised from the textbook. This included two levels of comparison: item-to-item and set-to-set. On the item-to-item level, the categories were used to determine if Mrs Fung had offloaded, adapted or improvised the content in her worksheet. On the set-to-set level, the structure of the worksheet and its contents as a set were compared with the entire of Section 6.4 to determine similarities and differences in sequencing. The usefulness of this dual-level of analysis will be made clearer in the next section of this paper.

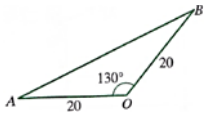
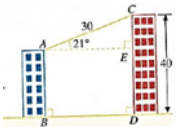
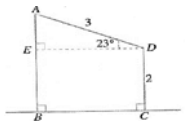
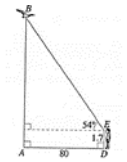
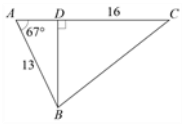
In the final stage of the analysis, the post-lesson interview and video-recording of Mrs Fung's enactment of the lesson were used to triangulate the decisions she made to offload, adapt, or improvise. We focus on her discussions about her lesson goals and beliefs which impacted her design decisions.

Findings and Discussion

Before implementing Worksheet 6.4, Mrs Fung played an introductory video for the students in the lesson to demonstrate how trigonometry could be used to solve contextual problems. Subsequently, she began implementing Worksheet 6.4 and asked the students to complete the first question by themselves. If time permitted, students would consult with their peers seated nearby, typically to check if their solutions were comparable. Mrs Fung neither encouraged nor discouraged students to share ideas with their peers but always requested that they initially attempt the problems by themselves. After the solution for the question was discussed by Mrs Fung, the class moved onto the next problem in a similar process.

Prior to Worksheet 6.4 within the Trigonometry unit, the students had encountered and solved problems using the Theorem of Pythagoras, learnt how to determine if a triangle was right-angled, and applied trigonometric ratios to triangles with acute angles to find unknown sides and angles. In the teachers' guide to the textbook (Chow et al., 2015b), the primary learning objective of Section 6.4 was to "apply the trigonometric ratios to solve some real-life problems" (p. 10). The analysis of Section 6.4 and comparison with the model solutions given in the teachers' guide resulted in three categorisations of problems: (A) *insert an auxiliary line to solve an angle/length;* (B) *two-step calculations to find an unknown length;* and (C) *two-step calculations to find an unknown angle* (see Table 1 for examples). Four worked examples (one of Type A and C, two of Type B) were first presented in Section 6.4, then a similar problem was subsequently provided for each of the corresponding worked examples for students to attempt. Afterwards, 19 exercise problems were given to be used by students for further practice.

Table 1
Summary of categories of problems from Section 6.4 in Chow et al. (2015a)

Type	Process	Order in TB	Order in WS	Examples	
A	Insert an auxiliary line to solve an angle/length	1	2	(A1) - See Figure 2 for full problem.	
B	Two-step calculation to find an unknown angle/length	2	3	(B1) - AB and CD are two buildings on level ground BD … Find the height of AB.	
		3	4	(B2) - The diagrams show the cross-section of a shed ABCD … The roof AD is 3m long … Find the height of the wall.	
		5	1	(B3) - B shows a bird flying above a point A on the horizontal ground, AD … Find the height of the bird above the ground.	
C	Two-step calculation to find an unknown angle	4	-	(C1) - In the diagram, ADC is a straight road. Town B is 13km away from A … Find the size of ∠BCD	

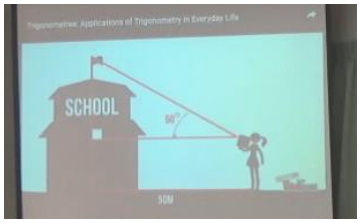
Note. The table does not include the complete list of problems from the textbook, only those relevant to the ones utilised by Mrs Fung. TB = Section 6.4 in textbook (Chow et al., 2015a), WS = Worksheet 6.4

In comparison, at the item-to-item level, Worksheet 6.4 consisted of four problems that were identical to four problems seen in Section 6.4. The representations of the problems were not altered by Mrs Fung in her worksheets, which would therefore suggest that she had *offloaded* her responsibility to design questions solely onto the curriculum resources and did not adapt or improvise in her Worksheet 6.4. In other words, Mrs Fung had copied the instructions and diagrams directly from Section 6.4 and did not include any modified or self-designed content.

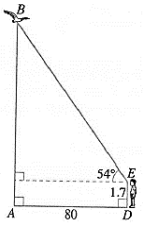
When comparing the two resources at a set-to-set level, it was evident that Mrs Fung had *adapted* from Section 6.4 to design Worksheet 6.4 by omitting and resequencing specific content. Firstly, Mrs Fung had only provided questions to students and did not provide any worked examples. Although the specific reasons for this omission were not explicitly discussed, Mrs Fung made several statements during the lesson and interview about how students had attempted similar problems before without the real-life context and that she wanted them to first attempt the problems individually. For the first question (B3), she provided students with "five minutes to try out on your own". Then she told the students, "instead of telling me, most of you are already quite good with your TOA CAH SOH. Try to read the content first, then they give you the diagram". As she roamed around the classroom, she prompted those students who appeared to have difficulty getting started to "just give [the problem] a go", to identify the appropriate sides and the relevant angles, and reiterated that she would like everyone to attempt the questions individually first before sharing or asking for help from neighbouring students. As she began to check their answers, she asked a student, "Shane, you saw [another student's] second part or you already know? You already know or after you seen his? You saw his, then you realized [what to do]?" She continued to prompt students individually who appeared to be stuck but never told them the solution. From these instances, it would suggest that one of Mrs Fung's goals was for students to learn to make sense of questions independently by drawing on their existing knowledge. By omitting the worked examples, students would be more likely to engage in the type of thinking that is typically expected in problem-solving activities (Henningsen & Stein, 1997) and experience some moments of struggle in solving these typical textbook problems.

Secondly, adaptations can be seen when comparing the sequence of problems. While Section 6.4 had presented questions A1-B1-B2-C1-B3 in this order along with worked examples preceding each question, Mrs Fung had chosen to present questions in the order of B3-A1-B1-B2 (Table 1). Aside from the absence of C1 in Worksheet 6.4, which was not addressed by Mrs Fung in the interview or the lesson, Mrs Fung had moved B3 to be the first question. As previously stated, Mrs Fung had begun the lesson with a short introductory video that provided scenarios for when trigonometry would be used in real life. In her post-lesson interview, she expressed that she wanted to show students the video to help them get a sense of "what is application of trigonometry about". As they had only ever encountered contextless problems, she was concerned that they would have language difficulties which would hinder their ability to understand and attempt the problems. In relation to her goal, Mrs Fung's awareness of her students' abilities and their previous understandings contributed to her decision of the first problem she chose. The example in the video and B3 involved similar representation of tasks and diagrams (Figure 1), and thus it would be productive to choose B3 as an introduction to solving applications in trigonometry if students were to initially try to solve the problem by themselves. Although B3 was *offloaded* from the textbook at the item-to-item level, when examining the differences at a set-to-set level, Mrs Fung's resequencing of questions demonstrated an *adaptation* of the textbook. This

adaptation was influenced by the representation for B3 and Mrs Fung's teacher resources, namely her pedagogical content knowledge and her goals to develop students' sense of how trigonometry is applied.



(a) Example from introductory video



(b) First question (B3) in Worksheet 6.4

Figure 1. Initial example and question given in Lesson 6

After most of the class successfully solved B3, Mrs Fung forewarned the students that the next problem, Al (Figure 2), would not be as "straightforward". In her post-lesson interview, she noted that her goal was for students to be able to solve problems involving two triangles but that she had anticipated that A1 would be the most challenging problem for her students - "majority of them don't know how to approach this question". As there were no worked examples of similar problems provided, nor had she included any problems that required adding an additional line to bisect the isosceles triangle in any of her other worksheets, it was unlikely that her students had encountered such a problem before and would know to draw the auxiliary line. Although she had intended to provide a hint for students, she wanted to "let them struggle a bit" first, suggesting that she held the belief that experiencing struggle was worthwhile and important for learning mathematics. In choosing to specifically include A1, Mrs Fung's decision was intended to provide an opportunity for students to grapple with the problem in search of a way to approach it, thereby deepening their skills and understanding of solving trigonometry problems with two triangles. Despite the appearance that her inclusion of A1 was merely an offload of Section 6.4, Mrs Fung's interview suggests this was a deliberate decision for both providing an opportunity for students to struggle and a resequencing with a consideration for students' learning progression.

13. In the diagram, OA and OB represent two positions of the minute hand of a clock, $OA = OB = 20$ cm and $\angle AOB = 130°$. Find
(a) the distance AB,
(b) the perpendicular distance from O to AB.

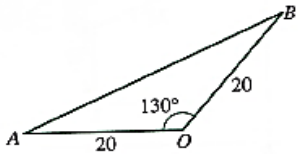


Figure 2. Question A1 in Worksheet 6.4, taken from (Chow et al., 2015a, p. 21)

Mrs Fung's goals and underlying beliefs which informed the decisions she made in offloading and adapting from the textbook can be described as an attempt to facilitate opportunities for *productive struggle* (Schoenfeld, 2017). As the worksheet became the main resource used in the lesson, to a large extent it replaced the textbook – a resource that would have an abundance of worked examples and hints that would have been useful for students. By omitting worked examples and asking her students to make sense of the problems

individually before providing guidance, Mrs Fung's adaptation through omission afforded students the opportunity to try several methods and to learn from those that did not work, rather than replicating a solution method from a worked example.

Secondly, while using an introductory video and B3 could ease students into solving application problems, Mrs Fung immediately followed B3 with A1 – a problem she was aware would cause some confusion. Mrs Fung discussed her concerns in the interview about ensuring students could eventually manage to solve the problem, but still insisted that students make an effort to think about how to approach the problem in the lesson. The selection and sequencing of A1 had the potential to cause students to become discouraged, especially those who had previously solved B1 easily and were now completely unaware of how to even approach A1. However, the nature of these adaptations also allowed her to act as a guide to coach students as she roamed around the room and supported students experiencing struggle. In comparison to the American teachers in Henningsen and Stein's (1997) study who avoided moments where students might experience struggle – despite knowing that they may be beneficial for learning, Mrs Fung actively tried to create these opportunities.

The use of the dual level of analysis prompted further investigation of Mrs Fung's use of the curriculum resources that was not accounted for by the DCE framework (Brown, 2009). Similar to Teck Kim from the study conducted by Leong et al. (2018), at first glance Mrs Fung's worksheet appeared to adhere with the previously mentioned assumption that Singapore mathematics teachers simply offload their responsibility to tailor content to meet students' needs, and instead select and use standard questions to develop procedural solving methods. At an item-to-item level, Mrs Fung's offloading of problems seemed to be consistent with this assumption. However, by examining Mrs Fung's worksheet on a set-to-set level, Mrs Fung's worksheet could be understood as a product of her interpretation of the curriculum resources and appropriation of the content with respect to her knowledge of her students' needs. She adapted from the textbook by omitting worked examples and re-sequenced problems, while also essentially replacing the need for the textbook during instruction. While Teck Kim adapted a textbook to create a worksheet to make concepts explicit to his students, Mrs Fung adapted the nature and sequencing of the textbook to facilitate students' exploration in solving. This study of Mrs Fung provides yet another step in the ongoing work of unpacking the complexities involved in Singapore teachers' design of instructional materials.

Concluding Remarks

The phenomena of teachers adapting curriculum materials is complex. At present, existing frameworks on teachers' curriculum use do not seem to fully capture what goes on through the materials teachers create and the invisible process of deciding how to adapt. Namely, while the DCE framework differentiates teachers' interactions with curriculum materials as offloads, adaptations, and improvisations, it does not address the potential for different grain-sizes of offloads, adaptations and improvisations. In this paper, Mrs Fung's interactions with the materials for designing her lessons were analysed on two levels which illuminated the different ways that she adapted from the textbook. From Mrs Fung's discussion of her lesson goals and observations of her enactment, her desire for students to grapple with problems and attempt to solve them independently were facilitated by these adaptations.

A limitation of the study is that the findings stem from secondary data gathered from a larger research project which focused on teachers' instruction, rather than their design of

instructional materials. Hence, the inferences which were made about her design decisions are restricted to the limited data available.

The case of Mrs Fung hopes to contribute to dispelling misconceptions about Singapore teachers' practices. In addition to Teck Kim, our findings suggest that when Singapore teachers interact with curriculum materials to design lessons, there's often more to the process than meets the eye. However, we also get the sense that we are just scratching the surface on what is an extremely complex phenomenon where several resources are all simultaneously involved. Furthermore, we propose that for Teck Kim and Mrs Fung, adaptations do not merely stop once the worksheets are created. Instead, they undergo an additional round of adaptations during instruction in response to students' reactions to the worksheets. Future research should aim to examine the implications of additional rounds of adaptations in comparison to a single round of adaptation.

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By teaching we learn: Comprehension and transformation in the teaching of long division

Ban Heng Choy
National Institute of Education, Nanyang Technological University
<banheng.choy@nie.edu.sg>

Joseph Boon Wooi Yeo
National Institute of Education, Nanyang Technological University
< josephbw.yeo@nie.edu.sg>

Jaguthsing Dindyal
National Institute of Education, Nanyang Technological University
<jaguthsing.dindyal@nie.edu.sg>

Despite recent calls to adopt practice-embedded approaches to teacher professional learning, how teachers learn from their practice is not clear. What really matters is not the type of professional learning activities, but how teachers engage with them. In this paper, we position learning from teaching as a dialogic process involving teachers' pedagogical reasoning and actions. In particular, we present a case of an experienced teacher, Mr. Robert, who was part of a primary school's mathematics professional learning team (PLT) to describe how he learned to teach differently, and how he taught differently to learn for a series of lessons on division. The findings reiterate the complexity of teacher learning and suggest possible implications for mathematics teacher professional development.

There have been recent calls to incorporate collaborative inquiry-based approaches embedded in teachers' practices to improve the teaching of mathematics. This has led to the adoption of collaborative professional learning activities such as video clubs (van Es & Sherin, 2002), Lesson Studies (Clea Fernandez & Yoshida, 2004), and collaborative lesson research (Takahashi & McDougal, 2016). However, it would be "wishful thinking" to expect that teachers would learn just because they gather "to talk about practice" (Bryk, 2009, p. 599). In Singapore, while there is extensive support for teachers to engage in learning communities for the purpose of working collaboratively to learn and improve their teaching, it is unclear whether and how teachers learn from these activities (Hairon & Dimmock, 2012). What really matters, therefore, is not the kind of professional development activities, but rather how teachers engage with these activities (Choy & Dindyal, 2019; Fernandez, et al., 2003). As claimed by Sherin (2002), learning from teaching occurs when teachers have opportunities to *negotiate* among three aspects of their teacher knowledge: understanding of mathematics, curriculum materials, and knowledge of how students learn. In this paper, we refer to Sherin's (2002) metaphor of teaching as learning to examine how a primary mathematics teacher, Mr. Robert, learned from his teaching through a dialogic process involving pedagogical reasoning and action (Shulman, 1987) as he worked with his colleagues on a series of lessons to teach division for Primary Three pupils (aged 9). The paper is framed by the following question: How does a primary mathematics teacher learn from his own teaching via his participation in a professional learning team?

Theoretical Considerations

Following Shulman (1987), we see that teaching "begins with an act of reason" and "continues with a process of reasoning" to culminate in a series of pedagogical actions, and "is then thought about some more until the process can begin again" (p. 13). In other words, with the aim of improving teaching, teachers need to learn to use their knowledge base for teaching to provide justifications for their instructional decisions through a process of

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 155-162. Singapore: MERGA.

pedagogical reasoning. This process involves taking what one understands about content and "making it ready for effective instruction" (Shulman, 1987, p. 14), through a cycle of activities involving comprehension, transformation, instruction, evaluation, and reflection leading to new comprehension. According to Shulman (1987), comprehension refers to how teaching first involves understanding the content and purpose. When possible, teachers should *comprehend* what they teach in different ways and relate these ideas to other ideas within and beyond the subject. The key distinctive of a teacher's work lies in how a teacher *transforms* his or her content knowledge into "forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (Shulman, 1987, p. 15). Transforming this knowledge involves preparation, representation, instructional selections, adaptations of these representations and tailoring the representations to specific students' profiles. Although comprehension and transformation can occur at any time during teaching, Shulman (1987, p. 18) sees these two processes as "prospective", occurring before *instruction*, an "enactive" performance in the classrooms. Moving on to a more retrospective process, Shulman highlights *evaluation* as the means to assess students' understanding and to provide feedback. But it is through *reflection*, by which a teacher looks back at the instructional processes and experiences, that a teacher learns from his or her experiences. This learning is encapsulated in the process of *new comprehension* where teachers have a better understanding of teaching and learning.

Shulman highlighted that *new comprehension does not necessarily follow through from reflection*. This explains that some teachers learn from their teaching experiences, while others do not. Hence, we argue that new comprehension of content, student learning, and teaching actions occurs when a teacher has a shift of attention, gaining awareness of new possibilities in teaching and learning (Mason, 2002), or simply when a teacher *notice* critical aspects of teaching and learning. These new insights expand the teacher's current cluster of resources, orientations, and goals (Schoenfeld, 2011), which in turn becomes the base from which the teacher make sense of instruction. Moreover, as Choy (2016) has highlighted, productive noticing can take place during planning, instruction, and reviewing of lessons. Consequently, new comprehension can occur during any of the activities of Shulman's model of pedagogical reasoning and action.

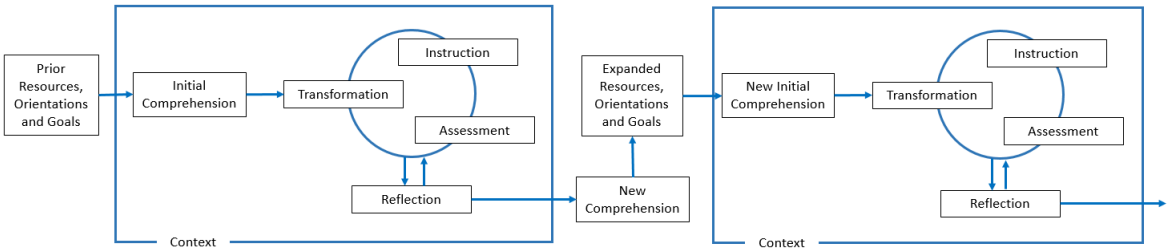

Figure 1. Adapted Model of Pedagogical Reasoning and Action.

Building on ideas from both Shulman (1987) and Schoenfeld (2011), we developed an adapted model of pedagogical reasoning and action to highlight the dialogic processes involved when learning from teaching. The strength of Schoenfeld's ideas lie in the fact that teaching is goal-directed, rests on a set of resources, and driven by a teacher's orientations. The orientations aspect is quite important as it explains why some teachers loop back to do happily what they have been used to doing and in doing so, submit to the exigencies of the context. Thus, in the model above, we show that teaching starts with some prior resources,

orientations and goals (ROG) and some initial comprehension. The teacher then transforms the initial comprehended ideas into a form suitable for teaching the students. The iterative and cyclical processes of transformation, actual instruction and assessment of learning feed forward to the reflection of the teacher (to different extents for different teachers). This process leads to some new comprehension, which may or may not lead to a new expanded set of ROGs and the cycle repeats. What this adapted model affords us is the opportunity to capture the complexity of the dialogic processes involved when teachers learn from their practice. On one hand, teachers comprehend new ideas about content and teaching to apply them in their instruction. On the other hand, they learn new ideas as they apply their new comprehension in their instruction. We shall now illustrate the dialogic nature of a teacher's learning from teaching through the example of Mr. Robert, who learned and applied new ideas about division as part a professional learning team.

Methods

The data presented in this paper were collected as part of a larger project which aims to develop the proof of concept for a new professional learning model for mathematics teachers. Drawing on current theoretical perspectives of teacher noticing (Dindyal, et al., 2021; Fernandez & Choy, 2019), we conceptualized professional learning sessions where teachers would have opportunities, in the context of a *community of inquiry* (Jaworski, 2006), to work and co-learn with us by:

1. Focusing on unpacking the mathematics in the curriculum documents;
2. Investigating how a topic may be unpacked in terms of a sequence of lessons, and a lesson as a sequence of tasks;
3. Teaching a sequence of lessons as part of a unit;
4. Observing and reflecting upon a sequence of lessons;
5. Articulating their learning from the observations; and
6. Suggesting possible changes to the sequence of lessons and tasks based on their learning.

As highlighted by Jaworski (2006), sustainability is often an issue with communities of practice and learning. To ensure sustainability and feasibility, we co-designed protocols to guide each professional learning session as teachers worked together to plan and teach a unit of work. As each session lasted about an hour and so, it was crucial that we built in specific focus for each session to facilitate more productive discussions. We also provided teachers access to relevant research and practice-based articles when requested, as well as templates to facilitate teachers' inquiry processes. Data collected include voice and video recordings of the discussion during the sessions, photographs of lesson artifacts such as lesson plans, discussion notes, and when available, samples of students' work.

In this paper, we report how Mr. Robert, an experienced primary mathematics teacher from Eunoia Primary School (pseudonyms), perceived and harnessed affordances as he worked with a team of nine other teachers to discuss the teaching of long division to Primary Three pupils (aged 9). The sessions were facilitated by a Lead Teacher, Ms. Mandy, who had extensive experience teaching in the primary school. We were present at the sessions as knowledgeable others to share new ideas for teaching. We did not insist that the teachers adopt any particular idea that we had shared. Instead, we left all the instructional decisions to them because we wanted to investigate their decision-making processes. The vignettes described here were developed from data collected from four discussion sessions and a video

recording of a short 20-minute segment of Mr. Robert's teaching. The voice recordings of the discussion sessions were parsed for segments related to discussions on the teaching and learning of long division. Notable episodes involving mathematically significant moments were marked for further analysis. Irrelevant incidents such as logistics and administrative matters were discarded. The marked segments were reviewed, and initially coded for processes related to our adapted model of pedagogical reasoning and action (See Figure 1). The reviewed segments were then transcribed before they were coded using a "thematic approach" (Bryman, 2012, p. 578) to highlight aspects of how Mr. Robert learned from his practice. We acknowledge that it is difficult to distinguish Mr. Robert's learning from the learning achieved by other teachers. Here, we assume that Mr. Robert, as an individual, can learn from his own teaching experiences, the ideas and experiences shared by his colleagues, as well as ideas we, as the research team, had shared with him. This corresponds to what Mason (2002) terms as the three worlds of experiences.

By Teaching We Learn: A Dialogic Process

Findings developed from our data suggest a dialogic process by which Mr. Robert had learned from his practice. First, we claim that he learned some new ideas about teaching division during the PLT discussions that offer opportunities to teach differently. Second, we propose that he taught differently by trying out some of the ideas learned, which in turn give rise to new comprehension. We will now describe vignettes of teachers' learning, focusing on Mr. Robert to highlight the dialogic process of learning from teaching.

Learning to Teach Differently

For the first two sessions, we worked with the teachers to unpack mathematical ideas related to division using the components of school mathematics as proposed by Backhouse et al. (1992), namely concepts, conventions, results, techniques, and processes. All the teachers were cognisant of the quotative and partitive notions of division and were fluent in performing the long division algorithm. They were also familiar with the key terms such as *quotient*, *remainder*, and *divisor* but not the term *dividend*. More specifically, they seemed to see quotient and remainder as part the answer to a division problem. For example, they would write $82 \div 4 = 20R2$, seeing 20 as the quotient and 2 as the remainder being the answer to $82 \div 4$. They did not think of other expressions that give the "same answer" as problematic. For instance, when we highlighted that $62 \div 3 = 20R2$, the teachers did not notice any issues with the notation. The usual way of writing the answer as "20R2" suggests that $82 \div 4$ is equal to $62 \div 3$. It appeared that the teachers did not notice this until we pointed out the issue to them. To highlight that the relationship between dividend, quotient, divisor, and remainder, we introduced the following "new equation":

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

For the teachers, this was something new and so we highlighted the relationship between division and multiplication, e.g., $20 \div 4 = 5$ is related to $20 = 4 \times 5$. More importantly, the equation involving dividend, quotient, divisor, and remainder was linked to how division can be demonstrated through manipulative, "splitting" the number into two or more components, and the long division algorithm. As an example, we showed how $82 \div 4$ can be visualised as distributing 80 items into 4 equal groups, with 20 items in each group; or seen as $80 + 2$, which can be rewritten as $4 \times 20 + 2$; and the long division which gives the quotient

20 and a remainder of 2 when 82 is divided by 4 (See Figure 2). The sharing of these new ideas provided opportunities for teachers to engage in comprehending the content and transforming their new-found knowledge to usable forms.

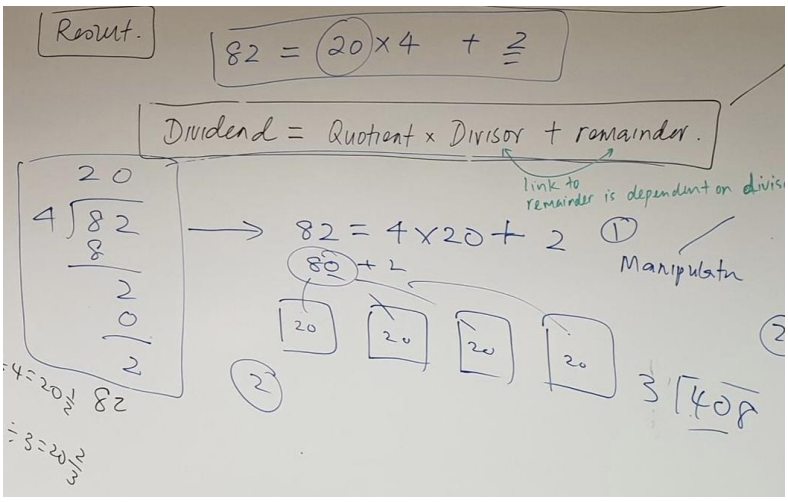

Figure 2. Snapshot of our sharing as documented on the whiteboard.

Teaching Differently to Learn

This "new" equation which highlighted the relationships between dividend, quotient, divisor, and remainder was taken up by Mr. Robert who tried to use this idea for his own teaching (Turn 15):

15.	Mr. Robert	I tried in my class, in fact I introduce in my class last week the quotient … like something like 9 = 4 + remainder something, you know the remainder thing? For the equation thing we did last week.
16.	Ms. Mandy	Dividend = Quotient × divisor + remainder.
17.	Teachers	[inaudible] remainder theorem.
18.	Mr. Robert	We did that last week. We could get the simple ones. But how you translate this to the long division working, it's still a disconnect.
19.	Researcher	Yea. So, they could get this, they can understand this kind of thing …
20.	Teachers	Small numbers [inaudible]
21.	Mr. Robert	2 digits they can get, 3 digits they are gone.
22.	Researcher	Ok, so they could get 2 digits but not 3 digits.
23.	Mr. Robert	Maybe at the start we just started with 2-digit number. In fact, once it goes beyond 20, they are a bit lost already.

Mr Robert's use of the "new equation" highlights how new ideas shared or discussed during PLTs can open up new opportunities to teach differently. As Mr. Robert *comprehended* these ideas for himself and *transformed* them into a sequence of examples involving 9, some 2-digit numbers, and even 3-digit numbers for his *instruction* (Turns 15, 21, and 23), he also began to be more aware of his students' thinking (Turns 18 and 21). He was able to *assess* that his students may be confused when the numbers went beyond 20. However, it was his *reflection* about the possible disconnect between this "new equation"

and the long division algorithm that opened up new threads of discussion and possibly opportunities to acquire new comprehension during the PLT.

Cycles of Learning to Teach Differently and Teaching Differently to Learn

Here, we begin to see how Mr. Robert's pedagogical reasoning and action had afforded opportunities for him to learn to teach differently. In the discussion that followed, we explored with teachers how students could make sense of division problems using different methods. For example, for 48 ÷ 3, students can do repeated addition: 3 + 3 + 3 + … = 48; or they can do repeated subtraction: 48 – 3 – 3 – 3 – … = 0. Students can also do skip counting: 3, 6, 9, …, 48; or reverse skip counting: 48, 45, 42, …, 0, amongst others. We also introduced the different chunking strategies (Putten et al., 2005), or what others refer to as partial quotients (Takker & Subramaniam, 2018), before we linked these informal strategies to the long division algorithm. For example, for 78 divided by 3, students may think of 3 × 10 = 30 and they will subtract 30 from 78 to give 48. Then they may subtract another 30 from 48 to give 18, and 18 divided by 3 is 6. Therefore, the answer is 10 + 10 + 6 = 26. This can be presented in this manner:

$$\left.\begin{array}{r} 6 \\ 10 \\ 10 \end{array}\right\} \quad 10 + 10 + 6 = 36$$

$$\begin{array}{r} \\ \\ \\ 3\,\overline{)\,78} \\ \underline{-\,30} \\ 48 \\ \underline{-\,30} \\ 18 \\ \underline{-\,18} \\ 0 \end{array}$$

Mr. Robert then explored and used these ideas in his own teaching. As seen from the snapshots taken from the video snippet of his lesson (see Figure 3), we see how he had tailored some of the ideas for his students. Although Mr. Robert decided not to write the "new equation" explicitly, he used the ideas to go through some of the informal division strategies with his students. Mr. Robert's decision to use the "7R1" notation could be in part due to how all the approved textbooks present the answers.

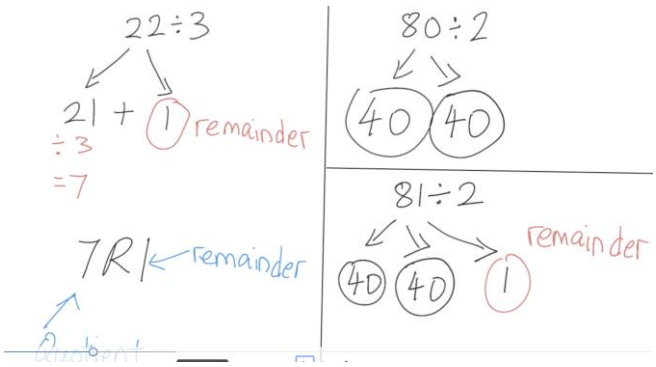

Figure 3. Snapshot of Mr. Robert's lesson to demonstrate informal strategies.

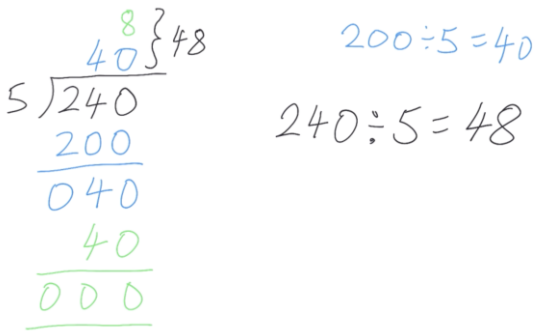

Figure 4. Snapshot of Mr. Robert's lesson to demonstrate the chunking strategy.

In another snapshot (see Figure 4), we see Mr. Robert demonstrating the chunking strategy (Putten et al., 2005) for his students. As seen from Figure 4, he used different colours to denote the different place values to make it clearer for his students. This use of colours was inspired by one of his colleagues in the same PLT who shared how the use of colours helped his students to grasp the importance of place value to understand long division. Here, Mr. Robert demonstrated the importance of learning new ideas from his colleagues and trying these ideas to see if they work. As we examine Mr. Robert's teaching and learning, we begin to gain insights into how he had learned from unpacking the mathematics, his colleagues, and knowledgeable others to be aware of different possibilities for teaching. But we also see how he had actually tried to teach differently in order to learn from his own teaching by assessing his students' understanding and reflecting upon the lesson.

Discussion

It was clear to us that the teachers in the PLT, including Mr. Robert, struggled with these ideas initially. However, it was also clear to us that teachers began to scrutinise these new mathematical ideas about division and explored the possibility of incorporating these ideas for their teaching. In other words, we argue that professional discussions involving experiences from different people, which focused on making connections between mathematics and pedagogy, have the potential for teachers to learn to teach differently. Nevertheless, for teachers' practices to change, it is necessary for them to try out these new ideas, as Mr. Robert had done, and reflect on their teaching to gain new insights. That is, for teachers to learn from their practice, it is necessary for them to learn about new ideas to teach differently and teach differently to learn these new ideas.

What Shulman (1987) implied in his model of pedagogical reasoning and action is that teachers can learn from their own teaching, or the idea of *docendo discimus*—by teaching, we learn. This idea aligns with the current notions of professional learning, which involve some form of job-embedded teaching inquiry activities, such as Lesson Study. However, implementing such teaching inquiry activities may be challenging due to time and resource constraints. There is a place and time for more elaborate teaching inquiry as part of a teacher's professional learning. But, what about the possibility of a teacher learning from his or her own teaching on a *day-to-day* basis? If we were to examine the processes of pedagogical reasoning and action, it became apparent that the model revolves around a teacher's day-to-day teaching activities. Hence, we propose two fundamental shifts in our thinking about professional learning. First, we see *every* teaching moment as an opportunity for professional learning. Second, we see pedagogical reasoning as the primary mechanism

to effect changes in pedagogical actions, and eventually changes in one's system of resources, orientations, and goals. As exemplified by Mr. Robert, every moment in teaching can provide affordances for teachers to learn from their own practice.

Acknowledgement

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Using interviews with non-examples to assess reasoning in F-2 classrooms

Kate Copping
The University of Melbourne
<kcopping@unimelb.edu.au>

The development of mathematical reasoning is a key proficiency for mathematics within the Australian Curriculum. However, reasoning can be difficult for teachers to assess, particularly with pen and paper tests. In this study, interview tasks were designed across three curriculum areas at three different levels to assess student reasoning through the use of examples and non-examples. Non-examples can be used to assist in building boundaries and deepening conceptual understanding. Through the interview, teacher and student dialogue can help students to demonstrate reasoning and clarify concepts through explanation and justification.

This paper examines the use of task-based clinical interviews to assess reasoning in the early years of school. The development of mathematical reasoning is considered a key proficiency within the *Mathematics Learning Area* of the *Australian Curriculum* and is described as a facility for "logical thought and actions" with "increasing sophistication" (Australian Curriculum Assessment and Reporting Authority, [ACARA] 2018a). This may be demonstrated, in part, through a student's ability to compare and contrast ideas, explain their thinking and justify conclusions made. In partnership with and addressed through the learning area foci of the *Australian Curriculum* are the *General Capabilities*, including *Critical and Creative Thinking*. Within this capability, students develop capacity to "generate and evaluate knowledge" and "clarify concepts and ideas", through "thinking broadly and deeply" and using reason and logic (ACARA, 2018b). These definitions are aligned to Kilpatrick's (2001) description of adaptive reasoning, where students think logically about conceptual relationships, reflect on their learning and justify their work. As an essential part of the curriculum, responsibility for assessing reasoning and critical thinking lies with the teacher.

Assessing students' capacity to demonstrate reasoning in mathematics can be challenging for teachers (Herbert et al., 2015). Formal, written pen-and-paper tests can be difficult for F-2 students (Foundation, the first year of school - Year 2) to complete. It has been established that this form of assessment may not accurately reflect students' conceptual understanding (Clements & Ellerton, 1995) and presents challenges to students at this level due to the reading and writing skills required, in light of the students' own developing literacy skills (Clarke, et al., 2006). One-to-one task-based interviews which are grounded in research are more effective at revealing students' conceptual understanding as well as their thinking and reasoning. For the purposes of eliciting and demonstrating mathematical thinking, interviews are well suited to early-years students (Cheeseman & Clarke, 2007). It is through the dialogue that happens between the teacher and the student that the student's reasoning becomes evident.

Task based interviews using non-examples, such as the 'triangles task' in the Early Numeracy Research Project, allow students to reason through justification (Horne, 2003). Similarly, Clements' (1998) discussion of interview tasks using examples and non-examples of 2D shapes, demonstrated that they allow students, through comparing and contrasting, to focus on the essential attributes of the shapes and promote critical thinking. Examples in mathematics generally fall into two categories: examples of a concept; or examples of the

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application of a procedure. Within these categories, examples can take the form of 'generic example', 'counter-example' or 'non-example'. Non-examples can help to clarify understanding by sharpening distinctions and deepening understanding of mathematical ideas (Bills et al., 2006). They provide an opportunity to reveal student thinking, and for students to apply reasoning and formulate justifications for why an example is correct or incorrect (Cavey & Kinzel, 2015). Teachers using non examples can assess students' conceptual understanding and reasoning using interview tasks designed to reveal misconceptions.

Methodology

Task-based clinical interviews were used to assess the reasoning of three students, at three different curriculum levels and in three different content areas. Task-based interviews were chosen for their utility as they are a valued tool for revealing student thinking, particularly for students in the early years of school (Clarke et al., 2006). Students are able to use discussion as a means of revealing understanding and therefore reading levels are not an issue (Bobis et al., 2005). Task-based interviews have developed from a background of Piagetian and Vygotskian theory, understanding that learning occurs in a social context. The interview process is centred around the dialogue which takes place between the child and the researcher, and the role of language is central to this. The researcher asks probing questions and the child clarifies meaning through explanation (Hunting, 1997).

Tasks were designed in consideration of research, including the development of conceptual understanding and common misconceptions, with one task for each level, at Foundation (number recognition, matching quantities and numerals to 'seven'), Level 1 (Counting on and counting back for early addition), and Level 2 (fractions, identifying 'quarters', demonstrating understanding of equal parts in a continuous model and fractions in a discrete model). Tasks were created with examples and non-examples for each content area, to expose conflicts in understanding which can arise through misconceptions (Zazkis & Chernoff, 2008). With non-examples, students can dismiss concepts that do not fit with their conceptual understanding however the dialogue within an interview can challenge this notion. Non-examples were intentionally included because they can be used to clarify boundaries for a concept, or where a procedure may not be applied, or fails to get a correct answer (Bills et al., 2006).

"Kye", aged five, "Cara", aged seven, and "Oliver", aged eight, (pseudonyms) attended an urban government school, where the need for assessing reasoning had been identified as an area for improvement within the school. The students were interviewed on site in a meeting room. Tasks were conducted with each student individually, and instructions, or questions were read to the students by the researcher. The students were then asked to explain their answers and why they had chosen (or not chosen) each answer. Each interview took approximately 10-15 minutes. The researcher recorded each answer and students' use of reasoning and justification were analysed from their responses

Tasks

Task 1

Task 1 (Figure 1) is a Foundation level task about number recognition. The Australian Curriculum lists the content descriptor for this as: "Connect number names, numerals and quantities, including zero, initially up to 10 and then beyond (ACMNA002)" (ACARA,

2018a). Key concepts for this task include Gelman and Gallistel's Counting Principles (1978) which state that meaningful counting relies on children knowing how to count and what to count. How to count includes: the one-to-one principle, where each item is counted only once, and assigned to a number as it is counted; the stable-order principle, where the number names are always used in the same fixed order; and the cardinal principle, where the last number counted or named is the total of the collection. What to count, relies on understanding the abstraction principle where anything can be counted including where the items in a collection are different, and the order-irrelevance principle where objects can be counted in any order.

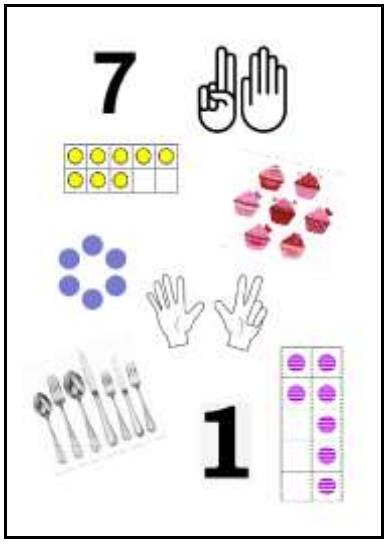

Figure 1. Foundation task

This task required the student to circle all the representations that showed 'seven'. Images chosen to represent familiar objects for Foundation students include: tens frames, counters, fingers, and common objects, as well as numerals. The types of images were chosen to reflect the counting principles, which are necessary for conceptual understanding. All items assess cardinality and the stable order principle. In addition, the cutlery assesses the abstraction principle, and the cupcakes and counters in a circle assess one-to one correspondence and order-irrelevance. The tens frames images assess order-irrelevance and could demonstrate knowledge of combining and partitioning (Clarke et al., 2006). Non-examples include the numeral '1', with extra 'tails' which could be mistaken by small children as the numeral '7'. The counters arranged in a circle represent 'six' but could be counted incorrectly by a student who is not able to create a start and end point for their counting. One set of tens frames and one set of hands are non-examples, displaying 'eight'.

Task 2

Task 2 (seen in Figure 3) is a Level 1 task about early addition and subtraction strategies of counting on and counting back. The Australian Curriculum (ACARA, 2018a) lists the content descriptor for this as: "Represent and solve simple addition and subtraction problems using a range of strategies including counting on, partitioning and rearranging parts (ACMNA015)". Research used to construct the task focused on counting stages (Steffe et al., 1983), particularly those at the initial number sequence stage or counting in verbal unit

items. Students at this level are able to hold a number in their head and have a conceptual understanding of the quantity that the number represents. Students are then able to count on a given amount of numbers to find a total. (See for example the top left column of Figure 3). This is a complex cognitive task requiring that the child understands the relationship between the symbolic representation of the task, as well as its relationship to process, numeration and quantity (Boulton-Lewis & Tait, 1994).

The first question demonstrates both a correct method, (top left column of Figure 3), and a common misconception for students who learn counting on as a process, (top right column of Figure 3). These students count on, but include the last number stated, lacking the conceptual understanding of the requirements of the task. Question two addresses counting back, which is often more challenging for children than counting forward (Steffe et al., 1988). A number line is provided for support, with the non-example showing a common misconception where the child counts marks on the number line, (top number line in Figure 3), and a correct example where a child draws 'jumps' on a number line, demonstrating counting back, (bottom number line in Figure 3).

Task 3

Task 3 (as seen in Figure 4) is a Level 2 task about fraction representations of quarters. The Australian Curriculum lists the content descriptor for this as: "Recognise and interpret common uses of halves, quarters and eighths of shapes and collections (ACMNA033)" (ACARA, 2018a). Key concepts for this task include the relationship between the numerical representation of a fraction and models to represent this. Due to the frequent use of 'pie' representations in the teaching of fractions, students can misunderstand the representation of a fraction in terms of a whole, particularly in a discrete model (Gould, 2005).

Representations of examples in the task include continuous and discrete models, equal parts, different shaped wholes, and an equivalent fraction. Common misconceptions for students include the understanding of equal parts in diagrams, and the relationship between wholes and parts of wholes, particularly in discrete items (Gould, 2005). Non-examples in this task include non-equal parts, images that represent one fifth in discrete and continuous models, and a whole that has been divided into quarters.

Results and Discussion

Task 1

Foundation student "Kye" completed the number identification task (Figure 2) and was immediately able to identify the numeral '7' as correct and the numeral '1' as incorrect, stating, "It's not seven, because it's a one". He then counted the seven fingers correctly, demonstrating one-to-one matching (Gelman & Gallistel, 1978) as he counted each finger. Kye counted the cupcakes as seven, again counting them with one-to-one deliberate matching, touching each cupcake as he counted. When drawing around the cupcakes, he recounted, drawing a line past each cupcake as he counted, resulting in an unusual 'circling' of the items. He then counted the bottom right ten frame (Figure 2) once and circled it. Kye once again relied on one-to-one matching, and did not demonstrate more complex understanding of number, which could perhaps have demonstrated part-part-whole number knowledge (Clarke et al., 2006), such as '5 and 2', or 'three empty spaces'. he then counted the second set of fingers as eight fingers and said he wasn't going to circle it, because it was eight fingers.

Kye had great difficulty counting the cutlery. He began counting and stopped halfway through and went back to the start twice. On the third attempt he said he was going to count them "carefully". He proceeded to count each item very slowly, but again stopped. He then said, "I'm going to count them at the bottom, and use my pencil". Kye counted the handle of each item, placing a pencil dot on the end of each cutlery item to count seven items. He then repeated the process before circling the items. The cutlery, demonstrating the abstraction principle (Gelman & Gallistel, 1978), were an obstacle that prevented Kye counting the items. His strategy was to count the items at the bottom, where the items were all the same. Kye also had difficulty counting the six dots in a circle and did not have a clearly identified beginning and end point for his counting. Kye counted seven dots, recounting his initial dot at the end, and immediately and confidently circled the group.

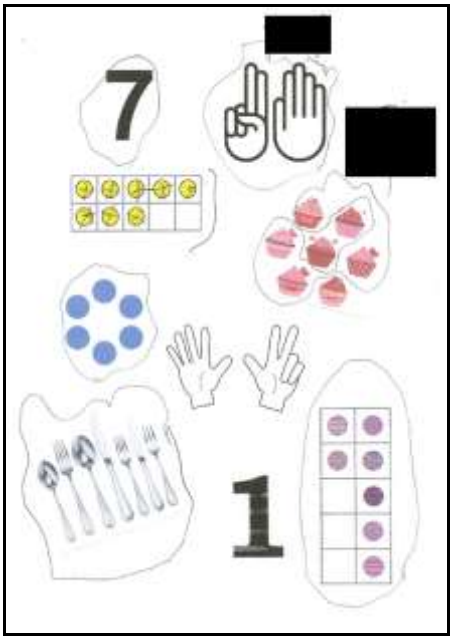

Figure 2. Kye's response to Task 1

Kye counted the final ten frame as eight and then started to circle the ten frame. He was asked, "How many did you say there were?" and responded, "Eight". He was then asked which ones he was circling, and he said, "the sevens" After discussion he decided he would recount the items. He recounted the dots, placing a cross on each to count eight and said he wouldn't circle them because there were eight and not seven. Kye was able to demonstrate one-to-one counting and some of the counting principles. His reasoning demonstrated an ability to justify why he believed something was correct. His critical thinking skills were used in his ability to adapt his counting skills with the cutlery counting to enable him to effectively count the items.

Task 2

Year 1 student "Cara" completed the addition and subtraction task (Figure 3). Cara was able to correctly answer both questions in the task, but interestingly only able to demonstrate reasoning in one part of the interview. In the addition question, Cara wrote her answer clearly stating that the incorrect answer was wrong "...you don't count the number your (sic) on."

When questioned, Cara said, “you already have 7, you don’t need to count it again, you have to count on the next number”. Cara has demonstrated a correct understanding of the procedure for counting on (Boulton-Lewis & Tait, 1994). She has also demonstrated an ability to think logically about the relationship between the concept of addition and the example and non-example provided (Kilpatrick, 2001). Cara has justified why one answer was correct, and why another was incorrect.

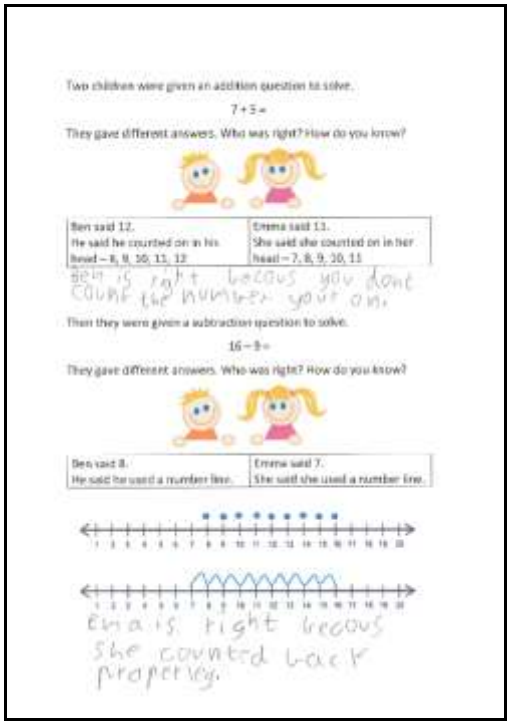

Figure 3. Cara’s response to Task 2

In the subtraction question, although Cara was able to answer the problem correctly, she was unable to demonstrate reasoning. When questioned on what she meant by “counted back properley (sic)”, she said, “That’s the way you’re supposed to do it.” On further prompting, she continued to talk about the “right way”. This was a procedural approach and her response demonstrated that Cara had a ‘rule’ for using a number line; however, she did not have a conceptual understanding of why this method was successful. Her inability to justify her response, or why the other answer was incorrect, revealed that although she could identify the correct solution, she could not articulate her mathematical reasoning.

Task 3

Year 2 student “Oliver” completed the fraction task (Figure 4). Oliver was able to demonstrate understanding of quarters in both a discrete and continuous fraction model (Gould, 2005). The task does not show discrete fractions with items of different sizes, which should be added to the task for future interviews. Oliver was able to articulate the reasons he provided to justify what was and what was not a representation of a quarter, including the need for equal sized parts in a continuous fraction model. He was able to clarify from the non-examples of fifths, what a quarter was: “This has five bits, but a quarter is one out of

four", and "There's five people, not four, so it can't be a quarter. It's a fifth." The non-examples sharpened his interpretation of quarters (Cavey & Kinzel, 2015).

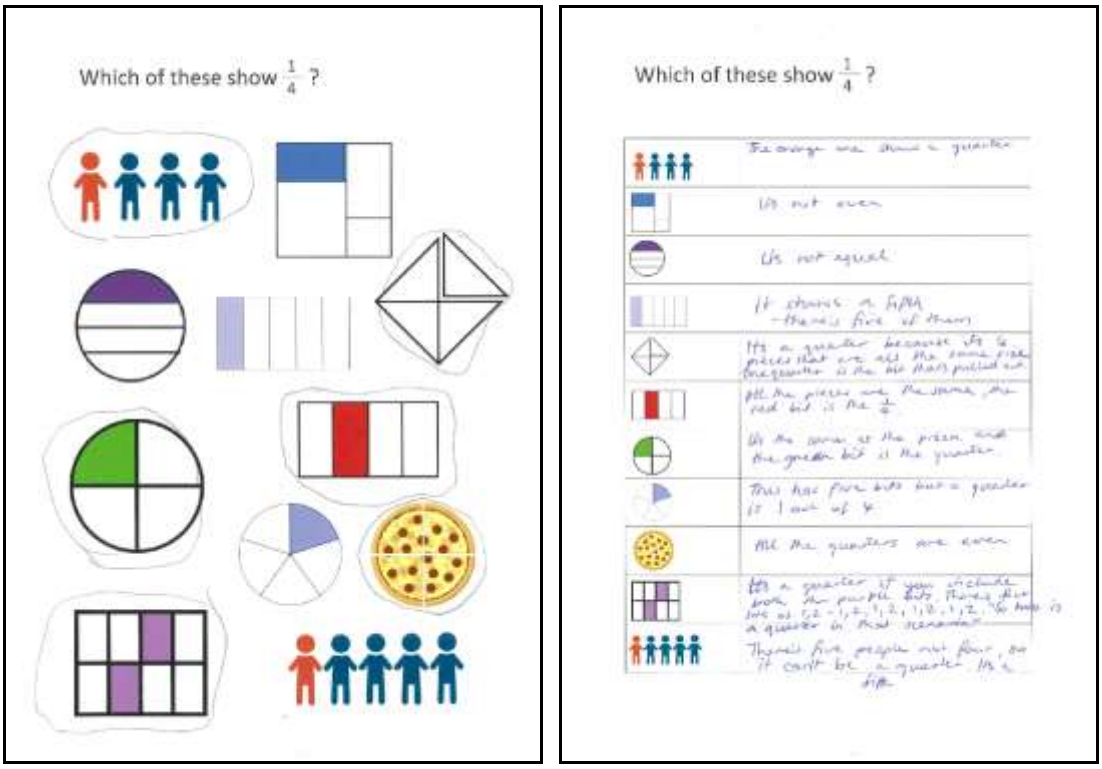

Figure 4. Oliver's response to Task 3 and interview notes

Logical reasoning is evident in Oliver's identification of two-eighths as a quarter. "It's a quarter if you include both the purple bits, there's four lots of 1, 2 – 1, 2, 1, 2, 1, 2, 1, 2. So, two is a quarter in that scenario". His justification and explanation of his ideas is a demonstration of clear reasoning and his current conceptual understanding (Kilpatrick, 2001). Interestingly, Oliver stated that the pizza showed a quarter as all the quarters were even. The interviewer said, "When I look at the pizza, I see four quarters, because they're all the same." Oliver responded, "You know, I think you're right, they are all the same." Initially the non-example had been dismissed by the student; however, the interview exposed this conflict in understanding, and enabled Oliver to more clearly clarify his understanding and create new boundaries for the concept of a quarter (Bills et al., 2006).

Conclusion

Although this was a small-scale study, only assessing one child within each identified concept, some conclusions can be drawn. Using examples and non-examples in a task-based interview situation allowed a teacher researcher to clarify conceptual understanding of three students, within specific topics of the mathematics curriculum area. The tasks required students to identify correct and incorrect examples of concepts and to justify their responses, in order to demonstrate mathematical reasoning. A task-based interview assessment allows for dialogue between the teacher and the student, to clarify the student's thinking, and provides an opportunity for the individual student to articulate conceptual understanding. Prompting questions from the interviewer can be used to seek explanations, with reasoning

and justification from the student; however, this relies on the pedagogical content knowledge of the assessor. Therefore, the need for carefully planned, research-based tasks is essential to the effectiveness of an assessment such as this, and can be useful to teachers, promoting the assessment of reasoning, rather than just assessment of a procedure, or ability to follow a 'rule'. This enables the teacher, as the assessor, to gain a deeper knowledge of the conceptual understanding of the student. The potential for a larger study, with a wider range of students, could be considered to better understand the possibilities of using task-based interviews to assess reasoning in a wider range of mathematical concepts.

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Spatial reasoning and the development of early fraction understanding

Chelsea Cutting
Royal Melbourne Institute of Technology
<chelsea.cutting@student.rmit.edu.au>

Young children are capable of engaging with ratio, measurement and operator meanings of fractions earlier than many national curriculum standards indicate, yet current trends in children's understanding of fractions in Australia, remain weak. Research suggests that spatial reasoning can positively influence mathematical knowledge; however, the connection between spatial reasoning and fraction understanding remains under-researched. This paper will present qualitative data from a Design Based Research study that examined a spatialised approach for teaching fractions to 6-and 7-year-old children. Findings indicate that spatial reasoning played an important role in helping children develop early fraction knowledge.

Examining the various perspectives of early fraction development reveals spatial reasoning may play an important role in the construction of such ideas. For example, research relating to young children's proportional and fraction understandings suggests that children engage in spatial scaling when reasoning in such contexts, which requires mentally shrinking or expanding spatial information to determine the relationships between the relative magnitudes (see Huttenlocher et al., 1999; Möhring et al., 2015) This work aside, Bruce et. al. (2017) state there are many 'gaps' in relation to what is known about spatial reasoning and its impact on mathematics education, including how different aspects of spatial reasoning may support young children's engagement with, and understanding of, early fraction concepts. To explore this phenomenon, the following research question was examined in a Design-Based Research (DBR) intervention study: *In what ways does the inclusion of a spatial reasoning approach to fraction instruction in the early years of schooling influence children's understanding of key fraction concepts?*

Background

Fractions are an essential building block of mathematical knowledge yet are complex because they are represented in multiple interpetations, such as *fraction as a relation* (ratio/rate/ proportion); *fraction as operator*; and, *fraction as a measure* (see Confrey 2008; Orbersteiner et al., 2019). Partitioning as an experienced based activity, provides the foundation for the development of children's understanding of fractions (Lamon, 1996; Siemon, 2003) including the closely associated concepts of unitising and equivalence. These concepts should be explored through the three aforementioned fraction contexts to enable flexible and sophisticated understandings to develop (Confrey, 2008). However, current research indicates that the key difficulties young children exhibit in developing early fraction ideas are concerned with making the connections between the concepts of partitioning, unitising and equivalence and the various representations and interpretations in which they are explored (Bobis & Way, 2018; Way et al., 2015).

A growing body of research indicates that young children can engage with these concepts utilising spatial reasoning (Congdon et al., 2018; Möhring et al., 2015). This research demonstrates that young children can adequately problem solve in ratio and proportional contexts when presented with spatial, non-symbolic representations. These fraction ideas are typically not introduced into the curriculum until upper primary and middle school years.

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Whilst this body of research is limited, it does provide the warrant to explore the impact spatial reasoning may have on helping children understand the relationships between early fraction concepts, contexts and associated representations, to mitigate the persistent challenges children exhibit in this area of mathematical learning.

Theoretical Perspectives

Spatial reasoning is defined using the National Research Council's [NRC] (2006) framework, which describes spatial reasoning as a problem-solving activity, involving the coordinated use of space, representation, and reasoning. For the purposes of this paper, the spatial reasoning constructs of spatial visualisation, spatial structuring and gesture will be of focus.

Spatial Visualisation

Lowrie et al. (2018) define *spatial visualisation* as "the ability to mentally transform or manipulate the visuospatial properties of an object…for example, visualizing a cube from its net or predicting a pattern on a piece of paper that has been unfolded" (p. 3). This spatial skill is the multi-step manipulation of objects generated or retrieved in one's mind. Given this definition, this skill involves visualising how different objects and contexts may be manipulated mentally to help develop ideas of partitioning unitising and equivalence within the three different meanings of fractions.

Spatial Structuring

Spatial structuring can be defined as "the mental operation of constructing an organization or form for an object or set of objects"(Battista & Clements, 1996, p. 503). This focusses on identifying objects' spatial components and their composites, and establishing what relationships exist between these elements. Fraction understanding is founded upon partitioning, unitising, multiplicative thinking, and patterning which are also foundational to spatial structuring (Papic et al., 2011).

Representations

Internal and external *representations* are key components of the spatial reasoning framework. Goldin (1998) describes internal representations as systems of verbal/syntactical representations, which describe the way a learner processes imaginative or mental images that include visual and spatial cognitive configurations. These representations involve children mentally organising a problem and mapping the processes for problem solving. In the context of fractions, the *external representations* such as concrete materials, pictorial and graphical representations, and language are central to this component of spatial reasoning and mathematics education. Additionally, gesture is considered an external representation which mediates mathematical meaning, particularly in learning and communicating spatial information (see Alibali et al., 2014; Bobis & Way, 2018) and is an important theme in relation to the present study.

Gesture

Gestures are described as the movement of a part of the body (typically one's hands or head) that is used to covey an idea or meaning. It can be used to connect, illustrate and exemplify complex mathematical ideas so that children develop a deeper level of

understanding and play a significant role in the cognitive processing of spatial information (Alibali et al., 2014). The visuospatial nature of gesture makes it suitable for capturing spatial information, in this case, information pertaining to early fraction ideas such as magnitude, as it brings the imagined or abstract spaces and objects into a more concrete form.

Research Design

This paper reports on a sub-set of data collected as part of a larger DBR study, that comprised of three cycles: a pilot cycle (Cycle 1) and two cycles of a teaching intervention which included an identical pre and post-assessment and unit of work that replaced the daily mathematics program for each class (Cycle 2 and 3), over a period of approximately three weeks (per cycle). This methodology was chosen based on the premise that the educational context is imperative for developing and extending theories for learning, and that "learning, cognition, knowing, and context are irreducibly co-constituted and cannot be treated as isolated entities or processes" (Barab & Squire, 2004, p. 1). Results presented for discussion in this paper are drawn from two tasks in Cycles 2 and 3: (i) a pre/post assessment item and (ii) a mapping-based task from the unit of work. Participating students did not receive any additional mathematics instruction during the intervention period. The participating classroom teachers also agreed not to teach their regularly planned fraction unit before their class participated in the intervention.

Participants

44 children aged 6-and 7-years participated in Cycles 2 and 3 of the intervention. The participating classes (Year 1-2 in Cycle 2; Year 2 in Cycle 3) were from separate, regional South Australian government primary schools. The teacher of each class did not teach any mathematics during this intervention; however, they acted as an additional researcher, by observing each lesson and recording their own reflections, interpretations and interactions with the children throughout each lesson.

Research Instruments

A 13-lesson unit of work was developed for this study. The unit of work was created and taught by the author of this paper. In Cycle 1, each lesson was piloted to determine its suitability for inclusion in the unit of work, and to determine the spatial skills and representations the children engaged with during each activity. Each of the lessons in the unit of work was approximately 50 minutes in length. An example of a task from this unit of work is based on a provocation developed from the picture book *Knock, Knock Dinosaur* by Caryl Hart: "The dinosaurs have escaped the house. They've decided to explore the neighbourhood. Help us find them!". Children were given clues and directions for where the dinosaurs had been 'seen' throughout the town. Using laminated maps and large carpet maps of fictional cities and towns, the children were asked to identify the locations of the dinosaurs, based on clues that contained fractional information (e.g., a quarter of the way along the train track; halfway along the bicycle path etc.). Many of the pathways chosen were not represented on the maps in a straight line, or were open to interpretation (e.g., negotiating which end of a path determined the 'start' of the measure). Thus, spatial reasoning was explicitly embedded into the anticipated problem solving strategies for this task.

An identical pre-and post-assessment was developed to assist in identifying the changes in understandings and strategies developed from the unit of work. The assessment was

administered in a one-to-one task-based interview format with the researcher. Each interview consisted of 24 questions relating to the children's whole number knowledge, fraction knowledge and their spatial reasoning abilities. A rubric was developed to assess each item and to make comparisons between children's initial and final understanding. Children's work samples were collected for analysis in this study and a journal for observations, interactions and reflections was maintained throughout the project.

Data Collection

Each child completed the pre- and post-assessment tasks within two days immediately before and after the unit of work was taught. The assessment took approximately 25 minutes for each child to complete. Children's work samples were collected and their dialogue, gestures and use of materials was documented by the researcher during each item.

The unit of work consisted of a 50-minute lesson each day for 13 consecutive school days. During each lesson, the classroom teacher and researcher kept separate journals of observations and interactions throughout each lesson. At the conclusion of each lesson, the classroom teacher and researcher held a de-brief about the perspectives of the learning.

Data Analysis

All data was analysed using Hybrid Thematic Analysis (Swain, 2018). The method of analysis chosen for this study enabled key themes and relationships to become visible, which were important for developing an understanding of the possible connections between spatial reasoning and fraction knowledge.

Analysis from two tasks revealed the relationship between spatial visualisation, spatial scaling and gesture. The first task was taken from the identical pre-post assessment. It was designed to explore how children conceptualised unit fraction magnitude when asked the following question: *Which is bigger, a third or an eighth? How do you know?* Children were asked to explain their reasoning with access to a range of materials including counters, popsicle sticks, strips of paper, and drawing materials made available (but not compulsory) for use. The intention was for children to demonstrate how they visualised and represented their understanding of magnitude.

The second task, "*The dinosaurs have escaped the house!",* taken from the unit of work, indicated the influence spatial reasoning had on children's understanding of fraction as measure contexts. This task invited children to explore partitioning and unitising with an emphasis on spatial visualisation.

Whilst both tasks had an intentional focus on spatial visualisation, the findings suggested that spatial structuring and gesture were deeply embedded in the children's conceptualisation and representation of their knowledge.

Results and Discussion

In the assessment task, *which is bigger, a third or an eighth? How do you know?* every child from Cycles 2 and 3 (n=44) answered this question with "an eighth" in the pre-assessment phase. The most common explanation to the second part of this question, *how do you know?* was "eight is bigger than three" indicating a reference to whole number magnitude understanding. Additionally, no child chose to use any materials for their explanation, nor use any gesture other than a shrug of the shoulders to indicate they did not know the reason for their answer. Conversely, in the post-assessment, 34 of the 44 children assessed within Cycles 2 and 3 not only answered correctly, but provided rich descriptions

supporting their answer that included gesture, evidence of spatial reasoning, and the use of materials to support their understanding of unit fraction magnitude. To exemplify, two responses from this question are presented (see Table 1) that are indicative of the interconnections between spatial and gestural elements evident in the majority of children's post-assessment responses to this item.

Table 1
Post-assessment interview (Pseudonyms assigned)

Speaker	Interview transcript
Adam:	It's a third, because look – if have this square paper *(A4 rectangular sheet)* and I imagine, like cutting in this way *(gesturing cutting the paper across two evenly spaced places, horizontally across the page)*, I get threes, each of these are a third. To get eight, you have to make more cuts and get more pieces, but the pieces get smaller and there's more of them, but they're heaps smaller – I can see them shrink. And it doesn't matter what size paper you use – a three is always bigger than an eighth.
Troy:	It's a third. When I see the parts in my head, I imagine a line and I can break it up evenly. Just…it's like… it's the more pieces or groups [of things] you need to make out of something, the smaller they get or less you have *(gesturing the forming of parts with hands, moving imagined objects to imagined groups in the air in an array like structure).*

Adam's response suggests some understanding of partitioning as he described how he was able to visualise the process, using gesture to communicate his claims. He demonstrated spatial visualisation through his description of visualising the units "shrinking" as he applied more partitions, which required holding multiple pieces of information in his mind's eye at once, whilst manipulating different components of the mental images (Lowrie et al., 2018). Adam's response indicates an understanding of quantitative equivalence in his discussion of relative magnitude, evidenced in his explanation of the relationship between the fractional units (i.e., a third is always bigger than an eighth regardless of the common whole) which demonstrates emergent multiplicative thinking. Adam's justification of this relationship suggests some abstraction about the essential foundations of fractional knowledge. These foundations include an appreciation for equal parts, and understanding that when the number of partitions increase, the size of the parts decrease (and vice versa) (Lamon, 1996; Siemon, 2003). Spatial visualisation, in addition to the use of gesture, appeared to assist Adam to communicate his understanding of fraction magnitude suggesting he is developing ideas of the relationship between partitioning (division) and multiplication. Additionally, Adam's explanation reveals there was an organisational structure to how he visualized the different partitions, by the way he gestured column and row structures when explaining how multiple unit fractions were created within the same whole. This suggests he was drawing on his internal representations of the patterning and the repeated units related to partitioning and unitising (Papic et al., 2011) which supported emergent multiplicative understandings and indicated an awareness of spatial structure.

Troy's response indicated a transfer of knowledge with reference to partitioning in continuous and discrete models. That is, Troy's response demonstrated an understanding of the measurement meaning of fractions by his description of a line that he mentally partitioned into thirds. Troy's response also exemplified the transfer of partitioning knowledge from

continuous to discrete contexts, by visualising and gesturing the unit fractions of a set. The transfer from continuous to discrete contexts is an important landmark in early fraction understanding, as these ideas require different cognitive demands (Confrey & Maloney, 2010). The demands include recognising that a continuous model is the formation of multiple, contiguous parts; and the discrete model involves the need to perceive a set within one entity. In this case, gesture appeared to be closely associated with how Troy structured and visualised multiple partitions of either discrete or continuous contexts. Alibali et al. (2014) argued that gesture is a vehicle for communicating spatial information, which was evident in his gestures regarding the size and orientation of the unit fractions (i.e., an array like formation). Moreover, Troy's description and use of gesture throughout this task suggests that the spatial composition of the of the unit fractions and the relationship to the fraction construct of measure, was an essential part of his understanding and ability to transfer such ideas across continuous and discrete models.

The second task used for this analysis provides further evidence to address the research question by explicating a connection between spatial visualisation, spatial structuring, gesture and fraction as measure ideas. For example, to introduce the set of dinosaur tasks described above, the following question was posed: *A T-Rex was spotted halfway between the central fountain and the duck pond – where would she be?* From observational data and work sample analysis, most children recognised the fraction as measure context for this activity and engaged in a spatial strategy to solve the problem. This was indicated by drawing straight lines 'as the crow flies' on the map (some children gestured paths with their hands) to determine how the paths could be partitioned between the landmarks to represent where the dinosaur was located. Several children (n=8) interpreted this task as finding the halfway point of the path the dinosaur *may* have taken from the central fountain to the duck pond. That is, the children drew non-linear paths from one landmark to another and then identified the half-way point, as Shaun's work sample illustrates (see Figure 1).

Figure 1. Shaun's work sample

Shaun's path has been marked 'no' at one location and the path marked with an 'X' (digitally enhanced for ease of reading) at another point. When the researcher asked him what the "no" meant, Shaun explained that he initially copied the location his friend had marked for determining half of the path, but Shaun soon realised that his friend was indicating the halfway point of a different path to what he had drawn. Shaun stated that he had to "straighten out the line [drawn path] in my head" (whilst gesturing pulling his hands apart) and when he considered the first mark ("no"), he realised this was "more like a three-part of the way [a third] (using their hands to gesture the three parts of the path), than a two part [half]". Shaun then placed an 'X' on the path (above the yellow car) as the halfway mark

instead. To paraphrase, Shaun stated that it did not matter how long the path was or in what shape/ orientation; to be half meant there were two equal parts of the concerning path. His recognition of the differing path lengths and its relationship to the target fraction demonstrates an emerging understanding of proportional thinking (fraction as relation). Although Shaun initially copied his friend's map, he recognised it could not be an accurate representation of the same fractional measure, as their paths were different lengths. Shaun stated he would have to mentally manipulate these paths (using spatial visualisation) to enable a comparison of measures. This type of thinking also suggests spatial structuring was an important component to his conceptualisation of the problem, particularly when combining components into spatial composites such as units of thirds and halves (Battista & Clements, 1996), to establish the relationships between these measures within his own representation and in comparison to his friend's path. Shaun demonstrated an understanding of relative magnitude, by explaining the differences of absolute measures through visualising and comparing the different paths and used gesture as a vehicle to demonstrate the iterative unit fractions of halves and thirds. Emerging proportional thinking as illustrated in this example was evident in 19 children's responses throughout the intervention, which highlights the abstraction and transfer of these concepts.

The relationship identified by this study between spatial visualisation, gesture and the concept of partitioning (in a fraction as measure context) extends Lamon's (1996) description of partitioning as being an 'experience-based activity'. The deep engagement between spatial visualisation with gesture forms an important part of this experience as it served as the vehicle for children articulating their experiences of partitioning. Moreover, spatial structuring was an important component in children's development of unitising and equivalence ideas that formed from their engagement with spatial visualisation and gesture, which in turn suggested it positively influenced the children ability to conceptualise the multiplicative nature of fractions in both discrete and continuous contexts. It is clear that the common multiplicative foundations spatial structuring and early fraction concepts share, influenced the way children visualised and used gesture when representing key fraction ideas.

Conclusion

The relationship between visualisation, gesture, spatial structure and fractions is an important finding and contribution to understanding how young children develop such ideas. Importantly, this study revealed that this relationship also contributed to children's abstraction and transfer of understanding of these concepts in both discrete and continuous models which is an essential component for developing conceptual understanding of fractions. Moreover, the results from this study go some way to addressing the persistent problems young children face in developing deep connections between the concepts and contexts in which fractions are explored and represented (Bobis & Way , 2018). These new understandings imply there are considerable benefits in adopting a spatialised approach to teaching fractions in the early years of primary school, because it can allow for a better exploration and understanding between the nature of young children's spatial reasoning, representations (internal and external) and the role these factors play in young children's development of fraction knowledge. The limitations of this study include the sample size of participants involved, and lack of video recordings for greater fidelity measures. However, future research directions could include a longitudinal study to provide greater insights into the connection between different aspects spatial reasoning and their impact on children's development of rational number knowledge more broadly.

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Secondary mathematics teachers' conceptions of assessment

Hem Dayal
The University of the South Pacific
<dayal_h@usp.ac.fj>

Teachers' conceptions play an important role in their instructional practices. In this study, the researcher explored a small sample of Fijian secondary school mathematics teachers' conceptions of assessment. Thirteen mathematics teachers from two case study schools took part in this study that utilised one-to-one interviews to gain insights into teachers' beliefs on the purposes of assessment. The findings further indicate that a majority of the teachers held contemporary conceptions of assessment. While they did value summative assessment roles, teachers tended to support the use of assessments to improve or support student learning.

The term assessment can be interpreted in different ways by different stakeholders. For example, while some teachers see assessment as an activity that is used to improve classroom instruction, others may value it as a means of establishing accountability within the school. Moreover, some may even perceive assessment as an activity that has no value at all (Brown 2003, 2004). In other words, varying conceptions of assessments can be placed on a continuum that has traditional conceptions on one end and the other representing contemporary conceptions. For example, teachers can, on one hand believe that assessments serve solely accountability purposes (and seen as irrelevant (Brown, 2004)), while on the other end of the continuum of conceptions, they may see assessments as purely an activity with a pedagogical aim. Educators can hold mixed beliefs, representing any point on the traditional–contemporary continuum. For the purpose of this study, we define conceptions following Brown (2004) as a guiding framework used by an individual to understand, respond to, and interact with a given phenomenon. In other words, teachers' conceptions of assessment can include their beliefs, attitudes, or perceptions (Harris & Brown, 2016).

Research suggests that such a continuum of teacher conceptions is likely when teachers are asked to list various purposes of assessment (Barnes et al., 2015). Apart from the 'purposes' category, assessments can be differentiated using other criteria such as nature of tasks used, cognitive demands associated with tasks, including frequency and grading of assessments (Wallace & White, 2014). The traditional-contemporary continuum of assessment can be seen as parallel to the commonly used summative-formative classification. Summative assessments are those that usually come in the form of standardized tests, measuring terminal performance while formative assessments represent any assessments that are designed primarily to support student learning (Wiliam, 2007, p.1053).

Teachers' conceptions play an important role in their instructional practices (Ashton, 2015; Buehl & Beck, 2015; Marshall & Drummond, 2006; Skott, 2015). Despite notable progress in re-thinking learning and assessment over the past two decades, there exists many different understandings of the term assessment and other associated terms such as formative assessment (van de Watering et al., 2008). For example, Popham (2014) explains that American educators usually see teachers' role in formative assessment as more important than students' roles in improving their own learning.

Differences in conceptions therefore could mean that teachers take relatively different perspectives on using assessment information. There is sufficient evidence that assessments, when developed and used appropriately, would lead to improved student learning (Black & Wiliam, 1998). In the Fijian secondary education context, assessments are generally

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conducted to prepare the learners for the external examinations at the end of the year. As such, the majority of the ongoing assessments take the form of written examinations that are similar in content and structure to the national examinations. In light of the relative importance of teacher conceptions, this study aimed to explore how a small group of Fijian secondary mathematics teachers' perceived assessments amidst an examination-oriented education system. While the study reported here was part of a larger study that aimed to explore how well mathematics teachers took up formative assessment practices (Dayal & Cowie, 2019), exploring teachers' initial conceptions about assessment was seen as an important part of the larger study's context. The following research question guided this study: *What are Fijian secondary mathematics teachers' conceptions of assessment? Are Fijian secondary mathematics teachers' able to conceptualise the contemporary purposes of assessment?*

After presenting the sociocultural framework used for this study, a brief review of literature is provided. This is followed by research methods, results, and discussion. Finally, some conclusions and implications are outlined.

Theoretical Orientation

Brown and colleagues have identified the following four teacher conceptions of assessment. These include assessment serving four distinct purposes: improving teaching and learning; holding students accountable for learning; making schools and teachers accountable for student learning, and assessment serving no legitimate purpose in schooling (Brown 2003; Brown 2004; Brown & Hirschfeld 2007). The first conception presents a rather fallibilist or humanist view of assessment. It sees assessment as learner-focused, for joint use by students and teachers to improve teaching and learning. This conception blends well with the idea of formative assessments or 'assessment for learning' loosely defined as any activity from which the elicited information is actually used to make changes to teaching and learning with the view to improve learning (Black & Wiliam, 1998). Formative assessments are in line with student-centred learning and Sheppard (2000) calls this the emergent assessment paradigm. Formative assessments are more about feedback that could be used to improve learning. Such a view of learning and teaching is consistent with the sociocultural theory that regards knowledge as fallible and a product of human creativity. This view of knowledge means that learning or knowledge creation is seen as a social process in which the learner is an active participant.

Conceptions not confined to this contemporary end of the assessment conceptions continuum would fall somewhere in between and would likely be represented by the other remaining conceptions identified by Brown and colleagues. At or near the traditional end, assessments serve rather authoritarian roles such as measuring how much learning has taken place, monitoring, recording and reporting students' performance, and holding individuals and institutions accountable for their actions. Toward this traditional end, knowledge is seen as objective and infallible (Sheppard, 2000; Wallace & White, 2014). Seen from this perspective, assessment's purpose is mainly for grading and certification. Such conceptions align well with the behaviorist ideas and sees assessment as merely measuring students' learning using quantitative methods. While realizing the important roles of assessment, this study took sides with Popham (2014) who claimed we must not rely only on traditional notions of assessment but should, instead, consider those conceptions of assessment that support effective teaching and learning. This paper conjectures that mathematics teachers would benefit a lot with a contemporary conception of assessment.

Literature Review

Hui and Brown (2010), in their study involving primary school teachers' in Hong Kong, revealed that these teachers were very well aware of the "improvement" purpose of assessment. In other words, the Hong Kong teachers generally held an 'assessment for learning' conception of assessment. The study did note, however, that some teachers also held accountability conceptions of assessment. Their data indicated that some teachers believed that the assessment tasks they designed were also valid for "accountability" and "examination" purposes. The study concluded that the prevalence of accountability as well as examinations conceptions of assessment among Chinese teachers may hinder the successful implementation of an assessment-for-learning policy.

In another study, involving Fijian pre-service and in-service teachers, Dayal and Lingam (2015) also noted that teachers held multiple conceptions of assessment. While a majority of the seventy participants' initial understandings aligned to a traditional conception that involved measuring students' performance, some of the participants agreed that assessments could have formative functions when they were asked to list down other major purposes of assessment. The study revealed that a higher proportion of pre-service teachers held an 'assessment of learning' conception of assessment in comparison to the teachers who had some years of teaching experience. This was revealed when both group of teachers were asked to choose from two different roles of assessment that they would favour: the master role, indicating 'whatever assessed should be given importance', against the servant role which suggested that 'whatever is important should be assessed'. Of the practicing teachers, 74% favoured the servant role, compared to only 30% of the pre-service teachers. The authors, however, showed concerns regarding a good number of in-service teachers still holding a narrower view of assessment. Dayal and Lingam's (2017) study utilized an open-ended questionnaire to explore pre-service and in-service teachers' beliefs about the two major purposes of assessment. Their findings confirmed that pre-service and in-service teachers could hold beliefs which are in support of summative assessment, formative assessment, or both types of assessment. Majority of the pre-service and in-service group gave explicit support in favour of formative assessments. None of the participants, however, noted that both forms of assessment are irrelevant, contrary to findings such as Brown (2004).

In terms of how secondary mathematics teachers perceive assessments, one notable, yet small study was that of Wallace and White (2014). The authors followed six pre-service mathematics teachers through what they termed a reform-minded teacher education programme in the United States. A notable feature of these programs was the inclusion of assessment ideas embedded in course assignments. The study's findings confirmed that pre-service secondary mathematics teachers initially held traditional perspectives on assessing student learning. The authors called this the 'test-oriented' perspective, characterised by assessment beliefs such as: assessments are tests, the purpose of assessment is to provide a grade, and assessment involves closed-ended tasks. The study noted that the pre-service teachers could modify their assessment practices by evolving through the 'task-oriented' and 'tool-oriented' perspectives on assessment. The latter represented the developmental phase where these pre-service teachers were able to see assessments as designing new ways that would help facilitate student learning.

The studies reviewed here and others such as Nisbet and Warren (2000) and Smith et al. (2014), confirm that both practicing and pre-service teachers have different conceptions of assessments. Some of these studies also point out that different assessment perspectives may have the potential to lead to different assessment practices, and the inherent need to explore

teachers' conceptions of assessment. Studies such as Wallace and White provide evidence that teachers can modify their assessment-related conceptions when given support. While this is not an explicit aim of the current study, exploring the conceptions of a small group of practicing Fijian mathematics teachers will add to our understanding of how assessments are perceived by mathematics teachers. The context of our study is presented next.

Context and Methods

The participants in this study were 13 mathematics teachers from the two case study schools: *Marau College* and *Kaivata College* (pseudonyms used). *Marau College* had nine teachers in the Mathematics Department, and *Kaivata College* had only four. The mathematics teachers had taught for an average of 9 years, ranging from 20 years to only three years. All of them had tertiary qualifications. For the five male and eight female teachers, real names are replaced by pseudonyms beginning with the letters A to M, the letters indicating the order in which the interviews were carried out. In order to elicit teachers' conceptions of assessment, one-to-one interviews were held at the teacher's respective schools. One-to-one interviews seemed suitable for two reasons. Firstly, it allowed the teacher participants to express freely their beliefs and experiences with assessments in mathematics. Secondly, the one-to-one interviews helped the researcher know the participants better, and this helped build positive relationships for the later phases of the study that involved teachers as key stakeholders in research (Kieran et al., 2013). On average, one interview lasted for fifteen minutes. The study utilised the following prompts for the interviews:

1. Think of the term *Assessment.* What comes to mind? List as many ideas as possible.
2. What is the main purpose of assessment? What are some other purposes of assessment?

All thirteen interviews were audio taped and transcribed. The interview data were analysed using traditional–contemporary continuum presented under the theoretical framework of the study. For example, upon transcribing the interviews, each response was read in full and the keywords or phrases that represented each participant's beliefs about assessment were highlighted and placed under either traditional or contemporary conceptions. For example, if the participants used the keywords or phrases that resembled traditional conceptions of assessment such as 'grading', 'passing an exam', 'measuring' or 'testing', these participants were classified as showing a traditional conception of assessment.

Results

This section presents the findings of the study.

Assessment Purposes

While all thirteen participants were able to define the term assessment, only five of the participants showed a narrower, traditional view of assessment. For these teachers, assessment essentially meant "testing students' knowledge" (Ella) , or "getting to know whether the students have got the content we have taught" (Cathy) , "to test whether students have understood and whether they are revising their work" (Fran), and "to know how much they know" (Bhim). A strong focus on answering the 'how much' question, coupled with ideas related to 'testing' or 'exams' revealed that, for this group, assessment meant answering the question 'how much does a student know?', thus reflecting a traditional,

measurement view of assessment. For example, Jenny, in her description of assessment said that assessment is "the test given to see how much students have learnt from something" and "it is an activity to grade the students". When asked about the major purpose of assessment, Jenny replied: "To rank". When asked to list a few other purposes, she said "to test and select the best". From Jenny's interview account, it could be said that she had a strong inclination towards an 'assessment of learning' view of assessment. This view of assessment has a strong leaning towards a testing culture, promoting competition, and using examination results to select students for placements. In her interview, Jenny revealed that she did not use assessments in a formative manner.

For the rest of the participants, assessment was more than 'testing'. For example, Kumar said that assessment meant "monitoring the performance throughout the year". Her definition viewed assessment as a continuous event, and not a one-off task. A similar view was given by Ledua, who said that "assessment is an ongoing process to see if the student is learning the concepts or not". Isha listed a number of ideas such as "exam, presentation, short test, assignments, tutorials, oral assessment, quiz and class-based assessment (CBA)" when talking about her views on assessment. She showed strong emotions against summative assessment – "sometimes assessment is like a 'torture' to students, especially the three- hour exams." Gavin showed an understanding that assessment not only concerned the students but also the teachers when he stated that "assessment is something which tells me how I have done in my class as a teacher". Overall, the majority of the teachers showed an expanded, contemporary view of assessment in their initial discussions on assessment. These views had elements of formative assessments such as views about having multiple forms of assessment; views about assessment as a continuous process; and views about assessment as informing the teachers on their work as well.

When asked to recall the major purpose of assessment, the teachers in this study exhibited the same tendency. Those who had initially shown a measurement view of assessment (Ella, Cathy, Fran, Bhim and Jenny) listed its summative function as the main aim of assessment. Examples of these included: "To test the students' knowledge" (Fran), and "to test students' ability" (Bhim). When asked to list any other purposes of assessment, three out of the five teachers were able to pick up some formative aims of assessment. For example, Cathy referred to teachers' teaching techniques and how assessment could help teachers know how they are performing. Ella stated that teachers could work on weaker students as a result of assessment. However, this group of teachers was still hanging on with their initial ideas about testing and examinations. As Ella noted, "if they have done a test, they have got low marks, it means we place more time on them." Only two teachers, Bhim and Jenny, in this group were unable to list any formative purposes of assessment. In their view, all purposes of assessment were summative in nature. Excerpts from Jenny's interview are shared below:

Researcher: In your view, what is the major purpose of assessment?

Jenny: To test the students' ability, to assess students and to know how much they know.

Researcher: Can you think of any other purpose?

Jenny: To pass exams and go to higher level?

Researcher: Any others?

> Jenny: Ummm…to see which students are, I mean good at which particular field, and whether they are supposed to go to tertiary institutions.

The other eight participants had listed formative assessment practices as one or more of the purposes of assessment. For example, Dan explained that the major purpose was for "us to know how well the students have learnt". He went further to claim that assessment "helps us to improve in our weak areas". For Ana, assessment helped provide feedback not only to the students but also to the teachers. She showed formative aims or purposes when she claimed that "tests are not always giving us all about learning." Mere claimed that the main purpose of assessment was to help students to learn. Apart from this, she added that assessment is used "to improve students' learning and teachers' teaching – when the activities I have given have not been done well, I come back and re-think about my teaching strategies." Another teacher, Gavin, held similar beliefs about assessment. His views about assessment reflected an inclination toward the formative view of assessment as well. He viewed assessment as something "which tells how I have done as a teacher". For him, good assessment meant that he had to "re-look at what students have given me and what I expected as the correct answer. If there were some differences, I have to do that again, or re-design my class and take another approach". These statements reveal that this group of teachers had strong views about the role of feedback in assessment. Their overall view of assessment could be classified as being more aligned towards assessment for learning.

Teachers in this group had also shown a combination of summative and formative purposes. For example, Isha mentioned "gaining certification" as the second purpose of assessment. In summary, majority of the participants were well versed in both summative and formative purposes of assessment. This group of teachers seemed to favour formative practices much more than summative or measurement purposes of assessment. Some even had strongly rejected the idea of "testing" alone. These sentiments are clearly visible in the accounts of some of these teachers: "the current assessment (three-hour exams) does not tell much as it is just a paper and pencil test – a lot of writing and recalling is involved. Learning/expressing is not there" (Isha); "assessment in mathematics can be very broad, in various forms. In my school, we just assess using paper work. We can assess by doing more practical work. There can be theory and practical assessment" (Haris). The views expressed by teachers suggest that they value formative assessment even more than summative assessment.

Discussion and Conclusion

Two types of assessment have been well distinguished in the assessment literature – summative and formative assessment, although these may not necessarily be mutually exclusive dimensions. A more productive view about assessment is the former and this is in line with the socio-cultural views of learning (Sheppard, 2000). Only five of the participants held a narrow, summative view of assessment. Two of these five (Jenny and Bhim) had very strong traditional conceptions of assessment, while the other three showed some support for formative assessments. This group of teachers tended to value the testing and grading function of assessment more. One reason for this could be that these teachers simply disregard the value of formative assessments. Another reason could be that they may not have used formative assessment practices well and thus may not have experienced any positive consequences of such assessments on student learning. The latter is more likely given the examination-oriented education system in Fiji.

Eight of the 13 participants held contemporary conceptions of assessment. While they did value summative assessment roles, teachers tended to support the use of assessments to improve or support student learning. Despite working in an environment dominated by the summative culture, it is interesting how this group of teachers supported the idea of formative assessment. It would be worth investigating how these beliefs are formed. Initial instincts, including understanding gained from sociocultural perspectives suggest that personal experiences with the use of summative testing may be one of the factors. As one teacher indicated, three-hour examinations are a kind of 'torture' to pupils' brains. From a social perspective, it can be argued that while summative examinations have been part of the Fijian education system from decades, teachers may have had bad experiences with summative assessments. It may also be inferred that the teachers in this study had seen that there are no real learning benefits from too much summative testing. It is interesting to note that a majority of the mathematics teachers do not render much support for traditional assessment practices, although they use such 'examinations questions' in their usual classroom teaching. Such a finding is consistent with previous studies like Dayal and Lingam (2015, 2017) that noted a relatively higher percentage of practicing teachers who favoured formative assessment practices. Ashton (2015) noted that belief systems rely heavily on evaluative and affective components. This may, to some extent, mean that a majority of the teachers in this study have negative feelings about summative assessments. In summary, it can be said that while cultural aspects may have affected the teachers' beliefs about the nature of mathematics, personal experiences, including external factors such as school and national policies may have had some impact on shaping the teachers' beliefs about learning, teaching and assessment.

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Secondary pre-service teachers' views on using games in teaching probability: An international collaboration

Hem Dayal
The University of the South Pacific
<dayal_h@usp.ac.fj>

Sashi Sharma
University of Waikato
<sashi.sharma@waikato.ac.nz>

Probability and *statistical literacy* is an important aspect of the school curriculum in many countries. In this study, we report on findings from a larger study that engaged pre-service teachers as key stakeholders in research in exploring teaching probability and statistics using a game-based teaching approach. The current study focuses on 23 pre-service teachers' views about game-based teaching and learning. Our sample of teachers were from two universities in the Pacific region. The findings strongly indicate that pre-service teachers can derive useful pedagogical knowledge by engaging in the game-based teaching intervention. All the pre-service teachers support the use of real-life based practical approaches in their teaching.

In a rapidly evolving world, there is a strong need to understand and be able to use mathematics in all aspects of life. One particular area of mathematics that we use or rely upon on a daily basis is probability and statistics (Koparan, 2019). The use of probability and statistics translates down to the need to understand and use data in almost all aspects of life, such as education, health, or predicting future events such as adverse weather conditions. This aspect of learning mathematics is termed *probability literacy* or *statistical literacy* (Jones et al., 2007). It includes having a working "knowledge and understanding of numeracy, statistics and data presentation" (Pierce & Chick, 2013, p. 190).

Given the importance of statistical literacy, many countries place probability and statistics in their core mathematics curriculum. For example, in the New Zealand school curriculum, probability is part of the three sub-strands in the curriculum document and viewed as critical in the learning of mathematics (Ministry of Education, 2007). In the Pacific education context, many educational jurisdictions have included statistical literacy as an important aspect from the early years of the school curriculum (Fiji Ministry of Education, Heritage & Arts, 2017).

Given the relative importance of probability and statistics in our curriculum, it is imperative that teaching of the probability and statistics curriculum aligns, to a higher degree, with our recent understandings of the term *statistical literacy*. Therefore, it is critical that teachers of probability and statistics are exposed to making use of lots of real world examples and activities in their teaching. One of the ways of doing this is through the use of games. In this study, we report findings about the usefulness of teaching probability and statistics using a probability teaching sequence designed by one of the authors (Sharma, 2015). The paper reports on benefits and challenges of using games from the perspective of our relatively small sample of secondary pre-service mathematics teachers from two different universities in the greater Pacific region. The research questions addressed in this paper are: *To what extent do the pre-service mathematics teachers find the probability teaching sequence useful? What are some of the benefits and challenges they foresee in adapting such games in their teaching?*

After presenting the theoretical framework, a short literature review is presented. This is followed by the specifics of the study's research design. Then, results and discussion are presented. A brief conclusion sums up this paper.

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Theoretical Framework

In this study, we utilised the socio-cultural theories of learning. The influence of socio-cultural context on a learner has been examined mostly from Vygotsky's frame of reference. The sociocultural environment incorporates use of a variety of tools such as language, sign, and cultural tools (artefacts) to assist with reaching higher mental models (Vygotsky, 1978). Given the aim of the study was to explore pre-service teachers' views about the benefits of using a newly introduced probability teaching sequence (reference withheld), it was important to see how they suggest they could make use of the ideas that they could have possibly derived from the teaching sequence. Given that we were exploring pre-service teachers' future intentions, it was critical that most of these are turned into productive actions when they begin teaching mathematics. In this regard, Valsiner's zonal theory (Valsiner, 1997), an extension of Vygotsky's zone of proximal development (ZPD), is seen as a useful framework for viewing teachers' thought processes as well as their actual actions (Goos, 2014).

Vygotsky defined ZPD as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). According to Valsiner's zone theory (Valsiner, 1997), one can assist a learner reach ZPD with the help of available resources and processes within the proximity to enable their *zone of free movement* (ZFM) and *zone of promoted actions* (ZPA) (Goos, 2014). ZFM usually includes contextual factors that limit pre-service teachers' thinking and actions, while ZPA includes all those activities that are designed by other adults, such as university lecturers, that are aimed at developing or promoting new skills. In this study, we focus on the pre-service teacher as the learner. Hence, it is important to critically review the contribution from each zone, in particular, focusing on what benefits and challenges pre-service teachers see in using the probability teaching sequence and how they intend to use the teaching sequence.

Literature Review

Two major interpretations of probability can be distinguished. The classical (theoretical) viewpoint assumes that it is possible to represent the sample space (all possible outcomes) as a collection of outcomes with known probabilities. For example, the probability of rolling a six on a regular six-sided die is one-sixth. In such a case, the theoretically derived probability is an estimate of the actual probability that is not known. Batanero et al. (2004) argue cases of equiprobability that arise in some simple game scenarios, such as rolling a die, may not be the same in complex everyday situations, such as weather predictions, risks and epidemics. On the contrary, the experimental interpretation assumes that the probability of something happening can be determined by doing experiments. A large number of identical trials (e.g., tossing two coins) are conducted, and the number of times a particular event (e.g. one head and one tail) occurs are counted. The greater the number of trials, the closer the experimental probability will move towards the theoretical probability of an event. By comparing inferences from their theoretical and empirical work students can evaluate and modify their hypotheses.

Students leaving school should be able to interpret probabilities in a wide range of situations (Jones et al, 2007). If students are to develop meaningful understanding of probability, it is important to acknowledge the different interpretations, and to explore the connections between them and the different contexts in which one or the other may be useful.

Games can provide a useful context for exploring different interpretations and contexts. Batenero et al. (2004) provide an excellent example of how different probability teaching contexts can be explored using games. They engaged a group of teachers in experiments involving different coloured dice. Although the authors did not specifically seek the participant teachers' views about the usefulness of such gaming experiments, they speculate that teachers do acquire knowledge that would be beneficial in their later professional work.

Research evidence suggests that teachers, including prospective teachers, find teaching probability and statistics difficult or challenging (Batanero et al., 2004; Leavy et al., 2013). For example, the findings from a small sample study conducted by Leavy et al. (2013) in Ireland suggests that prospective secondary mathematics teachers perceive statistics as a challenge due to, among other factors, the need to think and reason statistically. Anecdotal evidence suggests that teaching probability and statistics is also a challenge for Pacific Islands teachers. One possible factor could be the mismatch between the nature of probability and statistics, and the teaching approaches used by teachers. As reported by Dayal (2013), teachers from the Pacific Islands have a tendency to teach mathematics using traditional approaches such as relying heavily on notes and examples followed by routine textbook-type exercises.

The brief review of literature suggests that two different, yet not mutually exclusive, approaches to understanding the teaching probability and statistics are prevalent. This study hopes to add to our understanding of how pre-service teachers can derive potential teaching ideas for both theoretical and experimental aspects of probability and statistics. The literature seems to suggest general prevalence of teaching challenges as well as an acknowledgement of the potential benefits of teaching using games. The current study also aims to add to our understanding of pre-service teachers' perceptions of the degree of usefulness of games in teaching.

Research Design

To conceptualise our larger study, we drew on *design-based research theory* (Cobb & McClain, 2004). Design research is a cyclic process with action and critical reflection taking place in turn (Cobb & McClain, 2004; Nilsson, 2013). Further, all participants are equal partners in the research process (Kieran et al., 2013). Using a case-study design (Yin, 2014), our study itself involved cycles of three phases: a preparation and design phase, a teaching experiment phase, and a retrospective analysis phase. Both mathematics educators were involved in the whole research process. The role of researchers involved posing questions, and observing the research unfold with minimal interference. This paper reports on post-intervention findings, after our pre-service teachers had completed the teaching experiment phase. The teaching experiment, called the *probability teaching sequence*, involves a scenario where two people play a dice game. Each player throws a die and the difference of the two outcomes is calculated by subtracting the smaller number from the bigger number. If the difference is 0, 1, or 2, player A wins. If the difference is 3, 4, or 5, player B wins. The main question that pre-service teachers were required to think about when playing the game was whether or not the game was *fair* and to justify their reasons. From a socio-cultural perspective, the probability teaching sequence provides pre-service teachers an opportunity to 'think' and 'act' within their ZPD. For the full teaching intervention, see Dayal and Sharma (2020). The research context, participants and procedures are described in the table below.

Table 1
A summary of context, participants, procedures and instrument

Research Context	Research Participants	Research process	Research Instrument
The University of Waikato (UW) is located in Hamilton and operates from two campuses, Hamilton, and Tauranga, in New Zealand.	• 10 pre-service mathematics teachers completing their Graduate Diploma in Teaching programme • Equal number of males and females • Six New Zealanders, four international pre-service teachers • All teachers have mathematics as their teaching major. • Participants are represented using letter codes O – Y.	• The second author is the coordinator of the teaching methods course. • Upon completing the intervention, one-to-one semi structured interviews were conducted. • All ethical guidelines, as per UW research ethics approval, were adhered to. For example, each student gave their informed consent and were assured that their non-participation or withdrawal would not affect their performance in the teaching methods course. • All interviews were audio recorded.	Post intervention one-to-one interviews with the following prompts: • Think back on the activity we did today. Did you all like the activity? Why or why not? • Are there any probability teaching ideas that you can take to your classroom? Will you be using these ideas in your teaching? • Suppose you were to recommend this teaching sequence to a colleague. When will you suggest him or her to use it? • Do you feel there are some challenges in doing this activity? • What kind of support, if any, would you require?
The University of the South Pacific (USP) is a regional university that is owned by 12 member countries in the Pacific and is head-quartered in Suva, Fiji Islands.	• 13 pre-service mathematics teachers in their final year of the 4-year BSC GCED programme • Equal number of males and females • All teachers have mathematics as their teaching major • Ten teachers from Fiji, four from Kiribati. • Participants are represented using letter codes A- N	• The second author was not teaching the participants. • The intervention was held on a non-teaching day (Saturday) at the USP. • All teachers gave written informed consent and volunteered to be part of this intervention. • USP ethics approval was sought prior to the intervention. • Post intervention, participants reflected in a focus group set up. • All discussions were video recorded.	Post intervention focus group discussions using the above prompts. • Group 1: Participants A,C,E, H, I (Fiji) • Group 2: B,D, F,G, J (Fiji) • Group 3: K,L,M,N (Kiribati)

Findings and Discussion

The individual interviews and focus group discussions were transcribed and analysed by each author. The following sections present the common themes that arose after analysing pre-service teachers' opinions about the probability teaching sequence.

Affective and Cognitive Benefits

All the participants explicitly stated that they liked the probability teaching sequence. The reasons provided related to the teaching sequence being interesting "because it allowed us to think" (Participant A) about probability and "learn from their own mistakes" (Participant K) rather than learning probability using formulas. In addition, the pre-service teacher participants talked about affective reasons, such as "we liked the dice activity because it is better than giving notes from the textbook" (Participant K) or "this is a very creative way of learning probability" (Participant C).

Similar to the USP participants, UW participants also noted affective and cognitive benefits of the probability teaching sequence. Some of the responses included:

> "Open questions build student self-confidence because students can answer at their own level of understanding" (Participant Q)
>
> "It is different to most tasks with probability, so it will be good for students to get a change from routine" (Participant P)
>
> "The game makes students think logically to show all possible outcomes of rolling two dice" (Participant O).

Deriving affective as well as cognitive benefits and learning about probability teaching ideas was a common theme reported by a number of participants from both contexts when asked about whether or not they liked the activity. It is encouraging to note that the pre-service teachers were able to recognise such benefits and acknowledge that the teaching sequence provided another, interesting way to learn probability. This may be due in part to some of our participants, especially those from USP, being largely exposed to traditional approaches to learning during high school and university, such as completing routine textbook-type exercises (Dayal, 2013).

Deriving Teaching Ideas

In terms of learning about probability teaching ideas, the USP pre-service teacher participants could identify some holistic ideas as well as a number of specific topics that they could explore using this teaching sequence. The pre-service teachers' very general hints about teaching probability included comments such as "we learnt how to create good experiments using dice" (Participant L). In their discussions, the pre-service teachers from USP uttered various probability- and statistics-related terms (e.g., events, trials, chance of events, outcomes, skewness of outcome, expected probability, fairness, graphs, making predictions). In comparison, some USP pre-service teachers appeared to have some difficulty with identifying topic-related terms. For example, when asked to share the probability teaching ideas they could take into their classrooms, some participants in Group 3 stated general themes, such as "conducting experiments using dice" (Participant L) or "teaching probability" (Participant N).

It is worthy to note that these participants were all from Kiribati. In the Kiribati context, these participants mentioned that probability is introduced late in the school curriculum, only in upper secondary curriculum (Years 11 and 12). In contrast, in Fiji and NZ, probability and statistics is introduced from the early years.

Similar to the USP cohort, the UW cohort was able to list a range of teaching topics as well. For example, one participant mentioned the topic of sample space:

> "The lesson sequence allows students to explore sample space by using representations that make sense to them. For example, some students may use grid of numbers whereas others may use tree diagrams" (Participant P).

In addition to naming such probability teaching topics, it was encouraging to note that UW teachers were able to suggest many other pedagogical aspects from the probability teaching sequence, such as the sequence having a clear learning objective and a good range of questions that could promote student learning. For example:

> "The lesson sequence has clear objectives for student learning. The teacher can share these goals with students." (Participant O)
>
> "The sequence includes a range of questions. Asking questions can give teachers information about students' thinking" (Participant S).

Overall, the pre-service teachers derived a number of useful general teaching ideas, such as conducting experiments, as well as ideas about specific subtopics that are present in probability and statistics. The need to have practical activities using dice or coloured cubes, or even coconuts, were mentioned by USP and UW participants. The need for more real-life based activities were also mentioned:

> "It is important that students make connections to everyday life situations" (Participant U)
>
> "Students will be actually doing the thing. They will actually see what is happening by throwing the dice…and recording the data…" (Participant A).

As well as thinking about connections to real life experiences, participants thought about how the activity allows students to make connections to existing mathematics they may know. One UW participant noted:

> "It provides opportunities for students to make connections between probability concepts with everyday life and with other topics of study such as fractional number" (Participant Q).

Making connections to real-life and between different representations is critical in developing probabilistic understanding (Nilsson, 2013; Van de Walle et al., 2014). The findings suggest that the probability teaching sequence will likely benefit teachers as it provides them opportunities to ask students to play around with chance generating mechanisms, and use multiple representations such as tables, diagrams and graphs to explore probability concepts in a meaningful context. Since students can draw different representations to determine the theoretical probabilities, there is scope to make connections to real-life as well as among these different representations as reflected above.

Future Teaching and Challenges

All our pre-service teachers explicitly stated that they will be using this teaching sequence in their actual classroom teaching. When asked to suggest ways in which they would recommend this teaching sequence be best used, the groups seemed hesitant in providing specific answers. However, they stated a few specific scenarios, such as teaching a probability topic or as an assessment. Some responses included:

> "In conducting experiments about chance" (Participant L)
>
> "This activity would not be accessible at the start of a junior mathematics probability study, however it could be used as formative assessment" (Participant V)

Some participants reiterated general teaching ideas, such as:

> "teaching probability in a real-life situation" (Participant A)
>
> "use of experiments instead of textbooks, by using real-life context we can also help students learn probability more effectively." (Participant X)

In terms of teaching challenges, the major challenge noted by participants was the time factor. The reasons given by the USP cohort was mainly that the school teaching period was only of 1 hr and this activity could not be well implemented in an hour's time. Upon inquiry by the researcher if the teaching sequence could be broken down into smaller bits, the groups seem to agree that the time factor challenge could be overcome through this. Views such as giving a lesser number of trials was one of the ways suggested to overcome this challenge. In addition to time, the USP and UW participants mentioned class management as a possible challenge, "challenging and disruptive classroom environment that results in a lack of engagement" (Participant V). Some participants though stated that this challenge could be overcome by having smaller groups or by asking students to work in pairs.

In the context of mathematics lessons in Fiji and Kiribati, there is high importance placed on preparing students for external examinations. Hence, the limited lesson time, as mentioned by some of the participants, is a realistic challenge faced by many teachers. There was a consensus among the USP participants that covering the teaching syllabi well-ahead of time was critical for ensuring that ample time was left for students to attempt past-year examinations as part of their examination revisions. It was no surprise, then, that Participant I suggested that the use of these activities be reserved for "during a revision class", instead of part of the introduction of the topic or prior to revision.

Overall, our findings support, to a large extent, that some participants may use this probability teaching task or any shorter variant of it in their actual teaching. However, some may front load probability content first using more direct teaching methods, then use a game like this at the end of the unit to apply the learning. Seen from a socio-cultural perspective, the study provides evidence that our pre-service teachers' potential to learn new skills and develop (ZPD) is enhanced by engaging with the probability teaching sequence (ZPA), as well as via thinking and interactions with their peers in small group settings (ZFM).

Summary and Implications

While this study can be seen as a step forward in collaboration among teacher educators, it had its own limitations. One major limitation was that the two research contexts were quite different in terms of many factors, such as high school and teacher education curriculum. We negotiated such challenges by frequently discussing emerging issues through emails and Skype (e.g, the research process). Achieving exact consistency was not seen as critically important (Moss, 1994); instead, we made sure that an in-depth exploration was carried out while being within the ambit of our university learning and teaching regulations. The pre-service teachers registered an overwhelming support for the probability teaching sequence. They saw the probability teaching sequence as having affective and cognitive benefits for them, as well as the students. In addition, we noted a strong degree of support in terms of using this or a similar teaching sequence in their later teaching career. Lesson time constraints and class management were among the few challenges mentioned by the pre-service teachers if they were to implement the teaching sequences. Overall, our findings suggest that pre-service teachers find the probability teaching sequence useful and they could derive useful teaching ideas by engaging in this game-based teaching intervention.

From a socio-cultural perspective of learning, we note how our participants could challenge and modify their probability teaching ideas. Exposing pre-service teachers to such activities could be seen as extending their ZFM. However, only a few participants were able to suggest actual teaching ideas, yet suggesting some very general ideas which can be seen as a development of their ZPD. The fact that these pre-service teachers were cognisant of the teaching challenges suggests that while teachers may have noble ideas or intentions, not

all of it could be easily translated into action (Goos, 2014). In terms of future research, we intend to follow a small sample of our participants, with an aim to explore if and how they implement these ideas in their classroom.

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Language games in primary mathematics

Patrick Galvin
The University of Queensland
<p.galvin@uqconnect.edu.au>

This theoretical paper examines views about the role of language and mathematical discourse in learning mathematics. Current research is still addressing what constitutes a mathematical discourse. As new conceptions of the purpose of language use in mathematics are explored, and associated ontological and epistemic positions are revealed, one might ask: how are we able to reframe our view of language to support a social participation perspective? This paper proposes the consideration of Wittgenstein's philosophy of language games to shift our conception of classroom language use in mathematics to encompass broader contextual features such as participation, patterns of exchange and social norms.

This theoretical paper examines sociocultural theories and practices that considers language as central to learning mathematics. Underpinning these theories and practices is the notion of a strong connection between talking and thinking where social interaction impacts on learning (Barwell, 2018; Sfard, 2007; Vygotsky, 1978). Discourse practices recognise that there are many different factors that contribute to build meaning in a mathematical situation (Moschkovich, 2019). These factors may include the use of symbols or physical materials and written as well as verbal language (Moschkovich, 2019). Importantly for the theme of this paper, a mathematical discourse considers all uses of verbal language, or utterances, to support meaning. Informal language use is not disregarded. Research has demonstrated that particular discourse practices in mathematics assists students to engage more deeply in learning, building meaning, and knowledge in mathematics (Barwell, 2016).

Exploring conceptions of learning, meaning and knowledge relating to language can reveal the influence of an ontological perspective. Stretching the concept of language use to embrace a broader notion of what can be considered a mathematical discourse may involve finding new ways to see language. It is expected that the development of new forms of language use in learning mathematics can be supported by a corresponding shift in underpinning ontology (Murphy, 2015). Exploring Wittgenstein's notion of language games (1953) is a possible means of allowing such a shift (Standish, 1995).

This paper aims to examine how sociocultural theories influence a view of language use in the learning of mathematics; in particular, I attempt to reframe the view of language to support a social participation perspective. I propose that an interpretation of Wittgenstein's concept of language games, which is underpinned by social participation, can be helpful by providing a perspective of classroom language use that avoids seeing words as autonomous entities. Overemphasis on the use of specific words and terms can result in a narrow view of language use in learning mathematics (Barwell, 2016). This view prioritises the correct use of technical vocabulary or formal academic language. Instead, the idea of language games focuses attention on the broader contextual features in which talk occurs, such as participation, patterns of exchange, and social norms.

Wittgenstein's Language Games

Wittgenstein (1953) aimed to demonstrate that words are not defined by reference to the objects they designate, nor by mental representations one might associate with them, but by

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how they are used in the context of social activity. He challenged the idea that the meaning of words is anchored by invariable rules that can be demonstrated in acts of ostensive definition. Wittgenstein also opposed the notion that the rule for how to use a word can be abstracted from all particular uses. The meaning of the word is the use of it, which can only be learnt through such use with other language users. Wittgenstein questioned the idea that we can come to understand the essential meaning or essence of a word. He asked whether the word or concept of game has an essence that can meaningfully be defined in certain terms such as necessary or sufficient conditions (Wittgenstein, 1953):

> Consider … proceedings we call games, I mean all games, card games, board games, Olympic games and so on. What is common to them all? ... If you look at them you will not see something that is common to all, but similarities, relationships, and a whole series of them at that. (p. 66)

Wittgenstein puts forward the idea of language games to illustrate the point that without considering use in context it can be nonsensical to theorise about what words mean; that understanding and meaning are inextricable from the social contexts within which speakers interact. The notion of language games is used to help us to see that the rules that guide how words are used are embedded in the social contexts of such use; they are part of a "form of life" (Wittgenstein, 1953, p. 68).

The idea of a pragmatic theory of meaning contrasts with many commonly held views about how language operates in mathematics (Moschkovich, 2019). The notion that mathematical terms are tightly defined can result in such definitions being placed front and centre as a language feature in learning experiences (Strom et al., 2001). Rather than viewing the meaning of mathematical terms as fixed and the rules by which they are used as invariable we might seek to understand, instead, what are the norms or rules of the language games being played and in which contexts do language experiences support learning?

Wittgenstein's idea of language games does not provide a model of how mathematical discourse should look. Neither are language games part of a theoretical framework that can be mechanically applied. I am suggesting that language games are a way of seeing a mathematical discourse that looks beyond particular words and phrases and attempts to describe the overall purpose of the mathematical activity. The purpose is described in terms of social participation. For example, a language game could be one in which students appear to make a genuine effort to engage with others' ideas. The purpose of this game might be described as recognising other peoples' thinking. A language game could be one that involves trying to trump or better the previous speaker and the purpose is one-upmanship or winning. Yet another game could involve the teacher playing a catch-and-pass role. They chair a discussion by distributing contributions without comment or rephrasing. The purpose is to increase fluent exchange between interlocutors and support connection between ideas. There is not one type of language game, as there is no monolithic form of language (Moschovich, 2019). A description of language games is not intended to be definitive. Using a language games perspective aims to provide a way for teachers and researchers to look at a mathematical discourse that allows a connotation of meaning in terms of purposes for social participation.

References to Wittgenstein's importance for education often acknowledge his influence in providing an alternative view of the role of philosophy and note a corresponding shift in epistemological and ontological viewpoints (Standish, 1995). As new ideas for the purpose of education and the nature of learning are explored, a means of supporting the shift in ways of seeing, analysing, conceiving and acting as researchers and practitioners will also be required. For example, Wittgenstein's opposition to Cartesian conceptions of mind and understanding allows us to reframe our view of the nature of learning and knowledge

(Smeyers, 1998). A change of approach recognises that overvaluing the use of technical or formal mathematical language can be inhibiting rather than enabling and that informal or natural dialogue can be effectively blocked. Viewing the language of mathematics too narrowly can fail to allow the natural use of language to discuss, explain and reason. Such a view can hinder the process of inducting children into mathematical practices (Wagner & Andersson, 2018).

As young children are initiated into the practice of mathematics, they will already be exploring how they can engage in certain discourses to express and develop their thinking. Rather than constraining or obstructing natural use by maintaining too closed a view of how mathematical language should look, emphasis is placed on looking for natural use of language to develop. As a theoretical lens, the idea of language games allows a view of the broader contextual features in which a mathematical discourse occurs.

The following sections will consider sociocultural theories and practices in relation to developing classroom discourse for mathematics. It offers reflections of how Wittgenstein's language games potentially provide a lens for viewing the development of exploratory talk.

Exploratory Talk

A focus of research into classroom language use has been to distinguish between different types of talk. Talk that is rote learnt through repeated procedure or ritual can be considered essential to formative stages of learning (Sfard, 2007). In these formative stages, the role of a teacher is to model and shape how language is being used by students. However, highly practised forms of talk could be considered exemplifications for all classroom language use. Such a view can be normatively restricting. While ritualised forms of language use may be necessary for early initiation into new learning, it is thought that progression through later stages of learning requires more creative and generative uses of language (Sfard, 2007). Exploratory talk involves student-initiated language use that actively communicates about and negotiates meaning. As exploratory talk develops, patterns of classroom language use might be tentative, incomplete or fragmented yet allow for inventive purposes for talk. Overemphasis on polished forms of public speaking, or presentational talk, and on the correct use of formal language, can hinder opportunities for exploratory talk (Barnes, 1976).

The goal of supporting children as they develop use of exploratory talk has been researched on the difference between characterising mathematical language use as 'playing-with' and 'playing-at' (Fleener et al., 2004). Playing-with language use is seen as generative and employed by students to actively invent contexts to extend meanings. In contrast, 'playing-at' language use is considered to be evident when a student merely attempts to provide the teacher with an expected response. The development of exploratory talk requires that teachers are able to recognise and create opportunities for this form of language exchange. Having a tuned ear to help guide or shape verbal exchanges towards exploratory talk is an important skill, as outlined above by the various talk moves a teacher can employ. However, such hermeneutical listening is not easily achieved. To support exploratory talk, teachers are required to use interpretive listening to allow students to expand and relate meanings rather than narrowing them. Attempts to support 'playing-with' language uses will collapse into 'playing-at' games if the teacher appears to feel the need to seek closure to the learning episode and feels pressured to ensure that students have used acceptable mathematical terms and phrases (Fleener et al., 2004).

Using language games as a lens can provide a number of insights into the failure of 'playing-with' language games: It is difficult for a teacher to avoid authoritative control and

to use interpretive listening to guide their own participation (Fleener er al., 2004). The perceived need for students to use mathematical terms correctly can restrict opportunities for exploratory talk. There also seems to be a tendency for both teachers and researchers to focus on the use of specific words or terms rather than notice patterns of exchange or attempts to convey meaning using informal language.

Dialogic Pedagogies

Researchers have identified features of teaching and learning that support the development of dialogue (Hardman, 2019). Common to such dialogic pedagogies are talk-intensive practices that encourage students to engage in extended discourse to share and build a common understanding (Snell & Lefstein, 2018). Dialogic pedagogies are motivated by the idea often attributed to Vygotsky (1978) that regularly engaging in dialogue of a certain nature supports the ability to internalise a reasoning dialogue. An essential component of dialogic theories is the importance of learners interacting with others, including the teacher.

It has been recognised that the development of a dialogic pedagogy takes a certain skill set of the teacher (Khong et al., 2019). Research has aimed to explore and describe effective roles for teachers to provide practical support within classrooms. These roles include asking probing and clarifying questions, encouraging students to elaborate on their ideas, acknowledging and validating students' proposals, and encouraging sustained discussion (Sedova et al., 2019). Such 'talk moves' are designed to help teachers to interact with students and are also used to prompt and encourage peer-to-peer interaction. Different focuses of research into talk moves include: initial moves to engage discussion, moves to follow up ideas, moves to encourage students to interact with each other's ideas and moves to make student thinking visible (Ritchhart et al., 2011; Webb et al., 2014). Encouraging students to relate their thinking to a previous expression is an example of talk move that helps to build connections between ideas and prompt interaction.

Dialogic pedagogies emphasise the importance of collective participation and surfacing social norms that guide and shape the purpose of talk. Describing the purpose of a language game will also surface social norms. A language game could be one sided or balanced and interactive. A language game might prioritise authoritative use of technical language or allow novice attempts and informal expressions. A language game is a situated, social activity. Describing a classroom language game makes explicit the purpose, manner or intent of social participation.

Philosophical Positions

Opportunities for the development of exploratory talk may require more than teachers employing a set of techniques. It may also help if ontologies or epistemologies are reframed. The normative persuasion of a received ontology can imply that a shift in a teacher's views about knowledge is required to support the introduction of exploratory talk in mathematics (Murphy, 2015).

Ontological and epistemological views of mathematical knowledge will likely translate into different approaches towards engaging students in talk when learning mathematics. For example, a positivist perspective that sees mathematical knowledge as a set of stable patterns or universal invariants will likely influence teachers to lead students towards making correct interpretations (Radford, 2006). From this perspective, talk is more likely to be viewed merely as a means of reporting. For example, talk is used to allow students to report the

current state of their knowledge. Alternatively, a non-positivist perspective, which sees learning in mathematics as a generative process of meaning making or gaining understanding, will frame a view of knowledge in different terms. Exploratory talk is associated with the concept that knowledge is generated through collectively social activities (Barwell, 2018). So, increasing opportunities for use of exploratory talk in classrooms would appear to require that teachers are able to shift or reframe their epistemological perspectives.

How a teacher participates in mathematical talk with students could provide some insight into their views about mathematical knowledge. Using a language games lens, a teacher's influence on patterns of language use can be interpreted to uncover tacit beliefs about the purpose of language and the status of mathematical knowledge. If there is a causal connection, connections can be inferred between teacher ontology and observable features of classroom discourse. Increasing opportunities for exploratory talk may then require shifting a teacher's views about the nature of mathematical knowledge.

Learning-as-Participation

If interpersonal language use is seen to be necessary for the development of thinking then language exchanges and children's participation in such exchanges, with each other and with the teacher, are central to learning. Through our participation with other language users we become able to use language ourselves and develop our own thinking. This social participation approach sees learning mathematics as an initiation into using language in new ways. Learning is defined by participation in social practices rather than the acquisition of concepts or knowledge. Here the conception of learning and knowledge is reframed. The enterprise of learning mathematics is seen as becoming initiated into using a mathematical discourse and the goal is for students to eventually become participants in the use of exploratory talk (Sfard, 2007).

From this perspective, language is considered in much broader terms than just involving the utterance of words or phrases. As many features of a context are considered to give sense to the social activity in which language use occurs, it is no longer possible to examine language as an isolated or autonomous phenomenon (Gee, 2014). Ontological implications associated with the concept of discourse can appear to contradict commonly held views about the nature of mathematical knowledge. This conflict arises when the effect of background influences in shaping meaning appear in the concept of discourse. These background influences are often implicit, but powerful factors which are posited by sociocultural theories of language to shape the overall meaning and intention of a discourse.

Common to sociocultural theories of language is the idea that the terms of exchange take their meaning, intention or purpose from the contexts in which they are used. However, any attempt to pin down or isolate what it is about a particular context that conveys meaning to the discourse situated therein can seem impossible when considering a myriad of possible features (Gee, 2014). Further, the notion of context is not restricted to any particular instance of use, but extends to all previous uses. Terms of exchange have historical context: meaning has been shaped and formed through all previous uses and continues to be reshaped by each particular instance of use. In this view, language appears to be a fluid phenomenon with innumerable factors that influence meaning (Sierpinska & Lerman, 1996).

A language games perspective is consistent with a view of learning mathematics in discursive terms. Knowing mathematics is seen to be synonymous with being able to participate in a mathematical discourse (Sfard, 2007). However, viewing this participation and recognising forms of engagement does not necessarily require that we attempt to identify definitive sources of meaning. Wittgenstein suggests that philosophical theorising about

ideas such as certainty or meaning can lead us to have unrealistic expectations about language. The idea of language games is useful in allowing us to escape the trappings of theoretical dogmatism. That is, thinking that we need to pinpoint the meaning of terms used in a mathematical discourse is based on the idea that there are direct referents for the meaning of terms. A language games perspective is not based on this idea of objectivity. Using a language games lens involves looking in an adaptable and flexible way at the meaning of mathematical communication within social contexts.

Everyday Language and Mathematical Discourse

Proponents of a view of classroom mathematical language use that recognises a broad conception of contextual meaning emphasise that natural or ordinary language use allows for less complicated assimilation of practice (Moschovich, 2019). The ease of using everyday language can be contrasted with the difficulty of learning technical or formal language. A distinction between everyday language and academic language seems straightforward. However, some researchers argue that this distinction oversimplifies the complexities of relationships between language, communication, and learning (Moschkovich, 2019). It is then recommended that everyday and school mathematical practices are not presented as a dichotomous distinction (Gutierrez et al., 2010; Schleppegrell, 2010).

While cautioning us to avoid drawing impermeable lines between everyday and mathematical language uses, Moschkovich (2019) does see value in clarifying the differences between mathematical ways of talking and formal ways of talking mathematically. Here, we are asked to open our conception to a broader view of what an authentic mathematical discourse can be in a classroom. We are encouraged to move away from a simplified view of language framed in terms of words, phrases, vocabulary or a set of definitions and expand our view of the mathematics register. The proposed shift of focus is towards reasoning rather than accuracy and towards precision as an object of inquiry rather than a requirement of engagement: “instruction should move away from interpreting precision to mean using the precise word, and instead focus on how precision works in mathematical practices” (Moschkovich, 2019, p. 6). We are asked to share a progressive view of mathematical discourse that allows language use to flourish with attention on active negotiation of meaning within mathematical situations.

Likewise, avoiding an instrumentalist view that sees mathematics and language as sets of tools or competencies that provide a means to an end can allow us to see mathematics as a way of thinking or reasoning which is part of our general existence; “the capacity to think mathematically is inseparable from the capacity to reason in general and should be seen as an essential part of the latter” (Rider, 2017, p. 504). Rejecting the assumption that a child’s world is not in some way mathematical before they enter school helps to reframe our enquiry into practices of instruction; the problem of “how can mathematics instruction recognise the pupil’s experience?” is misconceived from the outset. The question should rather be “how can instruction make children recognise the mathematical in their experience?” (Rider, 2017, p. 511).

The question of what constitutes a mathematical discourse could be considered pivotal for theories that see learning in discursive terms. But rather than seeing the benefit of such theories hinge on a need to define what is meant by a mathematical discourse, they can be considered useful in providing a perspective for inquiry that explores this very notion. Using the idea of language games to see students as participants in discourse practices might reveal complexities, such as the relationship between everyday and mathematical discourses. This

perspective could help teachers and researchers shift away from oversimplified views of language (Barwell, 2016). Seeing learning mathematics in discursive terms is not an attempt to provide a definitive description of a mathematical discourse, but a way to view how classroom language is actually being used within rich social contexts as students grapple with new mathematical situations.

Conclusion

Learning can be seen as the change that takes place as students become participants in a mathematical discourse. A view of learning mathematics in discursive terms emphasises the importance of patterns of social interaction and recognises progression of learning in mathematics as a move towards more uses of exploratory talk (Sfard, 2007). Exploratory talk is thought to extend learning in mathematics by allowing generative and collaborative discourse (Murphy, 2015). The adoption of dialogic pedagogies may benefit this form of classroom talk. However, overemphasis on the correct use of formal academic language can impede the development of exploratory talk in learning mathematics (Barwell, 2016). In discursive terms, rather than seeing mathematical terms as autonomous and with objective referents, the broader context of a mathematical discourse is considered to give meaning and purpose to learning. Thus, Wittgenstein's idea of language games is suggested as a useful perspective for seeing learning mathematics in discursive terms. This perspective could be useful in providing insight into the influence of a teacher's views about mathematical knowledge on the development of exploratory talk. Language games could also support the development of an expanded view of a mathematical discourse.

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Insights into the pedagogical practices of out-of-field, in-field, and upskilled teachers of mathematics

Merrilyn Goos
University of Limerick
<merrilyn.goos@ul.ie>

Aoife Guerin
University of Limerick
<aoife.guerin@ul.ie>

"Out-of-field teaching" is an international phenomenon that seems particularly prevalent in mathematics. Our study is evaluating the impact of a national professional learning program for out-of-field secondary mathematics teachers in Ireland. Using the Productive Pedagogies framework, we compared the pedagogical practices of three pairs of teachers who were either upskilled, still out-of-field, or always in-field. The findings suggest that graduates of the upskilling program are developing pedagogical practices more like those of in-field teachers.

"Out-of-field" teaching is an international phenomenon that involves teachers being assigned to teach subjects that do not match their training or education (Ingersoll, 2002). This practice seems particularly prevalent in the teaching of mathematics. Out-of-field teachers of mathematics typically possess a teaching qualification but have limited advanced studies of mathematics and little or no specific preparation in mathematics pedagogy. There is growing recognition of the need for professional development programs that meet the particular needs of out-of-field teachers (Du Plessis et al., 2014). To date, however, there has been little research on the effectiveness of such programs (Faulkner et al., 2019). This paper reports on aspects of a larger study that is evaluating the impact of a long-term, large-scale, government-funded, nationally-consistent, university-accredited program offered to out-of-field teachers of mathematics in Ireland – the *Professional Diploma in Mathematics for Teaching* (PDMT).

Background to the Study

In Ireland, concerns about student performance in post-primary school mathematics at the beginning of the 21st century led to the introduction in 2010 of a new curriculum that shifted emphasis towards understanding and problem-solving and away from memorisation and procedures (National Council for Curriculum and Assessment, 2005). Concurrently, the Teaching Council of Ireland (2013) introduced new accreditation requirements for initial teacher education programs. In mathematics, fully qualified teachers must have a degree-level qualification with the specific study of mathematics comprising at least one-third of the degree. There are also minimum credit requirements in analysis, algebra, geometry, and probability and statistics, with additional credits to be obtained in a variety of optional topics. Despite these strict requirements, school principals in Ireland have autonomy in recruiting staff and assigning teachers to subjects and classes, thus leaving open the possibility of placing teachers in out-of-field positions.

Ní Ríordáin and Hannigan (2009) speculated that the phenomenon of out-of-field teaching of mathematics could be a possible obstacle to achieving the goals of the new mathematics curriculum. They conducted a national survey of teachers of mathematics in Irish post-primary schools, collecting data on respondents' teaching assignments, degree qualifications, and the subjects they were qualified to teach according to the requirements specified by the Irish Teaching Council. This survey established that 48% of respondents were teaching mathematics without the necessary subject-specific qualifications. In response to this finding, the Department of Education and Skills (DES) funded the PDMT to develop

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the content and pedagogical content knowledge of out-of-field teachers of mathematics to the level required by the Teaching Council. Six cohorts comprising 1078 teachers participated in the PDMT from 2012-2020.

The PDMT is a 2-year part-time postgraduate program with teachers' tuition fees funded by the DES. Delivery of the program is led by the University of Limerick in conjunction with a national consortium of higher education institutions. PDMT participants teach full-time in their schools while they undertake the program in the evening, on weekends, and during school vacations via a blended learning approach. Ten undergraduate mathematics modules are delivered online in 30-hour blocks across 6-week sessions, with additional face-to-face and online support. Two yearlong mathematics pedagogy modules are delivered face-to-face via weekend workshops and a one-week summer school. These pedagogy modules emphasise classroom practices that support problem-solving and promote conceptual understanding. One of the pedagogy modules also requires participants to complete a supervised action research project on their practice in the mathematics classroom.

An important aim of the PDMT is to develop out-of-field teachers' knowledge of mathematics content and pedagogy. The program additionally aims to support teachers in developing pedagogical practices aligned with the goals of the new mathematics curriculum in Ireland, and this is the focus of the present paper. To gain insights into the latter aspect of the PDMT, we compared video-recorded mathematics lessons taught by teachers who were currently, formerly, or never out-of-field in order to address the following research question: *What similarities and differences can be observed when comparing the pedagogical practices of out-of field, upskilled, and in-field teachers of mathematics?* (Upskilled teachers are those who have completed the PDMT.)

Conceptualising and Evidencing the Impact of Professional Development

Researching the impact of teacher professional development poses methodological and conceptual challenges. Desimone (2009) discussed the strengths, weaknesses, and trade-offs between observations, interviews, and surveys as the most common methods for studying teacher learning, and stressed the importance of choosing data collection methods to match a study's research questions. Adler et al. (2005) also pointed out that a personal investment in teaching makes it difficult for teacher educators to take a critical stance towards the research we do with teachers, and they suggested developing strong theoretical languages in order to distance ourselves from what we are looking at. In the present study, as the authors have the dual roles of researchers and teacher educators in the PDMT, we aimed to achieve this critical distance by situating our research within Desimone's (2009) conceptual framework for studying teacher professional development.

Desimone's (2009) framework has two components. The first component identifies the critical features that define effective professional development in terms of increasing teacher knowledge and skills and improving their practice. Drawing on existing empirical research, Desimone proposed that this set of critical features places emphasis on: (a) content focus, (b) active learning, (c) coherence, (d) duration, and (e) collective participation. The second component of the conceptual framework is "an operational theory of how professional development works to influence teacher and student outcomes" (p. 184). For this component, Desimone proposed a model with the following steps:

1. Teachers experience effective professional development (defined in terms of the set of critical features outlined above).
2. The professional development increases teachers' knowledge and skills and/or changes their attitudes and beliefs.

3. Teachers use their new knowledge and skills, attitudes, and beliefs to improve the content of their instruction or their approach to pedagogy, or both.
4. The instructional changes foster increased student learning. (p. 184)

Desimone (2009) acknowledged that other potentially important factors existed, but these were not incorporated into her model because they have not yet been the subject of much research on the impact of professional development. These factors might include, for example, professional identity (Hobbs, 2012), the role of the principal in providing opportunities for teacher learning (Du Plessis et al., 2015), and the role of curriculum materials and implementation (Remillard & Heck, 2014). Desimone also conceded that her model could be criticised as representing a positivist viewpoint. However, she maintained that the model could still be used in studies with different theoretical perspectives on teacher learning as a means of integrating the knowledge generated by empirical research with "the emerging consensus of what is good professional development" (p. 187).

Desimone (2009) noted that it is rare for a single study to investigate all four elements of her proposed model; in particular, there are significant methodological difficulties in designing evaluations that measure the effects of professional development on student achievement. Research conducted by our larger team has analysed the critical features of the PDMT program (Step 1 in Desimone's model; see Goos et al., 2020) and its effect on the teachers who participated in the program (Steps 2 and 3; see Lane & Ní Ríordáin, 2020; Ní Ríordáin et al., 2017). In this paper, we further examine the impact of the PDMT on teachers' *pedagogical practices* (Step 3) as a key element in Desimone's model of teacher change.

Research Design and Methods

We would have liked to investigate the effects of the PDMT on participants' classroom teaching approaches by observing lessons taught before and after the teachers experienced the program. However, this was not possible due to resource constraints and the demands of delivering a large, complex program involving 13 higher education institutions. Our research team's earlier analysis of PDMT participants' action research reports indicated that teachers perceived a shift in their pedagogical practices towards more student-centred approaches that emphasised conceptual understanding and problem-solving (Lane & Ní Ríordáin, 2019). To further investigate these teacher self-reports, we designed a cross-sectional study to compare the pedagogical practices of three groups of teachers: (a) those currently teaching mathematics out-of-field (n=2); (b) those who had been upskilled to fully qualified status by completing the PDMT (n=2); and (c) those who had always been fully qualified, in-field teachers of mathematics (n=2). These six teachers were recruited from six different schools.

Teachers were observed by the second author as they taught six junior secondary mathematics lessons in two blocks of three consecutive lessons. These lessons were also video-recorded for later analysis. Pre- and post-lesson interviews were conducted by the second author to obtain teachers' perspectives on lesson objectives, anticipated and actual challenges or successes, knowledge, and confidence levels. Surveys were also administered to the teachers to collect demographic information and data on teacher self-efficacy, job satisfaction, and preparedness for teaching topics in the secondary mathematics curriculum. All data collection was carried out by the second author. This paper draws only on teacher demographic data and the video recordings of lessons they taught.

The Productive Pedagogies framework was selected as a classroom observation instrument that has been theoretically and statistically validated in Australian research (Lingard et al., 2001). Although not specifically designed for mathematics classrooms, it has been used in longitudinal studies of mathematics teaching (e.g., Makar, 2011) as well as in

large-scale studies of primary and secondary school lessons in a range of curriculum areas. The 20 items of the Productive Pedagogies framework are shown in Figure 1. The framework has four dimensions, two concerned with the academic outcomes of schooling (left side of Figure 1) and two with the social outcomes (right side of Figure 1). The *Intellectual Quality* dimension emphasises the importance of all students being presented with challenging work. *Connectedness* makes learning meaningful by linking new knowledge to prior knowledge, other subjects in the curriculum, and the world beyond school. *Supportive Classroom Environment* foregrounds relationships and giving students a voice in the classroom, while *Recognition of Difference* provides students with the capacity to act as responsible members of a democratic society. A 5-point rating scale is used to provide an index of the variation in quality of classroom practice for each item.

Intellectual Quality Higher order thinking (HOT) Deep knowledge (DK) Deep understanding (DU) Substantive conversation (SC) Problematic knowledge (PK) Meta-language (ML)	Supportive Classroom Environment Student direction (SD) Social support (SS) Academic engagement (AE) Explicit quality performance criteria (EC) Student self-regulation (SS)
Connectedness Knowledge integration (KI) Background knowledge (BK) Problem-based curriculum (PBC) Connectedness beyond the classroom (CBC)	Recognition of Difference Cultural knowledge (CK) Inclusivity (I) Narrative (N) Group identities (GI) Active citizenship (AC)

Figure 1. Productive Pedagogies dimensions.

Before observing and video-recording lessons taught by the six teachers, the second author discussed the Productive Pedagogies scoring manual with the first author, who is an experienced user of the Productive Pedagogies framework. Both authors used the scoring manual independently to rate an online video of a junior secondary mathematics lesson, after which they compared their ratings and resolved any differences via further discussion. After the data collection was completed, the second author watched the video-recorded lessons, assigned scores for each item, and calculated mean scores on each dimension for each of the three types of teachers (out-of-field, upskilled, in-field). Similarities and differences between the teachers were further examined for each dimension by inspecting item scores.

Results

Demographic Data

Table 1 summarises the gender, years of mathematics teaching experience, and grouping (out-of-field, upskilled, in-field) of the participating teachers. Both out-of-field teachers were female and had taught mathematics for up to 10 years; the other teachers were male with mathematics teaching experience ranging from less than five to more than 16 years. Table 1 also shows the year in which upskilled and in-field teachers gained their mathematics teaching qualifications through the PDMT or initial teacher education program respectively.

Table 1
Teacher Demographic Characteristics

Characteristic	Teacher					
	T1	T2	T3	T4	T5	T6
Gender/ Group	M US	M IF	F OOF	M US	F OOF	M IF
Years teaching mathematics (year qualified)	16-20 (2018)	11-15 (1999)	<5 (n/a)	<5 (2018)	6-10 (n/a)	6-10 (2010)

Note. OOF = out-of-field; US = upskilled; IF = in-field

Pedagogical Practices

Table 2 presents the mean scores on the Productive Pedagogies dimensions for each group of teachers over the three lessons for which they were observed. Thus, each mean score is derived from six observations (two teachers × three lessons). One observable trend is that out-of-field, upskilled, and in-field teachers all scored highest on the dimension of Supportive Classroom Environment and lowest on the dimension of Connectedness. The same pattern was found in Makar's (2011) analysis of pedagogical practices in Australian primary school teachers' "regular" mathematics lessons.

Table 2
Productive Pedagogies Mean Scores

Dimension	Teacher Group		
	Out-of-Field	Upskilled	In-Field
Intellectual Quality	2.64	3.00	3.61
Connectedness	1.54	1.79	1.75
Supportive Classroom Environment	3.67	3.27	4.07
Recognition of Difference	3.10	2.23	2.57

Note. A 5-point rating scale was used. Each group comprises two teachers who were observed for three lessons.

Looking across the rows of Table 2 enables comparison between the three groups of teachers on each Productive Pedagogies dimension. In-field teachers had the highest mean scores for the dimensions of Intellectual Quality and Supportive Classroom Environment, while upskilled teachers recorded the highest mean score for Connectedness – although this was very similar to the mean score of the in-field teachers. Out-of-field teachers achieved the highest mean score for the dimension of Recognition of Difference. This may be because they were the only teachers in the sample who taught mixed-ability, rather than streamed, mathematics classes. These two teachers were observed to place particular emphasis on encouraging participation of struggling students, thus highlighting the element of Inclusivity (Figure 1) for this non-dominant group in their classrooms.

Because the PDMT is mainly concerned with teaching mathematics for academic outcomes, we next examine the detail of teachers' pedagogical practices in the corresponding dimensions of Intellectual Quality and Connectedness. Tables 3 and 4 show each teacher's score totals for the three observed lessons for each item of these dimensions. (Score totals

are displayed instead of mean scores for ease of comparison across multiple teachers and items.) Pedagogical practices that seem to characterise the greatest difference between teacher groups are highlighted for discussion.

Table 3
Intellectual Quality Score Totals

Item	Out-of-Field		Upskilled		In-Field	
	T3	T5	T1	T4	T2	T6
Higher Order Thinking	8	8	9	10	8	15
Deep Knowledge	9	9	11	13	12	15
Deep Understanding	9	12	10	10	12	12
Substantive Conversation	5	9	5	10	8	9
Problematic Knowledge	6	6	5	8	11	12
Meta-language	5	9	9	8	12	5

Note. A 5-point rating scale was used. Each teacher was observed for three lessons.

Within the dimension of Intellectual Quality, the greatest differences – equivalent to at least 6 points across three lessons, or a mean of 2 points per lesson on the 5-point observation scale – occurred on the items representing Higher Order Thinking, Deep Knowledge, and Problematic Knowledge (Table 3). The general trend is for the scores to increase from out-of-field to upskilled to in-field teachers. Also notable is the high Meta-language score for in-field teacher T2, who regularly provided help in the use of mathematical terminology for students who had been identified with low literacy skills.

Figure 2 provides examples of questions posed by Teacher 5 (out-of-field), Teacher 4 (upskilled) and Teacher 6 (in-field) that illustrate differences in the quality of their pedagogies for promoting Higher Order Thinking. According to the Productive Pedagogies classroom observation manual, Higher Order Thinking requires students to manipulate information and ideas in ways that transform their meaning and implications, for example by synthesising, generalising, explaining, or arriving at a conclusion or interpretation. This level of thinking is evident in the question asked by Teacher 6, and to some extent by Teacher 4. However, Teacher 5's question only requires students to rehearse procedural routines.

T5 (OOF) Solving equations	T4 (US) Simultaneous equations	T6 (IF) Introducing simultaneous equations
Q. Now what happens if I have the scales and I take 8 away from 12 on the RHS? Q. What did she do to both sides of the equation?	Q. How do I get this $-3x$ to become an x? Q. Can you explain to me what you did?	Q. Which of these equations do you think is the hardest to solve? Why? 1) $95x^2 - 2x + 105 = 0$ 2) $3x + 2y = 8$ 3) $9x^4 - 39x^3 + 9x^2 - 90x + 3035 = 0$ 4) $\sqrt[4]{9x^4} - \frac{9}{2}x^3 - 4^{5x} - 87x = 0$

Figure 2. Examples of teacher questions illustrating variation in promotion of Higher Order Thinking.

For the dimension of Connectedness, the differences between teacher groups were less pronounced – perhaps as a consequence of the lower scores across all three groups (see Table 2). The greatest difference – equivalent to at least 3 points across three lessons, or a mean of

1 point per lesson on the 5-point observation scale – occurred on the item representing Problem-Based Curriculum (Table 4). In line with the Intellectual Quality dimension, the trend here is for scores to increase from out-of-field to upskilled to in-field teachers.

Table 4
Connectedness Score Totals

Item	Out-of-Field		Upskilled		In-Field	
	T3	T5	T1	T4	T2	T6
Knowledge Integration	3	3	4	3	3	3
Background Knowledge	6	7	7	6	6	6
Problem-Based Curriculum	6	6	7	9	8	10
Connectedness Beyond the Classroom	3	3	4	3	3	3

Note. A 5-point rating scale was used. Each teacher was observed for three lessons.

Figure 3 shows examples of tasks presented by Teacher 3 (out-of-field), Teacher 1 (upskilled) and Teacher 2 (in-field) that illustrate differences in the quality of their pedagogies for promoting a Problem-Based Curriculum. The Productive Pedagogies classroom observation manual defines a problem as a task with no specified correct solution that requires knowledge construction on the part of students. In keeping with the mathematics education research literature, we re-interpreted this definition to mean that a *mathematical problem* is a task for which the student does not know, and needs to construct, the *solution method* (National Council of Teachers of Mathematics, 2000). There is some evidence that this kind of knowledge construction is called for in the tasks offered by Teacher 2 and Teacher 1; however, the task set by Teacher 3 instead requires using well-defined algorithms for algebraic manipulation.

T3 (OOF)	T1 (US)	T2 (IF)
Expanding and simplifying $4(x+2)$ $6(a+4)+2(2a+3)$	Introducing patterns x, y, z, x, y, z, x, y, z, … What letter is in the 63rd position?	Introducing Pythagoras' Theorem (Counting up boxes in the squares of the sides) … How many boxes should be in here (the square on the hypotenuse) based on what we did earlier?

Figure 3. Examples of tasks illustrating variation in problem-based lessons

Conclusion

In this paper, our focus was on the extent to which the PDMT encouraged teachers to take up pedagogical practices that emphasise conceptual understanding and problem-solving, in line with Ireland's new secondary mathematics curriculum. Because it was not possible to collect longitudinal data on PDMT participants, we instead designed a cross-sectional study to identify similarities and differences between these upskilled teachers and other teachers of mathematics who were still out-of-field or had always been in-field. This design does not allow us to make claims about causality in relation to the PDMT, but it does illuminate some interesting comparisons between these three groups of teachers. The groups were *similar* in that out-of-field, upskilled, and in-field teachers all scored highest on Supportive Classroom Environment and lowest on Connectedness, a finding that aligns with

previous research using the Productive Pedagogies protocol (Makar, 2011). Some of the *differences* between groups suggested that upskilled teachers (PDMT graduates) might be adopting pedagogical practices more like those of in-field teachers than those who are still teaching mathematics out-of-field, especially in relation to promoting Intellectual Quality and Connectedness. These conclusions can only be tentative, given the small sample, but they suggest that structured lesson observations can usefully supplement upskilled teachers' self-reports of changes in their pedagogical practices arising from participation in a targeted professional development program. In addition, such structured lesson observations may be useful for informing the design of programs to develop out-of-field teachers' (and also pre-service teachers') *knowledge* of mathematics and pedagogical *practices*, particularly in pinpointing specific items within the academic outcomes of schooling that require further consideration (e.g., knowledge integration and connectedness beyond the classroom).

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Noticing structural thinking through the CRIG framework of mathematical structure

Mark Gronow
Central Queensland University
<m.gronow@cqu.edu.au>

Structural thinking skills should be developed as a prerequisite for a young person's future mathematical understanding and a teachers' understanding of mathematical structure is necessary to develop students' structural thinking skills. In this study, three secondary mathematics pre-service teachers (PSTs) learned to notice structural thinking through the CRIG framework of mathematical structure. The framework consists of Connections, Recognising patterns, Identifying similarities and difference, and Generalising and reasoning. I report here on how the CRIG framework helped the PSTs' notice structural thinking.

To develop an ability to notice structural thinking, teachers must first of all be aware of *mathematical structure*. Mason et al. (2009) defined mathematical structure as "the identification of general properties which are instantiated in a particular situation as relationships between elements or subsets or elements of a set" (p. 10). Stephens (2008) described structural thinking as an awareness of how mathematical properties develop into generalisations. Furthermore, Mason et al. (2009) promoted structural thinking as understanding the concepts and knowing procedures to use and when solving mathematical problems.

Varied theories exist about structure; as mathematical structure or structural thinking. Wertheimer (1945) proposed that mathematical structure is knowing how a formula is connected to a mathematical concept. Hiebert and Lefevre (1986) combined conceptual and procedural knowledge as 'proceptual' thinking across mathematical processes. Stephens (2008) defined 'structure' as synonymous with relational thinking (Skemp, 1976). Schwarz et al. (2009) proposed that structural thinking is knowing the relationships and connections between mathematical concepts.

The concept of structural thinking in mathematics is not clearly understood by many teachers of mathematics (Richland et al., 2012). Mason et al. (2009) stated that teachers' awareness of structural relationships transforms students' thinking and disposition to engage, they believe that teachers need to focus on structure so students can think structurally. Research in teachers' awareness of mathematical structure or structural thinking is limited. Gronow et al. (2020) explored secondary mathematics teachers' understanding and use of mathematical structure. Their study investigated how teachers used mathematical structure and encouraged structural thinking through components of mathematical structure: Connections, Recognising patterns, Identifying similarities and differences, and Generalising and reasoning. The four components, known as CRIG pedagogical framework of mathematical structure developed during Gronow et al.'s (2020) study found teachers' identified with structure but were not aware they were using it in their teaching. The CRIG framework, in this study, is introduced to PSTs as an effective mechanism for learning to notice structural thinking. The four components of the CRIG framework are detailed next.

Connections. Vale et al. (2011) recognised connections between mathematical representations as fundamental to structural thinking. Making connections between contexts or concepts allows learners to develop mathematical understanding. Mathematics teachers

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make connections between prior, present and future learning, and in real-world contexts of mathematical representations.

Recognising patterns. Patterns are essential in children's mathematical development which begin with their observations of the natural world. Children recognise, observe and generate patterns before reaching school and learn patterning in formalised learning situations that develop structural thinking processes that lead to a deeper understanding of abstract mathematical concepts. Mulligan and Mitchelmore (2009) found that children's structural thinking, identified in patterning awareness, is essential for mathematical concept formation in future learning.

Identifying similarities and differences. Learners develop structural thinking through noticing the differences in mathematical representations. Mason (1996) believed structural thinking is noticing similarities and differences in mathematical relationships. Mulligan and Mitchelmore (2009) discovered that children who found similarities and differences in patterns were involved in structural thinking.

Generalising and reasoning. Mason (2008) described this as an activity that develops a more in-depth experience of mathematics. Mathematical thinking that eventuates into a generalised fact is structural thinking, it connects mathematical relationships from concrete representations to abstract ideas. Mason et al. (2009) wrote that appreciation of structure involves the experience of generality. Stephens (2008) applied structural thinking to designing arithmetic questions. He asserted that children who could articulate a generalised principle underlying a whole problem were thinking structurally.

The framework of noticing also supports the process PSTs learning to notice structural thinking. Scheiner (2016) identified how noticing is not restricted to a single process. Mason (2002) asserted that "every act depends on noticing" (p. 7), he used the term "awareness" to characterise the ability to notice, referring to noticing as an awareness of what one is attending to. In this study, noticing structural thinking implies an awareness of understanding and use of mathematical structure.

By adopting Mason's (2002) approach to noticing, the development and use of mathematical structure has emerged as a form of directing PSTs' attention to their mathematical thinking. Mason studied what he noticed when doing mathematics and called what he noticed the structures of attention of how one thinks mathematically. The aim of this study is for the PSTs to notice structural thinking through learning the components of the CRIG framework of mathematical structure. The PSTs use of the CRIG framework provides an opportunity to detect their awareness of structure, thus answering the research question: *How does the CRIG framework help PSTs to notice structural thinking?*

Method

Context and participants

PSTs in their final year Bachelor of Education/Bachelor of Arts (secondary mathematics) degree at a Sydney university were invited to participate in this study. Three PSTs, referred to as Ms K, Ms M, and Mr T, volunteered to participate in the study during their professional experience placement. Each PST taught mathematics at a secondary school in metropolitan Sydney. Ms K taught an accelerated Year 9 class, Ms M taught a top streamed Year 8 class, and Mr T taught a mixed ability Year 7 class. The PSTs were familiar with the concept of

mathematical structure through the content of courses studied in their undergraduate degree; however, they had no prior knowledge of the CRIG framework.

Study design, instruments, and data collection

The study design comprised of three cycles of: professional learning workshops (PLWs), which were audio recorded. Video recordings of PSTs' mathematics lessons and a noticing reflection audio recording of PSTs reviewing a recorded segment of their mathematics lessons.

Analysis

The audio recordings of the PLWs and noticing reflections were all transcribed to a word document and uploaded to NVivo (QSR International, 2017). The videos of the mathematics lessons were also uploaded to NVivo. NVivo was used to code the data from the PLWs, mathematics lessons and noticing reflections for PSTs' utterances and comments that identified a CRIG component. The data were analysed for evidence of the PSTs' noticing of structural thinking through the PSTs attending to the CRIG framework. The videos acted as the main source of evidence for identifying the PSTs noticing structural thinking through their use of the CRIG framework when teaching. The PLWs and the noticing reflections provide further evidence of the PSTs attention to the CRIG framework.

Results

This section presents a summary of the data collected for each PST from the three cycles of PLWs, mathematics lessons and noticing reflections. An outline of the results from the PLWs are given, followed by exemplars of each PSTs' utterances from the mathematics lessons and comments made during the noticing reflections in Tables 1, 2 and 3, coded to a CRIG component.

During the PLWs, the PSTs were taught to notice structural thinking through the CRIG components. The first PLW began with a presentation on the CRIG framework, followed by a viewing of a video titled Related Problems: Reasoning About Addition (Teaching Channel, 2017), where a teacher used the CRIG components to teach addition to a Year One primary class. Ms K *Recognised patterns* in the teacher's instructions to students. Ms M also *Recognised patterns* as a teaching strategy to engage the students. Mr T noticed that the students used *Similarities and differences* to make generalisations.

In PLW 2, the PSTs viewed a video recording of a child attempting several different arithmetic problems, they were asked to examine the child's mathematical thinking when solving the problems. Ms K noted the child relied on calculations and did not *Identify Similarities and Differences* between the numbers. Ms M noticed the child was using *Generalising and Reasoning* in her structural thinking when she recognised that the problem could be solved another way. Mr T stated the child "Got it after the CRIG prompt, meaning she has structural understanding."

In PLW 3, the PSTs considered how the CRIG framework could be applied to teaching the expansion of binomial products. Ms K made *Connections* to the distributive law and expanding the expression using the FOIL method. Ms M was *Identifying Similarities and Differences* when changing numbers, pronumerals, signs and coefficients in the binomial expression. Mr T stated that *Generalising and reasoning* was identified as a way to summarise the process of expansion and apply it in other mathematical contexts.

Table 1
Exemplars of Ms K using the CRIG Framework to Notice Structural Thinking

Cycle	Mathematics lesson	Noticing reflection
1	*Topic: Simultaneous equations* *Connections* to the relationship between the graphs' intersection points and solving the equations simultaneously. *Recognising Patterns* of the power of x to determine the curve's shape. *Identifying similarities and differences* "What is different about the line's shape?"	*Connections* between the equation and the graph. "I think to show how the y^2 and the x^2 is giving us part of the circle, that relationship." *Identifying similarities and differences* between graphs and equations: "So, they could see that all of them had a square except the last one."
2	*Topic: Angle sum of polygons* *Connections* to prior learning "How did we prove the angle sum of the quadrilateral?" Angle sum of a polygon formula: *Recognising patterns:* "Can you find the pattern of what is going on between the number sides and triangles?" *Generalising and reasoning:* "Calculate the interior angle sum of any polygon."	*Recognising Patterns* to develop the formula: "They understood it better with the pattern." *Identifying similarities and differences* different patterns helped students' thinking. "I had the triangles meeting at a point. I adjusted it as I saw the pattern they were working out."
3	*Topic: Quadratics* *Connections* "Quadratics and parabolas go hand-in-hand. The visual representation of a quadratic is a parabola." *Identifying Similarities and Differences* of the x^2 expression in an equation "This is not of degree two; it is a power of negative two. So, this is not a quadratic." *Generalising and Reasoning* relationships between the equation and the graph.	*Connections*: "I was connecting it to when we did the non-linear simultaneous equations." *Recognising Patterns*, "Rather than drawing random graphs, I'd link them to recognise any patterns from factorised quadratics." *Generalising and Reasoning* "Generalising the solutions of when crossing the x-axis."

Table 2
Exemplars of Ms M using the CRIG Framework to Notice Structural Thinking

Cycle	Mathematics lesson	Noticing reflection
1	*Topic: Circumference of a circle* *Connections* to a real life example of a pizza as a sector of a circle. *Recognising patterns* in the ratio of a circle's circumference and diameter. *Similarities and Differences* comparing the circle's radius and diameter.	*Generalising and Reasoning* through students' discussion when dividing the circumference by the diameter. "I'm looking at what they just did. I'm asking them to contribute what they found and see what they conclude from what they've done."

2	*Topic: Area of composite shape* *Identifying Similarities and differences* to explain the formula of the area of circles. "Area equals πr^2 which is the same as saying $\pi \times r \times r$." *Generalising and reasoning* "How come we have π for every circle? Because the circumference divided by the diameter was always equal to π."	*Recognising patterns* "asking them how to figure out the area. That could have been kind of recognising patterns." *Identifying Similarities and differences* "How to write something in exact form and not exact form *Generalising and reasoning* "Asking them questions they can conclude."
3	*Topic: Volume of a cylinder* *Connections* of a real-world problem: "This is a picture of the sinkhole. What shape does it look like?", *Generalising and reasoning* "What do we need to know to solve this problem? What are we trying to find in the end?"	*Connections* "How they could use previous things they've learnt." *Recognising patterns* "By helping them recognise patterns to work mathematically." *Generalising and reasoning* "Recognising the meaning and interpreting the information."

Table 3
Exemplars of Mr T using the CRIG Framework to Notice Structural Thinking

Cycle	Mathematics lesson	Noticing reflection
1	*Topic: Ordering fractions* *Connections* to a real example "What is one-third of my chocolate bar." *Identifying Similarities and Differences* in ordering fractions "When you look at this, which one's bigger? Or, which one's smaller?" *Generalising and Reasoning* defining a rule "The size of the parts needs to be the same."	*Connections* "I should have reworded the question because this was what we did last lesson." *Recognising Patterns* "What do you notice I'm doing with these numbers?" *Identifying similarities and differences* "Show the diagram of shaded fractions not symbolically."
2	*Topic: Adding and subtracting fractions* *Identifying Similarities and Differences* "What do you notice about the numerators?" *Generalising and Reasoning*, using a whole number method to add fractions. "So, if $1 + 1 = 2$, then, if I use the same thing, for a $\frac{1}{2} + \frac{1}{2}$, is $1 + 1 = 2$, and $2 + 2 = 4$, so it's over $\frac{1}{4}$. Right?"	*Recognising patterns*: "I tried to set up some patterns and then asked them to recognise the patterns." *Generalising and Reasoning* "I've tried to incorporate generalisation in terms of asking them, 'What do you think would be the next pattern?'"

3	*Topic: Stem and leaf plot graphs* *Similarities and Differences* between graphs and stem-and-leaf plots. "Now what were the things we compared. What's similar?" *Generalising and reasoning* to analyse stem-and-leaf plot data. "Take a look at your graph and talk to the other person and tell them what the graph tells you?"	*Recognising patterns*: "So I should have put one number on so the students to see a pattern." *Identifying Similarities and Differences* "I should have asked about the placement of these three numbers: "How are they different?"

Discussion

During this study, the PSTs' noticing of structural thinking developed through their learning of the CRIG pedagogical framework of mathematical structure. Noticing of structural thinking was evident in their references to the CRIG framework drawn from the statements made during the PLWs, utterances in their mathematics lessons, and noticing reflection comments. Exemplars given demonstrate the PSTs' noticing structural thinking through the CRIG components.

The PSTs use of the CRIG components were identified in varied pedagogical strategies. Ms K encouraged students to use a pattern to find the rule for the angle sum of a polygon, Ms M used real world examples for each of her lessons to connect students understanding to the mathematical concept and Mr T used the CRIG components in his questions.

The PSTs' teaching accommodated the CRIG framework and supported their understanding of the mathematical content. Ms K considered other patterning approaches to finding a rule for the angle sum of a polygon and Ms M noticed similarities and differences in binomial expansions. The PSTs' pedagogy focused on a structural thinking learning environment, Ms K promoted students' thinking by challenging them to connect the equation to a graph, Ms M connected mathematical concepts to real-world examples and Mr T asked questions so students would notice patterns, and similarities and difference. In their noticing reflections, the PSTs stated how the CRIG framework supported their teaching. Ms K, was thinking of her future teaching: "If I were to do this again, I'd teach the patterning way, and I would incorporate the CRIG more." Ms M stated CRIG helped her understand student thinking "They're trying to understand the difference between volume and capacity." Mr T reflected on how CRIG improved his explanations. "I should have made it more explicit, by connecting to their prior experience." The CRIG framework in these cases supported the PSTs' noticing of structural thinking.

Prescott and Cavanagh (2007) found that secondary mathematics PSTs tended toward a traditional teaching pedagogy. Awareness of the CRIG framework encouraged the PSTs in this study to move beyond traditional teaching pedagogy. The PSTs were more inclusive of student learning, as noted when asking CRIG component focused questions. Mr T's questions promoted students' structural thinking. He challenged students' thinking about why using a whole number method when adding fractions was incorrect. "So, if $1 + 1 = 2$, then, if I use the same thing, for a $\frac{1}{2} + \frac{1}{2}$, is $1 + 1 = 2$, and $2 + 2 = 4$, so it's over $\frac{1}{4}$. Right?" The PSTs diverse pedagogical strategies also saw them use the CRIG components when instructing or communicating with students. In her second mathematics lesson, Ms K used *Recognising patterns* to help students develop the angle sum of a polygon formula. As the students had discovered a different pattern, one that was not considered by Ms K, she acted

in-the-moment and noticed the students' new approach, she encouraged her students to continue with their strategy and asked one student to explain it to the class. Ms M promoted student involvement in her lessons by arranging students in groups to complete activities, many of which had a real-world experience, such as, here final lesson of finding the volume of a cylinder as a sink hole.

The professional learning program to understand and use the CRIG framework helped the PSTs' to notice structural thinking. Ivars et al. (2018) identified the need for a specific framework for PSTs to have effective noticing. The CRIG framework provided this focus. The ability of the PSTs to understand the CRIG framework and to use it demonstrated its simplicity as a practical and useful tool for teachers of mathematics. The PSTs' content knowledge was established from their extensive mathematical background in their university studies. The CRIG framework, however, deepened the PSTs structural understandings of mathematical relationship, for example Ms K's students finding an alternative approach to finding the angle sum of a polygon.

The PSTs' lack of professional experience before this study could have influenced their fundamental understanding of the CRIG framework and their ability to notice structural thinking. However, having more teaching experience in the future will provide continual opportunities notice structural thinking through the CRIG framework when doing mathematics and when teaching. The PSTs' teaching experience was restricted to their university professional experience program. Researchers have identified how PSTs' limited experiences influence what they attend to when teaching. Star and Strickland (2008) found that secondary mathematics PSTs were not good at noticing mathematical content. Mason (2002) also asserted that PSTs lack experience in recognising and using classroom interactions effectively to promote mathematical understanding. Contrary to the results of these studies, the PSTs in this study produced mathematics lessons that engaged students with activities, instructions and questions that focused on developing students' structural thinking through using the components of the CRIG framework. The PSTs effectively demonstrated an ability to learn and apply the CRIG framework as a new pedagogical skill to mathematical content that they had not taught before. The introduction of structural thinking through the CRIG framework could be regarded as an extra burden for the PSTs to consider when teaching. Nevertheless, the evidence indicates that the PSTs were comfortable with identifying and including the components of the CRIG framework in their lessons and were able to notice structural thinking.

The PSTs were able to articulate the benefits of the CRIG framework to notice structural thinking they indicated that the CRIG framework had shaped their noticing structural thinking and had changed their teaching. Ms K stated that thinking structurally helped her make sense and explain mathematical concepts. In the final PLW, Ms K stated, "You structure your practice to facilitate deeper thought as to what and how things made sense."

Conclusions and Further Research

The CRIG framework proved to be useful for helping PSTs to notice structural thinking. The CRIG framework provided the PSTs with a foundation for teaching mathematics that helped them focus on developing their understanding of mathematical structure. Moreover, this provided PSTs opportunities to notice structural thinking.

Mason (2002) introduced the concept of noticing into the lexicon of mathematics education, and with his colleagues (Mason et al. 2009) the notion of teachers' noticing of structural thinking has emerged as a significant contribution to mathematics teaching. PSTs noticing of structural thinking as the focus of this study has demonstrated, as evident from

the results, that there is potential to advance the discourse of mathematics teaching in this area.

The introduction of mathematical structure in the teaching and learning of mathematics and the noticing of structural thinking has implications for future research in mathematics teaching. Future research could consider how developing noticing structural thinking through the CRIG framework may benefit practicing teachers of mathematics (e.g., primary, secondary, pre-service, novice, experienced, and out-of-field teachers).

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Spatial ability, skills, reasoning or thinking: What does it mean for mathematics?

Danielle Harris
University of Canberra
<danielle.harris@canberra.edu.au>

Spatial reasoning is identified as a Numeracy general capability in the Australian Curriculum, and more globally as a significant precursor to mathematics proficiency. Currently, the literature surrounding mathematical-spatial relations remains largely removed from classroom practice. This paper provides a reflection on the spatial cognition field as it relates to mathematics. The focus of the review is to find points of connection between psychological notions of spatial skills and spatial reasoning as it stands in curriculum and assessment.

Spatial reasoning as an instinctive, vital, human capability has been demonstrated throughout history (e.g., locating the source of the Cholera epidemic in London; supporting the discovery of DNA; NRC, 2006). At a global level it refers to proficiency in mentally representing and transforming objects and their relations (Mulligan, 2015). At a local level, spatial reasoning is ingrained in daily activities, such as the ability to locate our keys, the process of parking a car or packing a suitcase. Although these different skills are often taken for granted and fall under the label *spatial reasoning*, it may not be the case that being good at one type of skill ensures aptitude for another (Newcombe, 2010). Spatial reasoning as an umbrella term has been deemed so closely related to mathematical proficiency it no longer makes sense to explore whether the two are related (Mix & Cheng, 2012). Whole books (Mix & Battista, 2018) and mathematics research journal special issues (Resnick et al., 2020; Sinclair & Bruce, 2015) have been dedicated to the theoretical positing of mathematical-spatial relations. Despite the decades of analysis, the gap between cognitive theories of the mathematical-spatial relationship, and classroom promotion of spatial reasoning remains vast (Lowrie et al., 2020). This paper presents a review of some of the different spatial understandings brought about by differences in terminology, and how these link to the current state of spatial instruction in mathematics classrooms. The aim of this paper is to identify connections across the fields about mathematical-spatial relations, with a view to providing a common conceptual framework on which to build future empirical studies.

Spatial Vocabulary

Spatial terminology varies across discipline, country of origin and research intent. One reason may be that the richness of our mental imagery is poorly articulated by our linguistic capabilities (Hayward & Tarr, 1995). Consequently, a range of terms have been used to define spatial concepts with little consistency. Here I seek to define key spatial vocabulary to provide a shared conceptual framework that is currently lacking in the literature.

The term *ability* is often used to differentiate students in education and has been defined as a "salient psychological attribute" (Wai et al, 2009, p. 817), implying it is stable over time. By contrast, spatial *skills* suggest the opportunity for growth and change (Uttal et al., 2013). More generally, spatial *reasoning* invokes thoughts of non-verbal problem-solving while spatial *thinking* conjures up images of a habit of mind or more holistic spatial sense (Whiteley et al., 2015). These terms are distinct from the mental processes that occur during spatial tasks. *Visual imagery* (imagining a referent object(s); Presmeg, 1986) and *spatial relations* (relative position or movement between objects; Hegarty & Kozhevnikov, 1999)

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are often used to describe stable spatial characteristics. By contrast, spatial *manoeuvres* are the dynamic mental processes undertaken when performing tasks (Ramful et al., 2017). The accuracy and usefulness of these processes may vary depending on spatial aptitude and task demands (Hegarty & Kozhevnikov, 1999; Presmeg, 1986).

Never is the lack of consistency in terminology more evident than when searching keywords in the literature. For example, in conducting their spatial training meta-analysis Uttal et al. (2013) searched 14 different terms yet failed to include spatial reasoning or thinking. To move the field forward there needs to be consistency in the meaning and use of spatial terms. A proposed conceptual model for spatial terminology is presented in Figure 1.

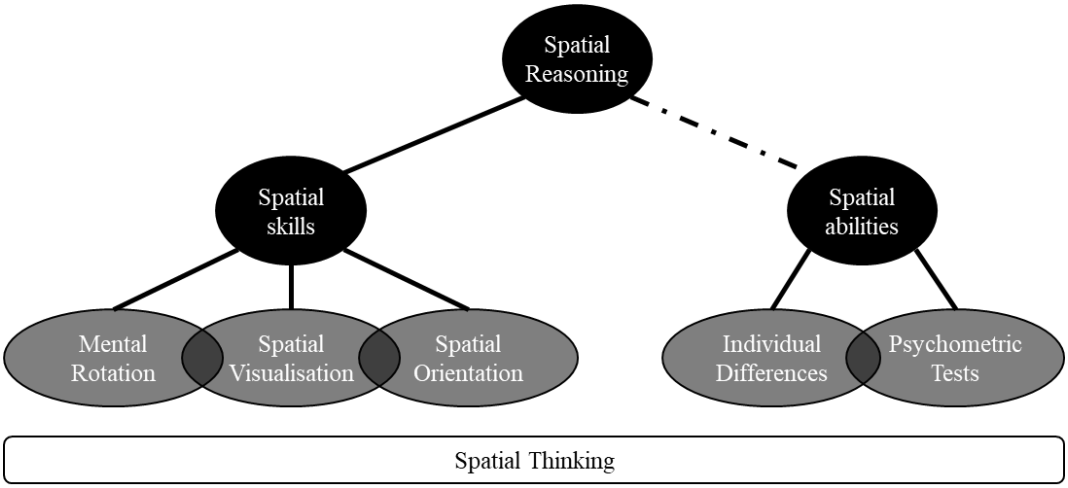

Figure 1. Conceptual model of spatial terminology.

Spatial abilities

Spatial ability was described as an intelligence distinct from verbal ability almost 140 years ago (Galton, 1883). The measurement of spatial ability was predominantly conducted with instruments developed by psychologists (Hegarty & Waller, 2005). Despite research now indicating that spatial aptitude is not fixed (Uttal et al., 2013), there are individual differences that show trends in spatial abilities. Generally, males perform better on spatial ability tests (Hegarty & Waller, 2005). However, research is emerging to suggest that gender differences may lie in strategy choices, thus calling into question some of the long-held beliefs about gender factors in spatial ability theory (Newcombe, 2020).

Piaget and Inhelder (1967) proposed that although infants show evidence of spatial coding, mature spatial reasoning does not emerge until age 9 or 10. Congdon et al. (2018) report evidence for pre-schooler's awareness of spatial properties, but it is not until a few years into formal schooling that language and conceptual understanding develop. Separate spatial abilities also seem subject to different developmental trajectories. Crescentini et al. (2014) found that the ability to perform object-based spatial tasks emerged earlier than tasks requiring awareness of one's body and environment. This may be largely due to children's exposure to activities and environments that support the development of spatial reasoning at the different scales (Newcombe, 2002; 2020). Understanding the developmental path of spatial abilities may assist educators and researchers in providing appropriate experiences for children to foster their spatial reasoning. The dotted line between spatial abilities and spatial reasoning in Figure 1 above signifies that spatial capacity (or ability) exists for

everyone, however, education, experience and environmental interaction are most influential in the development of the more holistic notion of spatial thinking (Newcombe, 2002).

Spatial skills

Spatial skills are the quantifiable factors comprising spatial reasoning that are distinct, yet related (Carroll, 1993). The structure of these skills, much like the overarching theme itself, remains under some debate. Newcombe and Shipley (2015) proposed a typology of spatial tasks categorised by the characteristics of the referent object(s); whether they remained still (i.e., static) or moved (i.e., dynamic) and whether spatial relations were within (i.e., intrinsic) or between (i.e., extrinsic) objects. Such a framework could provide researchers with the foundations for linking the mental manoeuvres undertaken during spatial tasks and skills in other fields, such as mathematics. However, the typology proposed by Newcombe and Shipley is largely based on psychological tests and has yet to be supported by further research or in applications beyond lab-based studies (Mix et al., 2018).

Measuring spatial skills. The idea of assessing different spatial skills emerged in the field of aptitude testing for occupations (Hegarty & Waller, 2005). As psychometric tests measuring spatial skills continued to evolve, correlations with other skills and outcomes emerged. For example, spatial skills were the strongest predictor of Science, Technology, Engineering and Mathematics education success and career choice in a 50-year longitudinal study (Wai et al., 2009), above verbal ability and mathematics proficiency.

Spatial task performance has been related to mathematics outcomes in correlational (Gunderson et al., 2012; Mix et al., 2016) and intervention studies (Cheng & Mix, 2014), leading to categorisation of spatial skills based on test affordances. For example, object-based spatial skills are considered in a different category to egocentric skills, where one's perspective becomes the reference point (Sorby, 1999). This distinction is a consequence of test design and the intentional promotion of specific strategy use (Hegarty & Waller, 2004). This psychological approach results in cognitive theories limited by the measures used in empirical studies and may be counter-productive to the development of robust models of mathematical-spatial relations that are based on applications of spatial skills.

Ramful et al. (2017) adopted a different methodology in the development of their spatial reasoning instrument (SRI). They defined spatial constructs (as opposed to skills) by the spatial manoeuvres found in the Australian Numeracy curriculum; namely, mental rotation (i.e., imagining an intact 2D shape or 3D object in a different orientation; Cheng & Mix, 2014), spatial visualisation (i.e., performing complex, multi-step manoeuvres that change the form of the referent object; Hegarty & Waller, 2005), and spatial orientation (i.e., imagining different perspectives, navigating, or taking different orthogonal views; Newcombe & Shipley, 2015). These constructs correlate with psychological tests of spatial skills but provide opportunities to explore links with mathematical-spatial processing in a more applied way. This measure is the first of its kind but there are calls for more measures of spatial reasoning that consider real world spatial problem-solving (Mix, 2019).

Spatial Reasoning

Spatial reasoning, as a foundational component of Numeracy, requires an awareness of space, the ability to imagine objects and relationships, and use this information to reason and problem-solve (ACARA, n.d; NRC, 2006). Spatial reasoning manifests differently across situations (such as the constructs identified by Ramful et al., 2017). For example, mental rotation involves imagining an object's position within its direct environment, while spatial

visualisation exists in isolation, where the environment is less important than the relations within and the ability to visualise and transform the object's form. Spatial orientation requires imagining dynamic interaction with an environment on a larger scale.

Spatial reasoning is not easy to quantify, and researchers look to spatial tests (Mix, 2019) or to the most explicitly spatial aspects of curricula (i.e., geometry; Battista et al., 2018) to make inferences about its underlying structure. Spatial reasoning in education goes beyond success on spatial tests. Educators need to be equipped with the tools to recognise and foster students' awareness of space in the mathematics classroom, and to encourage them to notice spatial relations in their interaction with the world.

Spatial Thinking

Spatial thinking is less well-defined by literature, except where used interchangeably with spatial reasoning. Newcombe (2010) used the term spatial thinking to describe Albert Einstein's unique way of seeing the world, that is, in pictures and relations. In this paper, I propose a distinction between spatial reasoning, the application of spatial skills during problem-solving, and spatial thinking, the tendency to visualise non-verbal aspects of objects and relations, separate to mathematical thinking (Newcombe, 2010; Whiteley et al., 2015).

In the National Research Council (2006) report, spatial thinking was defined as an "amalgam of three elements: concepts of space, tools of representation, and processes of reasoning" (p. 3). Figure 1 shows spatial thinking as underpinning all forms of spatial representation and assessment discussed above. This model positions spatial thinking as a habit of mind that guides communication, reasoning and problem-solving. Therefore, promoting spatial thinking across education, provides students with strategies when faced with new or complex materials (Uttal & Cohen, 2012).

Visualisation

Much like the absence of spatial terms in Uttal et al.'s (2013) literature search, Figure 1 did not capture all aspects of spatial vocabulary. One missing element is visualisation, which is critical for spatial thought (Battista et al., 2018). Visualisation occurs differently for those with varying spatial skill levels. Strong spatial thinkers tend to generate mental images that facilitate problem-solving and concept development, poor spatial thinkers tend to produce mental images that, while detailed, offer little in their affordances for problem-solving (Hegarty & Kozhevnikov, 1999; Presmeg, 1986).

Mathematics and Spatial Reasoning

A complete review of the mathematics-spatial literature is beyond the scope of this paper and well captured in Mix and Battista's (2018) edited book. Here, I focus on the connection between cognitive theories of mathematical-spatial relations and spatial reasoning in mathematics curricula and assessment based on Ramful et al.'s (2017) three constructs.

Mental Rotation

Mental rotation is one of the most extensively studied spatial skills in the mathematics literature. In fact, 3D mental rotation training by Cheng and Mix (2014) was found to lead to improvements on missing term addition and subtraction tasks. Furthermore, mental rotation is thought to support geometric reasoning by providing the mental models on which to examine geometric properties (Battista et al., 2018). For example, to perform mathematical rotation tasks on a coordinate grid, one must first be able to correctly visualise

the rotation of the referent object. The disconnect between the psychological and educational notions of mental rotation are evident in these two lines of thought. While one is focused on repetitive, comparison tasks that rely on speed to force rotation (psychology; Hegarty & Waller, 2005), the other advocates for mental rotation processes in building conceptual knowledge for geometric understanding (mathematics education; Battista et al., 2018).

Apart from small-scale studies (e.g., Bruce & Hawes, 2015; Cheng & Mix, 2014), the development of mental rotation in mathematics classrooms remains largely incidental as a result of engagement with geometry material (Lowrie & Logan, 2018). Lowrie et al. (2018) provided a pedagogical model for developing mental rotation beyond curriculum learning through a classroom-based spatial intervention. However, the unique contribution of mental rotation to mathematics improvement was not addressed.

Spatial Visualisation

The complex, multi-step manoeuvres that constitute spatial visualisation are evident within mathematics curricula in geometric concepts of symmetry and net to solid conversions. Furthermore, psychological measures of spatial visualisation such as paper folding have been found to relate to multiplicative and algebraic thinking by reflecting students' ability to map folds to parts (Empson & Turner, 2006). Lowrie et al. (2019) trained spatial visualisation skills, which led to improvements in geometry and word problems. They concluded that the impact on geometry tasks was reflective of students' increased ability to manipulate spatial properties, while the word problems were evidence for improvements in representing information spatially during problem-solving (Hegarty & Kozhevnikov, 1999).

Rittle-Johnson et al. (2019) found strong relationships between patterning, mathematics and spatial visualisation. They found that spatial visualisation at the beginning of the pre-school year was a significant predictor of later numeracy performance (a subset of the mathematics assessment) in that same year. They also found that initial patterning skills were a significant predictor of later mathematics, over and above prior mathematics knowledge and a composite spatial measure. Their findings shed light on the complex relationship between spatial skills, patterning and mathematics. It is possible that spatial visualisation is helpful when developing mathematical understanding but is less influential long term when content knowledge increases.

Spatial Orientation

Few psychological studies have explored the direct role of spatial orientation in mathematics (Newcombe, 2010) but mapping tasks and orthogonal perspectives are explicit elements of the Measurement and Geometry strands of the Australian Curriculum (ACARA, n.d.). Two longitudinal studies have examined the unique role of spatial orientation in mathematics performance. Frick (2019) found that spatial orientation skills measured in kindergarten predicted performance in quantity, magnitude and geometry tasks, but not arithmetic in year 2. Mix et al. (2016) found significant contributions of spatial orientation to a general mathematics measure in years 3 and 6. Spatial orientation skills such as understanding scale and magnitude are critical for performance on mapping tasks as well as development of number line knowledge (Gunderson et al., 2012) and proportional reasoning (Möhring et al., 2016). Given these preliminary findings, I propose that this is a spatial construct that should be included in empirical studies of mathematics-spatial relations.

Spatial Intervention

Many studies to date have analysed correlational data, providing valuable insight into areas where spatial intervention may support student learning (Mix, 2019). Spatial skills have been found to be malleable and responsive to training (Uttal et al., 2013). In their meta-analysis of spatial training studies Uttal et al. (2013) found no difference in effect sizes as a result of the form of the training (i.e., instructional courses, video games or spatial skills training) on spatial outcomes. However, when examining potential transfer to mathematics the range of outcomes has produced variable results (Stieff & Uttal, 2015). Stieff and Uttal acknowledge the difficulty in conducting classroom-based studies on a large-scale but when done successfully, they have the greatest potential for effecting change.

To progress the field and transfer theoretical understandings to practical, student benefits we need to shift the focus from performance on cognitive tests to how spatial reasoning manifests in mathematics. Lowrie and colleagues have demonstrated reliable transfer to mathematics achievement (Lowrie et al., 2017; 2019; in press) in ways that others in the spatial cognition field have not (Cheng & Mix, 2014; Hawes et al., 2017). They achieved this through the integration of spatial skills in a pedagogical framework, delivered by classroom teachers (Lowrie et al., 2018). In their intervention studies, ranging in length from 3 to 10 weeks and focusing on mental rotation, spatial visualisation and spatial orientation (Lowrie et al., 2017; in press) or spatial visualisation alone (Lowrie et al., 2019), Lowrie and colleagues consistently demonstrated mathematics improvements with effect sizes ranging from .38 – .40 (Cohen's d) compared to business as usual control groups.

There remains a gap in the literature. To date there have been no systematic studies of spatial skill interventions to determine their contribution to mathematics. Studies remain either isolated (e.g., 3D mental rotation; Cheng & Mix, 2014; spatial visualisation; Lowrie et al., 2019) or combined (e.g., Lowrie et al., 2017; in press), making it difficult to identify the unique contributions of spatial skill development to mathematics. Similarly, the effect on mathematics has been too broad, leaving the field still speculating about the mechanisms that result in improvements in mathematics based on spatial training (Stieff & Uttal, 2015).

Limitation and Conclusion

The focus of this review has been to highlight some of the connections between cognitive theories of spatial skills, emerging from lab-based studies, and applied spatial reasoning, in education. This review could not be exhaustive and there remains a considerable absence of spatial terminology as well as spatial concepts such as transformation, and representation. These were excluded based on the goal of seeking common ground across psychological and educational domains, as these terms often have different meaning in the two fields. For example, in mathematics transformations are functional in problem-solving (Battista et al., 2018), while in psychology transformations refer to mental manoeuvres (Frick, 2019).

To progress the field in practical and constructive ways the focus on spatial reasoning in mathematics needs to be situated within real world applications (Lowrie et al., 2020). Spatial instruction needs to be explicit, not merely fostered through the more spatial content within the curriculum. To bridge the disconnect between cognitive theories of mathematical-spatial relations and classroom practice, there needs to be shared meaning and studies need to be conducted at scale with teachers instrumental in the process. Finally, experimental design needs to allow for conclusions to be drawn about the mechanisms that connect spatial thinking, reasoning and skills with mathematics understandings to ensure sustainable and positive outcomes for students.

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Contextualising space: Using local knowledge to foster students' location and transformation skills

Danielle Harris
University of Canberra
<danielle.harris@canberra.edu.au>

Tracy Logan
University of Canberra
<tracy.logan@canberra.edu.au>

Tom Lowrie
University of Canberra
<thomas.lowrie@canberra.edu.au>

Location and Transformation skills are critical tools for navigating the world and establishing foundational steps for geometric reasoning associated with co-ordinate grids and the Cartesian plane. The contextual nature of using local landmarks to understand students' mental representation of large-scale space has the potential to enhance these skills. This paper examines a classroom activity that draws on students' local knowledge when representing their environment. Factors such as geographic distance and isolation, and incorporation of spatial relations are explored. Recommendations are made for educators to incorporate the sophisticated local knowledge when building mathematical understanding.

From the Foundation year of school, the Australian Curriculum identifies *Location and Transformation* as critical elements of mathematics (ACARA, n.d.). This content sits within the general capability of spatial reasoning. Although identified in the Australian Curriculum, educators are left with little support for incorporating spatial instruction in their teaching (Lowrie & Logan, 2018). Engaging with position and movement provides a novel opportunity to embed learning into tangible, real-world, contexts for students. Rather than abstract notions of mathematical content confined to a page or screen, teaching about large-scale space affords students the opportunity to be active participants in their learning. Physical exploration has been linked to greater accuracy and flexibility when estimating landmarks and distances compared with abstract (i.e., virtual) experience (Richardson et al., 1999). This embodied approach to spatial reasoning has been found to be effective in mathematical and cognitive learning models (Nathan et al., 2020; Tversky, 2009). To address the problem of how to bring spatial instruction into the classroom in an accessible, contextualised way, we explore engagement with a spatial task that drew on the local knowledge of students from culturally and geographically diverse schools.

Location and Transformation

Location and Transformation are interwoven throughout the Australian mathematics curriculum. In the early years, the focus is on position and movement to assist with simple directions. As students develop, they are taught increasingly complex mapping skills as a foundation for the introduction of the Cartesian coordinate system (ACARA, n.d.). Despite the inherently spatial nature of this content, concept development often fails to consider the opportunities of promoting spatial representations to provide students with a fallback strategy when content difficulty increases (Lowrie, Logan & Patahuddin, 2018).

Location is a broad term spanning Measurement and Geometry, ranging from descriptive language (i.e., *behind* or *next to*), to pictorial (grid representations), and symbolic (co-ordinate systems). This learning progression was identified by Lowrie, Logan and Patahuddin (2018) as critical for development of sound mathematical understanding. They

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posit that student experiences support language growth and engagement with pictorial representations (i.e., concrete materials, gesture, maps, pictures). It is these foundations that foster development of symbolic understanding and further applications to more complex mathematical concepts.

Large-scale Spatial Representation

Mapping skills sit at the nexus of numeracy and spatial cognition. Numeracy (via *Location and Transformation*) and Geography curriculums emphasise the development of mapping skills throughout schooling (ACARA, n.d.), while psychologists explore the relationship between mental representations of real and virtual environments to understand the development of navigation skills (Keil et al., 2020; Richardson et al., 1999).

Drawing on student experience is critical when developing mathematical and spatial thinking (Lowrie, Logan, Harris et al., 2018). Connecting new learning to students' knowledge provides the foundation for language development such as directional and relational language (e.g., the park is south of school, I go past the corner store on my way). Although language alone is not sufficient for developing spatial thinking (as this would undermine the non-verbal nature of the concept), language can be critical for directing attention and building towards more complex spatial concepts (Newcombe & Stieff, 2012). Experience and language lay the groundwork for developing increasingly sophisticated large-scale spatial representations and map understanding (Larkin & Kinny-Lewis, 2017). These tools transcend cultural boundaries and provide access points for all students when building content knowledge.

Large-scale spatial representation has traditionally been thought to reflect a cognitive map incorporating Euclidean space, landmarks, and routes (Tversky, 2003). Although cognitive maps develop through exposure to both physical space and maps, the notion that the representations themselves are map-like is a topic of some debate (Foo et al., 2005). Some researchers have argued that mental representations of large-scale space may be more like graphs, with spatial locations represented as nodes, connected by familiar routes but flexible enough to account for changes in orientation and task demands (Peer et al., 2021).

Spatial Relations

Landmarks serve two main purposes in spatial representations (Presson & Montello, 1988): 1) as navigational cues, and 2) as reference points for determining spatial relations (Clements & Battista, 1992). Here we focus on spatial relations, however the salience and organisation of landmarks in the spatial representation can be highly contextual. For example, a student may recall passing the park and shops on their journey to school, but it does not necessarily help them position the locations from a birds' eye perspective.

Scale adds an extra element to the notion of spatial relations. The structure of large-scale space is divided into regions that, even in the absence of language, can be thought of in terms of distance and direction (Kuipers, 1978). By removing physical boundaries, students are free to reveal the scale and relative position of the landmarks as they exist in their mental representation. It is through this physical enactment of their mental representation of space that we can gain insight into their awareness of their local environment, including scale and relative position, and use this as a springboard for developing further content knowledge.

The Context of the Study

Research has shown that a great deal of curriculum content is established in a city-centric style that leaves students in regional and rural communities at a disadvantage (Roberts, 2017). However, recent work has highlighted the incredibly sophisticated local knowledge possessed by students outside of city centres (Lowrie et al., 2021). It is this contextualised knowledge we propose provides curriculum accessibility for all students in developing *Location and Transformation* understanding.

When performing tasks relating to their local environment, visual prompts allow children to recall and represent a greater amount of information than free recall alone (Matthews, 1985). Therefore, by providing students with physical stimuli we can explore children's representation of space using familiar landmarks (Peer et al., 2021). Tversky and Hard (2009) argued that the mere presence of an individual in a spatial perspective task alters the interpretation of spatial relations. In this study, while all students were oriented to face north, relations between landmarks were relative to the school or position of other landmarks (as determined by the student).

This study is situated within an Australian Research Council Discovery Project exploring spatial reasoning in children from culturally and geographically diverse communities. Specifically, this study examined students' large-scale spatial representations, with a focus on factors such as geography, distance, and spatial relations, with the goal of analysing the efficacy of using local knowledge to foster foundational spatial concepts.

Method

Participants

Thirty Grade 5 students from three NSW schools participated in this study. The sites represent vastly different geographic locations and population density: an urban site in Western Sydney, a rural site (population < 1,000), and a regional site (population > 30,000).

Procedure

Students were shown a collection of local landmark sites (such as parks, shops, prominent town features) and asked whether they recognised the site. They were asked how often they visited or travelled past the site, whether they had positive or negative feelings about the location, and how familiar they were with the site.

Students were seated facing north and given a piece of A3 paper with a dot representing the school in the centre. As each site varied significantly in terms of geography and density, the school was chosen as a central point as it was familiar to all students, and consistent within and between sites. Students placed the photos of landmarks they recognised around the school point from a bird's eye perspective over their local area. Students performed this task twice on consecutive days with different landmarks. The photos were large compared to the school marker and the A3 paper. There were no constraints on the way students were able to complete the task and all photos were provided to the students at the same time.

Scoring and Analysis

We analysed student representation according to three criteria, and then made site-based comparisons using Analysis of Variance, and Nonparametric tests (chi-square) to explore distributions within sites:

1. *Landmark recognition* = proportion of the possible landmarks recognised
2. *Landmark accuracy* = landmarks positioned correctly relative to school
3. *Spatial relations* = the scale and relative position of landmarks
   a) *Scale* = some photos placed further than others
   b) *Relative position* = clustering of photos

Results

Landmark Recognition

A 3x3 mixed factorial ANOVA revealed significant main effects in landmark recognition across distance categories (within-groups) and site (between-groups), and a significant interaction, $F(4,54) = 3.85$, $p = .008$, partial eta^2 = .22. All students recognised a larger proportion of near landmarks, $F(2,26) = 19.59$, $p < .001$, partial eta^2 = .60. Between sites, rural students recognised a significantly larger proportion of landmarks than urban students, $F(2,27) = 4.13$, $p = .027$, partial eta^2 = .23. Means are presented in Table 1.

Table 1
Average percentage of sites identified in each of the distance categories

	Near (<1 km)	Intermediate (1-5 km)	Far (>5 km)	Total
Urban	56%	57%	56%	56%
Rural	94%	76%	57%*	76%
Regional	100%	50%	43%	64%

**Note.* All far landmarks in the rural site were located in neighbouring towns roughly 40-50km away.

Urban students recognised half of all landmarks across distance categories, while regional students were familiar with all locations within 1 km of school, dropping to half the sites beyond 1 km. By contrast, rural students identified a large proportion of landmarks in their own town. Despite the distance of the far landmarks, rural students still identified more than half the possible landmarks.

Landmark Accuracy

A 3x3 mixed factorial ANOVA revealed significant main effects in accuracy by distance (within-groups) and site (between-groups) and a significant interaction, $F(4,54) = 2.88$, $p = .031$, partial eta^2 = .18. Landmarks in the near range were positioned most accurately, $F(2,54) = 13.60$, $p < .001$, partial eta^2 = .34. At the school level, rural students were more accurate than urban students, $F(2,27) = 8.21$, $p = .002$, partial eta^2 = .38. At the urban site there was no difference in performance based on distance categories while rural and regional students experienced decreasing accuracy as distance increased. Regional students had a sharper decline with increasing distance than rural students (see mean percentages in Figure 1).

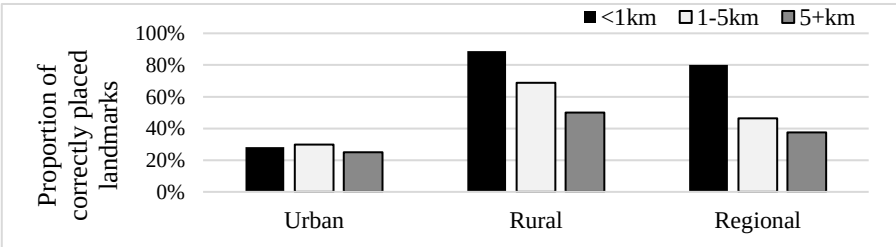
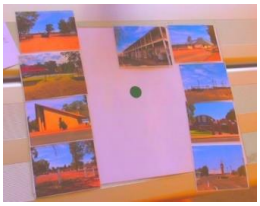


Figure 1. Mean percentage of landmarks in correct position relative to school site

Spatial Relations

Students used different strategies to demonstrate their mental representation (examples in Table 2). We analysed the final position of the photos as not all students verbalised their thinking during the task. The most distinct differences were in orientation and structure. Some students kept all photos facing themselves while others rotated the photos to reflect how the landmark would appear when journeying from school. The structure students chose when arranging the photos varied between grid-like and relational. The relational structure accounted for the scale and relative position of landmarks, or a combination of both. These differences are discussed further in the next section.

Table 2
Representation categories

Orientation	Upright	Rotated
		
Structure	Grid-like	Relational
		

Scale and relative position. A third of all students demonstrated elements of scale and relative position, however, this had no significant connection with accuracy. One exception was for those that demonstrated scale, these students were more accurate when placing near landmarks, $F(1,29) = 6.81$, $p = .014$. There were no significant differences for the other distance categories. Table 3 includes sample arrangements of the four categories.

Table 3
Sample representations, and student numbers per category

		Scale	
		Yes	No
Relations	Yes	 N = 10	 N = 8
	No	 N = 7	 N = 5

Non-parametric analysis (cross tabulation using chi-square statistics) revealed a difference by site in the representation of scale, $\chi^2(2) = 8.69$, $p = .01$, but not relative position, $\chi^2(2) = 46$, $p = 79$. A large proportion of students in the rural and regional sites represented scale compared with only one urban student. Despite the distance category parameters remaining constant, students at the urban site appeared less sensitive to the distance when arranging the landmark photos.

Discussion

Recognition and Accuracy

Landmarks within 1 km were most recognisable and positioned with the most accuracy (with the exception of urban students). Regional students were incredibly familiar with their local area, within 1 km, recognising all the possible landmarks and accurately positioning 80% of those. More progressive, rural students were able to recognise most landmarks within their town and still more than half of the landmarks in towns 40-50 km away. Even at this distance, rural students correctly placed roughly half of the landmarks, which was more than the urban or regional students whose far landmarks were roughly 5 km away. Tversky (2003) talks about key landmarks when referring to cognitive maps. In towns like the rural one in this study, the geographic size and relatively low density may contribute to students being aware of all landmarks. By contrast, the density of the urban environment makes competition for landmark memory much higher. For example, most students in the regional town were able to identify something as routine as a street sign, while at the urban site only a McDonald's and a movie theatre were consistently recognised. The regional students similarly recognised a local McDonald's but were also able to identify local parks, shopping centres, petrol stations and hardware stores. It is possible there are fewer of these to compete for attention, or the nature of children's lived experience drives their memory for these locations. This finding has implications for classroom practice, the richness of local knowledge demonstrated by rural and regional students can be drawn upon when introducing concepts such as scale. When verbalising their thinking, those students who demonstrated scale and relative position were able to clearly articulate the relations between the sites, and often drew on these relationships to help them position less familiar photos.

Presson and Montello (1988) discuss the importance of context when it comes to spatial memory for location. Our results highlight the impact of student context in mental and physical representation of their local environment. We argue that the sophisticated local knowledge in rural and regional areas should be harnessed when building understanding around *Location and Transformation*. Similarly, it would benefit urban students to engage more with their local environment, for example through community walks or mapping exercises, to provide foundational experiences and develop directional language before building towards more abstract representations of space. Educators are well-placed to draw on student strengths and experiences when building mathematical knowledge – this task is one example of how local knowledge can be used.

Spatial Relations

The open-ended nature of the task allowed students to reveal the diversity of their mental representations of the local environment. While some students kept all photos upright, others rotated the images to align with their view as they mentally traversed the journey. This latter approach may be indicative of the graph approach (Peer et al., 2021), with students

connecting nodes (i.e., locations/landmarks) via their well-travelled routes. Anecdotal evidence from some students' reflections suggested that these differences may be due to map-like (i.e., bird's-eye) versus route-based strategies. Future research may benefit from exploring these distinctions further.

Relative position. Despite the body of work discussing relative position as a critical component of the accuracy of spatial representations (Peer et al., 2021; Presson & Montello, 1988), our findings did not establish a link between students who demonstrated relative position and their accuracy in positioning landmarks. The difference between our study and those before are that we drew on the local environment in selecting landmarks, whereas previous studies have focused on new learning. In these instances, the locations (or nodes) under consideration are determined by the researcher. In our study it may be that students were drawing on knowledge beyond what we presented to them, for example a third site (such as home) may have helped them triangulate locations (Foo et al., 2005).

Scale. By contrast, representation of scale did show significant connections to accuracy and context. Those that demonstrated scale by positioning the photos at varying distances from school were more accurate in their placement of near landmarks. It is one possibility that these students had a robust mental representation of their local area and then used this to extrapolate to the larger area. In newly learned environments nearer landmarks have been shown to be associated with greater salience and accuracy (Keil et al., 2020).

Consistent with the notion that context is critical when examining *Location*, rural and regional students were more likely to represent scale. The nature of their interaction with their local area appears to have a bearing on their awareness of the scale of the environment. Many rural students travel long distances by bus to school while many regional students reported not travelling very far beyond their local community in their daily lives. Both environmental conditions may contribute to students' sense of environmental scale (Presson & Montello, 1988). Scale and magnitude are foundational numeracy skills, our findings suggest that where city-centric teaching models may disadvantage some students (Roberts, 2017), the opportunity to draw on students' local knowledge and experience may make abstract mathematical concepts more accessible for all students.

Future Directions

This task provided some insights into the different ways students represent large-scale space. The factors explored in this paper were broad in terms of geography and assumptions about student experience of both the sites and town structure. Future research may look at more individual factors, such as students' freedom to roam, means of transport, and family culture. Although we explored the use of relative position and we did not analyse the order in which students placed the photos, it is possible more in-depth analysis of the students' actions and thinking may give insights into key landmarks (or nodes) around which their spatial representations were built.

Conclusion

Much of the spatial research examines lab-based or abstract notions of spatial reasoning which often leaves students in regional and rural areas at a disadvantage. We have visited sites with different social, geographical and cultural contexts. We have chosen to examine the question of spatial representations with a different lens. Our results indicate that the engagement with the local environment afforded by rural and regional living has provided

students with an advantage in representing their familiar space. We suggest that this embodied, contextualised spatial knowledge is a strong foundation for building mathematical knowledge around *Location and Transformation* as a springboard for more complex mathematical skills.

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Curriculum development and the use of a digital framework for collaborative design to inform discourse: A case study

Ellen Jameson
University of Cambridge
<Ellen.Jameson@Cambridgemaths.org>

Rachael Whitney-Smith
Australian Curriculum, Assessment and Reporting Authority
<rachael.whitney-smith@acara.edu.au>

Darren Macey
University of Cambridge
<Darren.Macey@Cambridgemaths.org>

Will Morony
Australian Curriculum, Assessment and Reporting Authority
<Laurie.Morony@acara.edu.au>

Anne-Marie Benson-Lidholm
Australian Curriculum, Assessment and Reporting Authority
<Anne-Marie.Benson-Lidholm@acara.edu.au>

Lynne McClure
University of Cambridge
<Lynne.Mcclure@Cambridgemaths.org>

David Leigh-Lancaster
Australian Curriculum, Assessment and Reporting Authority
<david.leigh-lancaster@acara.edu.au>

This paper reports on a new initiative of collaborative work between the Australian Curriculum, Assessment and Reporting Authority (ACARA) and Cambridge University as part of the 2020-21 review of the Australian Curriculum: Mathematics Foundation – Year 10. The ACARA mathematics curriculum development team worked with the Cambridge Mathematics team using the Cambridge Mathematics Framework, which incorporates summaries of the research literature, to inform the review of Statistics and Probability in the mathematics curriculum as part of ACARA's program of research.

The Australian Curriculum, Assessment and Reporting Authority (ACARA), during the 2020-21 review of the Australian Curriculum: Mathematics Foundation (pre-Year 1) to Year 10, identified an opportunity to trial a new approach to coherent curriculum design. A team of curriculum specialists incorporated the *Cambridge Mathematics Framework* (CMF) into the Statistics and Probability areas of the curriculum review as an analytical tool for examining content revisions, making decisions, and providing justification to other stakeholders based on consolidated interpretations of relevant research. Teams from ACARA and the University of Cambridge developed ways of incorporating the CMF which led to areas of validation and areas of change in the curriculum and recommendations for use and support of the CMF for the Cambridge team to apply in the future. This paper presents an outline and some details of this new initiative and discusses implications for the Australian Curriculum, the CMF, and curriculum review more broadly.

Challenges for domain coherence in curriculum design

Learning mathematics has been described as the process of building a scaffold from the ground up, a rising and expanding network of ideas supported by the synthesis and consolidation of ideas students have already developed (Tall, 2013; Thurston, 1990). Day to day in the classroom, this process is non-linear, as teachers and students visit related ideas back and forth, retracing steps, making connections, bringing new ideas to bear on old ones, and vice versa. A coherent mathematics curriculum seeks to provide a substantive

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 235-242. Singapore: MERGA.

progression within key organising constructs, structuring the process in time across years of study while supporting the underlying conceptual structure of the domain (Jameson et al., 2018; Schmidt et al., 2005).

The extent to which this is possible depends on what knowledge can be brought to bear about the underlying structure of the domain. Each teacher, curriculum designer, and researcher in mathematics education, from their own education and professional experience, has developed a sense of the ideas and the relationships between them that make up parts of this scaffold, though perspectives on some areas will be based on more information than others due to individual specialisations. However, opportunities for sharing these perspectives to assemble a larger coherent picture are often limited.

The importance of connecting research and practice is well recognised in mathematics education, but there are challenges to making these connections successfully (Flessner, 2012). These challenges stem in part from how research is designed and the investment it takes to bring professional judgment from practice and research together. First, much of this research is structured around developing particular theories of learning and understanding of surrounding issues, and produces knowledge in a very different framework to pedagogical knowledge (McIntyre, 2005). Each study is intended to address a specific gap in knowledge, to make a unique or complementary contribution with respect to existing research and experience. This means that studies typically do not result in unambiguous recommendations for practice individually, and the collective picture can be even more complex.

Secondly, in order for research to contribute to practice, teachers and educational designers need practical access to it. Some barriers to access are physical or financial, while others have simply to do with the time it takes to find, read, and synthesise reports of multiple studies, and the study or training required to be familiar enough with research practices and strands of work in the field for critical analysis (van Schaik et al., 2018).

Another challenge is that curriculum design involves agents and stakeholders who are members of different communities of practice (Pinto & Cooper, 2018; Remillard & Heck, 2014), with differences between their priorities and perspectives on mathematics. Pinto and Cooper (2018) reported that in curriculum design discussions between different types of stakeholders, people with backgrounds in more than one camp act as knowledge brokers - people who can translate between perspectives and help the group to make decisions based on shared understanding. Shared objects of discussion can also help. However, discussions which are not successfully mediated may not end with meaningful agreement, whether about structuring principles or scope and sequencing.

Lastly, a challenge lies in the compressed selection of objectives which occurs distinctively in every curriculum due to time and resource constraints. Different decisions guide this selection under different circumstances, but it always involves trade-offs – for example, depth and breadth, this set of key ideas or that set of key ideas, ordered along in this sequence or that sequence. It is not possible or even necessary to include everything, but the choices which are made affect the coherence of mathematical experiences in the classroom and opportunities for teachers to develop a more connected perspective of the domain (Schmidt et al., 2005). Whatever selection is made, the curriculum aims to have its own sense of completeness, coherence, consistency, correctness and relevance, in particular as it is developed to provide access to educational entitlement for students.

Conceptual mapping has been used in multiple instances to address curriculum challenges. Confrey et al. (2017) have designed “learning maps” based on learning trajectories, which are empirically supported conjectures of the network of constructs students experience as they build understanding of mathematical concepts. Learning maps

are designed to show details which help teachers to provide learner-centred instruction (Confrey et al., 2017). Koch et al. (in press) have developed a network representing teacher knowledge of mathematical topics for middle grades in Canada, derived from empirical work with teachers rather than students. The CMF has some similarities with each of these and also key differences. It allows maps to be generated from a network of mathematical ideas which, similar to Confrey et al.'s (2017), represent concepts building on one another, but these concepts in the CMF are derived from interpretation and synthesis of research literature. They represent not professional knowledge itself, as in Koch et al.'s (in press) work, rather what the reviewed research suggests is useful for designers to know about students' conceptions.

Context

Review of the Probability and Statistics component of the Australian Curriculum

The current F-10 Australian Curriculum review process began in June 2020 when Australian education ministers through the Education Council agreed to the terms of reference, and a guiding paper, *The Shape of the Australian Curriculum*, was developed. From there, content review began, as well as consideration of how the proficiencies could be further developed and incorporated with this revised content. The Cambridge Mathematics team were introduced to the project in June 2020 and began working with the team of curriculum specialists tasked with reviewing content in the Statistics and Probability strands, with both teams using the CMF to explore questions and inform regular discussions.

The review was structured around the organising ideas of Mathematising, Structure, and Approaches and took place in four steps: (1) identifying core concepts at the Learning Area level, (2) identifying core concepts at the Strand (branch) level, (3) using identified core concepts to curate essential content for the learning area and identifying any gaps, redundancies or imbalances, and (4) organising content with embedded proficiencies into strands using core concepts and/or core concept organisers within the wider Mathematics scope and sequence, also relying on an initial programme of research. Once this process was initially completed, the result was sent out for feedback from teacher and curriculum specialist reference groups. The next stage in the process is public consultation.

The ACARA team had in place its own programme of research which made them aware of key issues they wanted to look at further in Statistics and Probability. However, work with outside groups, like the Center for Curriculum Redesign, and drawing on Australian research in the field (Bargagliotti, 2020; Callingham & Watson, 2005; Callingham & Watson, 2017; Franklin, 2007; Watson & Callingham, 2020), led them to seek additional feedback on aspects of the work. Their two guiding questions for the collaboration were: (1) *In what way would engaging with the CMF and the Cambridge team support/validate the revisions to the Statistics and Probability strands of the revised curriculum?* And (2) *If adjustments/additions are made based on engagement with the CMF, what led the ACARA review team to make these changes?*

The Cambridge Mathematics Framework (CMF)

The CMF is a tool for conceptual mapping in educational design which supports research-informed design decisions in mathematics education. It consists of a searchable network of key mathematical ideas and the relationships between them in the domain of school mathematics, along with a set of tools for exploring and analysing the network and

descriptions of what these ideas look like in the classroom. These ideas are ordered in relation to their interdependence, not tied to year ranges, and this provides the opportunity for designers to make choices of their own with respect to temporal sequencing.

The network is derived from interpretation and synthesis of mathematics education research carried out by the Cambridge Mathematics team. The ideas in the network are linked to underlying research sources and can be accessed in the form of dynamic maps which are presented with corresponding Research Summaries, which tell and reference the stories of the map representations with respect to the research sources. External content, like curriculum statements, tasks or assessment items can be linked to the network to help designers to analyse how the ideas underlying their work depend on each other, as was the case with the ACARA collaborative work.

The goals of Cambridge Mathematics involve domain coherence at different levels of educational design, and the CMF is intended to inform design work at different scales: national, regional, and school-level curricula, resources, and even lessons in some contexts. All levels are important for optimal impact, but opportunities to trial the CMF are more frequent for smaller resources. The Cambridge team viewed this collaboration as a valuable contribution to its current formative evaluation goals. In this case, they wanted to examine whether the CMF as a reference tool was meaningful, trustworthy, useful, and usable for curriculum design spanning a range of years in school mathematics.

The CMF situates statistics education as learning how to understand variability in data (Macey et al., 2018). This variability is expressed through the concept of a distribution and exploration of its graphical and mathematical representation. Figure 1 shows an example of this and illustrates the materials the ACARA team was working with; the map shows the highly connected waypoint "knowing simple distributions" which draws together the sometimes-disparate ideas that underpin the concept of a distribution, and establishes a stepping point for more advanced statistical concepts that rely on it.

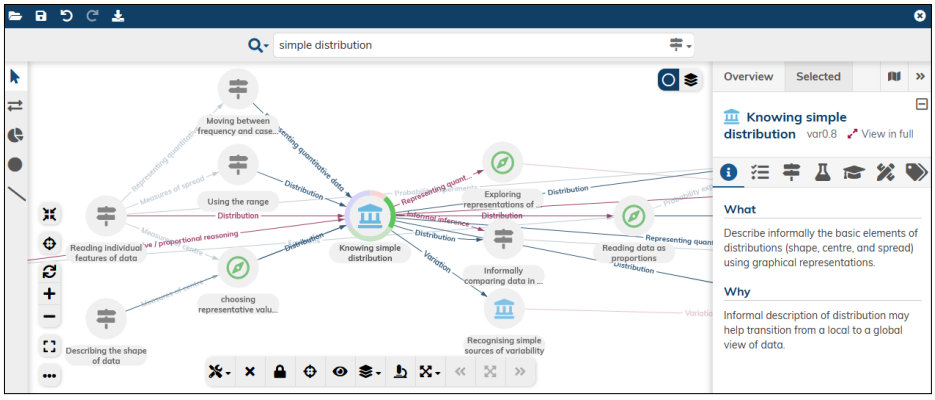

Figure 1. A view of a portion of a map within the CMF

Methods

The collaboration between the ACARA and Cambridge teams took place primarily in and between seven meetings from June - August 2020. After an orientation meeting in which the two teams discussed the context and established mutual goals, they met again for the Cambridge team to introduce the features of the CMF and demonstrate how to search and how to work between maps and detailed descriptions. The Cambridge team linked ACARA's original curriculum statements to mathematical ideas expressed in the CMF and produced

underlying maps of ideas and relationships which they provided to the ACARA team for consideration. Having previously piloted the CMF in the design of the UNICEF Learning Passport mathematics curriculum (Oates et al., 2020), which spanned a wide year range, the team was able to apply ideas from that project to the ACARA review.

The ACARA team kept diaries and notes on a weekly basis as they worked with the CMF. The Cambridge team used the diary-interview method, adapted from Zimmerman & Wieder (1977) to develop a detailed picture of their activities. One ACARA team member kept a running diary, while others kept notes, and in each joint discussion the ACARA team would raise issues which had come up in their work over the past week, having to do with the content, use of the CMF, or both. In the final meeting before the revisions went out for initial review, the ACARA team debriefed the Cambridge team on the full diary and their sense of how things had gone overall relative to their interests and expectations.

Outcomes and discussion

Ways of working with the CMF

The ACARA team identified the location of core concepts in the CMF and explored similarities and differences in the way these concepts were represented and the landscape of other connected ideas. This process helped them to clarify what they thought the core concepts were and how things could be structured around them for students to approach and investigate. To do this, they used search features and structural cues in CMF maps. After reflecting on this process, they noted that "there was sufficient detail" in the map "to provoke further exploration of ideas but without predicating the outcome, so it can be a tool for critical inquiry". It was possible to find and recognise "big ideas writ small" and then continue to the next issue.

The higher-level core concepts, structure, approach and mathematising, had already been transformed to key organisers for a larger set of core concepts so that these could be revised and restructured more usefully. From this process, what it means to reason stochastically became a structural focus. Proficiencies like problem solving and reasoning are always embedded in specific content areas, and the ACARA team reported that the CMF helped them to do this more meaningfully, integrating content statements with proficiencies and bridging between the statements and the bigger picture.

Within the timeframe for the review, the ACARA team found themselves choosing what to pay attention to in the CMF based on what most surprised them based on their expectations and prior understanding. When they identified areas requiring particular attention, they used not only maps but some of the more detailed information in the CMF, including descriptions of ideas, rationale for structure and examples of what it looks like in the classroom when students are working with them. They referred to the research summary level of the CMF as applicable for more detailed investigation, however, as these summaries had already been reviewed by external researchers in general, they trusted that research had been reasonably and robustly interpreted.

Use of research synthesis for validation and change

The ACARA team found that research synthesis in the CMF provided further validation for many of the revisions they were planning based on the research they had already consulted. They found there was a high level of consistency with their existing

understandings, but that some things stood out as being particularly surprising, and it was these that drew their attention for further investigation.

There were a few notable areas in which the ACARA team decided to adjust content and sequencing based on the implications of research synthesis in the CMF; four examples are given below.

- Before: There was initial concern that the pairings of measurement and geometry, statistics and probability was restricting development of other connections – the ACARA team knew there were connections between measurement and statistics which weren't being explored.
- After: Some of the connections they found in the CMF led to rich discussions around how connections between the mean, error and measurement could be made and actively furthered in the curriculum presentation on the website.

- Before: Summary statistics, which are introduced close together in the current curriculum, sometimes leading to students being unable to distinguish between mean, median and mode later on, as well as to 'procedural approaches' that lacked understanding of what the measures are and why they'd be of interest.
- After: The separation of these statistics as distinct ideas with distinct relationships to other topics in the CMF which built up to them prompted the ACARA team to make several changes. They moved mean and median around to get at deeper conceptual understanding of each and to introduce them at different times, shifted from frequency to mode, and introduced ordinal data, which wasn't included previously, so students would engage with these concepts sooner.

- Before: The notion of distribution was mentioned 11 times in content and achievement standards across 10 year levels, but nevertheless seemed procedurally driven and not conceptually connected for the ACARA team.
- After: After discussing research implications which were apparent in the CMF, they shifted to embedding expectation of reasoning about representations, conceptual understanding, and connections. Distribution is now mentioned only twice but it is richer in that it points to how to talk about distributions in terms of their characteristics (spread, skewness, etc.).

- Before: The ACARA team felt that some connections between probability and statistics were not being made.
- After: The idea in the CMF that probability estimates are the result of narrative frequency was used as a way to bring statistics and probability together more explicitly.

Use of maps as shared artefacts in discussing decisions with stakeholders

The ACARA team felt the maps they were working with would be a useful contribution to discussions with reviewers in which they might need to provide justification for their decisions. Not only did the maps link to research sources and research summaries (synthesis documents), but they also showed what some of the key sequencing decisions were as a result and allowed the conversation to focus on these areas. The full consultation with teachers has not yet begun, but the curriculum and teacher reference groups have provided initial feedback. Qualitative feedback from the combined teacher and curriculum reference group indicated they had seen a positive development in the statistics strand from the original version of the material.

Drawing on this, the ACARA team identified instances where the CMF was used to provide justification for decisions in a way that reference group members agreed was clear and helpful. In one example, CMF maps were used to illustrate the reason for separating statistics and probability as different strands. Some reference group members with a particular focus on statistics felt that the "end-game" or big picture was more apparent, and that it helped see the purpose and meaning of particular decisions. The ACARA team felt it gave them more confidence in laying out their perspective, knowing their reasons had research behind them and they could trace choices back to this in a discussion.

Formative evaluation

Just as the ACARA team found the CMF useful for analysing gaps, ordering, and coverage, the Cambridge team found the reverse was also true. The ACARA team's critical engagement with the CMF as curriculum designers provided valuable formative feedback on the representation of mathematical ideas in the CMF, the tools available for working with relevant information and how these could be efficiently accessed and effectively used. Several points from the Cambridge team's evaluation themes are below:

1. *Meaningful*: Overall, the ACARA team recognised within the CMF concepts which they were working with, realised implications, and made meaningful decisions. There were particular areas in which it became clear during discussion that some implications were not explicitly represented in the CMF. In such cases, CMF content was further refined and possibilities for other supporting documents were raised.
2. *Trustworthy*: The ACARA team themselves felt the CMF provided them with good justifications for their curriculum revisions. Other stakeholders agreed.
3. *Useful*: (a) Because the CMF is a dynamic digital online tool, the collaboration demonstrated that it was productive for two teams across the world from each other to interact virtually around the same artifacts. (b) A theme running throughout the joint discussions was the notion of perspectives from research being represented explicitly vs. implicitly; the Cambridge team realised some perspectives needed more explicit and actionable support, either in the network or the guidance documents. Discussions like this are useful to identify whether other assumptions about what is implicit in design need to be made more real for designers.
4. *Useable*: (a) From the ACARA team's perspective, the CMF "made the research usable" and "did the heavy lifting in a limited time frame". They noted the CMF helped them to overcome time and resource constraints to bring new and well-synthesised research influences into the review. (b) The ACARA team found their first exposure to the CMF mapping environment to be demanding, but it progressively became more comfortable and they felt it had been worth getting over the initial familiarisation hump. The Cambridge team could provide additional support to streamline this process. (c) The ACARA team concluded that using the CMF was not a shortcut in terms of time spent, but they felt the output reflected a broader range of research and was more coherent, helping them meet review goals.

Conclusions

The ACARA team entered the collaboration seeing potential in the CMF as a tool for validation, conceptual insights, construction and exploration, and they agreed that these goals had been met. The process that worked for them involved using the CMF for a combination of individual exploration, group decision-making and justification activities,

providing some evidence that the design of the CMF supports active professional decision-making. Reflecting on the outcomes, the ACARA team identified opportunities where the CMF could be used in other strands beyond statistics and probability. The Cambridge team continues fine-grained analysis of interview data which can inform refinement and future use of the CMF for curriculum design, and is in the process of following up on suggestions which emerged from the process. This collaboration demonstrated the value of the CMF as a map-based design tool to support mathematics curriculum design, and processes emerged which will streamline its use in future versions.

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The development and efficacy of an undergraduate numeracy assessment tool

Andrea Knowles
University of Otago
<andrea.knowles@otago.ac.nz>

Chris Linsell
University of Otago
<chris.linsell54@gmail.com>

Boris Baeumer
University of Otago
<boris.baeumer@otago.ac.nz>

Megan Anakin
University of Otago
<megan.anakin@otago.ac.nz>

This paper describes the development and efficacy of an online tool for assessing the numeracy of undergraduate students. The tool was designed to be easy to administer, provide immediate feedback to students on whether they had the required level of numeracy, and to be consistent with other measures of adult numeracy. When used with students taking a mathematics or statistics course, we found a significant correlation of r = 0.45 between their numeracy score and final mark in their enrolled course. Students who had a numeracy score less than our threshold had a 30.6% probability of failing their course, whereas students who had a numeracy score of at least our threshold had a probability of failing of only 8.0%.

We define *numeracy*, in an undergraduate university context, as having the knowledge, skills, and confidence to use mathematical tools in a range of disciplinary contexts. Tertiary educators may expect students entering their programmes to have the prerequisite numeracy to successfully complete their quantitative courses. However, student performance does not necessarily align with these expectations (Parsons, 2010). Students lacking numeracy skills are less likely to continue with a course when they are faced with difficulties with quantitative material (Matthews et al., 2009). Large scale numeracy assessment tools such as the Literacy and Numeracy Test for Initial Teacher Education (LANTITE) (Australian Council for Educational Research, 2016) and the Literacy and Numeracy for Adults Assessment Tool (LNAAT) (Tertiary Education Commission [TEC], 2008), have been developed to provide detailed feedback to individuals about their numeracy competency. Such tools are aimed at measuring the level of numeracy demonstrated by an individual rather than establishing if that person has a sufficient level of numeracy to be successful in a particular situation. Therefore, we sought to develop an *undergraduate numeracy assessment* (UNA) tool that could be used specifically for identifying if students have the prerequisite level of numeracy to enable them to be successful in their quantitative courses.

Background

The New Zealand Ministry of Education (2009) cautions us on using educational assessment as a sole means of assessing numeracy capability because high school students with high levels of success in formal qualifications may often present with low levels of numeracy. Since expectations from lecturers about students' mathematical competence does not necessarily align with numeracy entry levels (Parsons, 2010), high school leavers who are not identified by their teachers as having problems with numeracy may be identified subsequently in adulthood (Bynner & Parsons, 2006). Furthermore, the teaching of mathematical and statistical knowledge within courses of a quantitative nature does not necessarily link directly to a students' mathematical qualification (Gnaldi, 2006; Taylor et al., 1998).

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We built upon descriptions of students' numeracy difficulties that were generally anecdotal or restricted to mathematical content (Taylor et al., 1998). We identified important underlying numeracy constructs for undergraduate students that included proportional reasoning, understanding of rational numbers, and multiplicative thinking (Galligan & Hobohm, 2015; Linsell & Anakin, 2012; Linsell et al., 2017). These constructs can be found in the large-scale numeracy assessment tools, such as the LANTITE and LNAAT. However, there are limitations when using these tools to assess the numeracy of undergraduate university students. First, students with high attainment take longer to answer questions than students with low attainment (TEC, 2017). Thus, students and education practitioners may feel that the time taken to complete a robust adaptive test across a six-step progression may be arduous or unnecessary. Second, assessment feedback provided to a student describes individual strategies, strengths, and knowledge (Hall & Zmood, 2019; TEC, 2008) but not a level of numeracy competency. Third, the New Zealand TEC has aligned numeracy progression benchmarks in the LNAAT to levels of the mathematics and statistics in the New Zealand Curriculum and to National Certificate of Educational Achievement (NCEA) standards for numeracy assessment (Thomas et al., 2014). A LNAAT score of 605 (Step 5) approximates to the NCEA numeracy standard as required for university entrance. However, further work is needed to confirm whether LNAAT is well aligned and represents numeracy competencies that adults require to be successful in society. Further study is also needed to investigate numeracy competency, to predict success in quantitative courses at the university level. One way to address the limitations of the large-scale assessments is to carefully frame assessment items. We define framing in three ways. First, assessment items need to be encased in appropriate and meaningful contexts (Mason et al., 2009). Second, items must allow for authentic user responses. Third, items must assess conceptual knowledge alongside procedural fluency (Hiebert & Carpenter, 1992). With well framed assessment items, educators may be able to establish a student's numeracy competence and predict their readiness to succeed in quantitative courses.

Development of Assessment Tool

Our aim was to produce a dependable assessment tool that was easy to administer, gave immediate feedback to students on whether they had the required level of numeracy, and that was consistent with other measures of adult numeracy. We decided that an online assessment would be necessary for facilitating marking and giving immediate feedback to students. We had previously used the LNAAT for investigating numeracy of undergraduates (Linsell & Anakin, 2012; Linsell et al., 2017). The LNAAT has been aligned with other measures of numeracy (Thomas et al., 2014) and we therefore decided to benchmark our tool against this.

We wanted to determine whether students had a particular level of numeracy, rather than measure what level of numeracy students had. Therefore, it was unnecessary to set questions that could be answered with lower levels of numeracy than our requirement. Our previous work (Linsell et al., 2017) had indicated that Step 6 of the LNAAT numeracy scale was necessary for success in undergraduate quantitative courses. Furthermore, detailed examination of the responses of students to the LNAAT numeracy questions suggested to us that a score of 740 was necessary, considerably higher than the 690 threshold for Step 6 (Casey & Knowles, 2018). Step 6 includes requirements for students to:

- solve addition and subtraction problems involving fractions, using partitioning strategies;

- solve multiplication or division problems with decimals, fractions and percentages, using partitioning strategies;
- use multiplication and division strategies to solve problems that involve proportions, ratios, and rates;
- know the sequences of integers, fractions, decimals and percentages, forwards and backwards, from any given number.

Our assessment consisted of 20 questions on the topics of fractions, decimals, ratios and proportions, and percentages. Students were required to answer five questions, which covered a range of sub-topics, in each topic.

Using a question format similar to that of the LNAAT, our assessment made use of meaningful contexts, previously unseen by the students, to determine whether the students could use mathematical tools to solve problems. This use of contexts ensured that conceptual knowledge (Hiebert & Carpenter, 1992), rather than just procedural knowledge, was required to solve the problems. Contexts were chosen that reflected the experiences of undergraduate students but that were not specific to any particular academic subject. Figure 1 shows an example of a question that requires students to make use of their knowledge of operating with fractions (this sample question is for illustrative purposes only and was not used in any assessments). The format for this question was multiple-answer, while other questions made use of numeric answers, fractions (both proper and mixed), multi-choice and drag-and-drop formats.

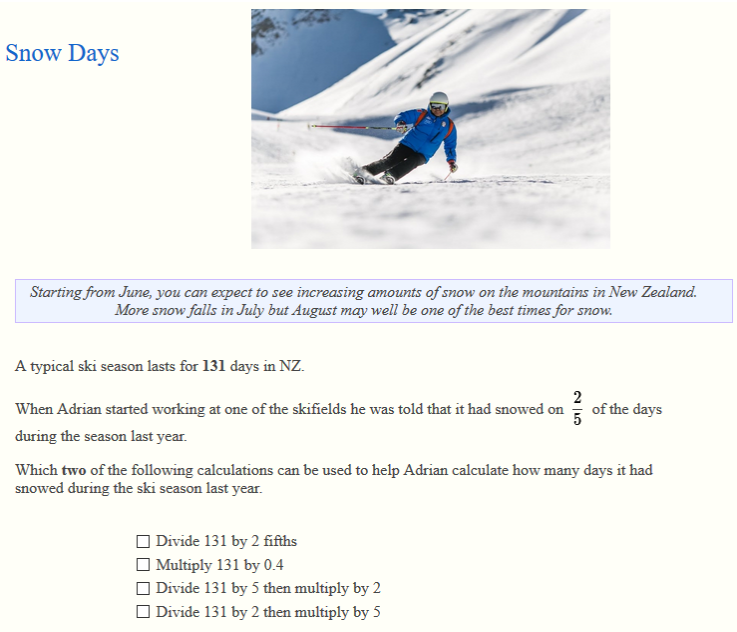

Figure 1. Snow Days question employing multiple answer format.

To ensure authenticity of students' work when sitting the assessment in computer laboratories, we designed the assessment to make it unlikely that nearby students would be answering the same question, or that one student's answer would be useful to another student sitting the assessment later. The assessment used a number of levels of randomisation. In addition to randomising the order of questions, contexts were randomised (e.g., for multiplying fractions the context of recipes was randomised with the context of student allowances) and pictures accompanying the questions were changed accordingly, names of

people, objects, places and courses were randomised (e.g., quantity of flour to quantity of sugar), and the numbers used in each question were randomised. When randomising numbers, it was important to select values that did not alter the level of difficulty of the question (e.g., in the Snow Days question only the fractions 2/5, 2/10, 3/5, 3/10 were used and the number of snow days was randomised between 131 and 139 excluding 135).

The platform we used was adapted and further developed from an online system for assessing first-year university students of mathematics and statistics at the University of Otago. Question presentation was simplified, fractional and drag-and-drop answer formats were added, and the reporting of feedback expanded. The development of the question bank and its benchmarking took multiple iterations of setting the test, analysing answers (e.g., too easy, too hard, misleading etc.), improving questions, and adding questions. The test was first administered in MATH151 *General Mathematics*, and the success rate for questions was found to vary between 28% and 89%. Possible reasons for the range of difficulty were identified and questions were revised. Next, two parallel versions of the test were developed and used in EMAT198 *Essential Mathematics for Teaching*. Again, questions that were particularly easy or hard were identified and modified if necessary. Students taking EMAT198 (n = 67) also sat a LNAAT assessment, which was used for benchmarking. There was a strong correlation of $r=0.45$ ($p<0.001$) between EMAT198 students' scores on UNA and their LNAAT results (see Figure 2). Regression showed that a LNAAT score of 740 corresponded with a UNA score of 14.

We combined all questions (modified if necessary) from iterations 2 and 3 for use in STAT115 *Introduction to Biostatistics* in the second semester. For this fourth iteration the success rate for questions was found to vary between 49% and 92%. This variation is likely to be due to general gaps in students' conceptual knowledge rather than assessment item difficulty. In total, there were five iterations of question development and improvement to develop a test for use in the following year.

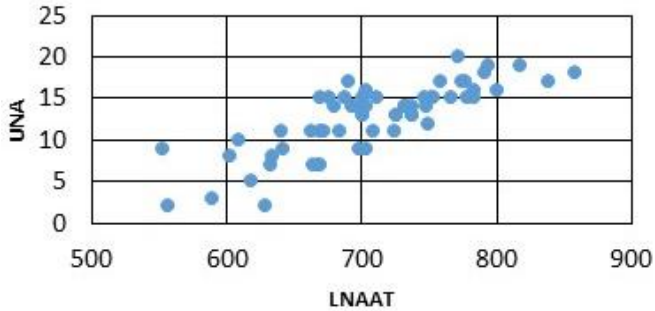

Figure 2. Correlation of UNA vs LNAAT assessment score in EMAT198 (n = 67)

Numeracy of Undergraduates

For students taking MATH151 *General Mathematics*, the UNA numeracy assessment was administered during tutorials in the third week of Semester 1 2019. The test was carried out under exam conditions. Of the 142 consenting students taking MATH151, 131 sat the UNA test, with the remaining 11 students not attending the tutorial in which the test was administered. Students scored between 1 and 20 on the 20-item test ($M=13.3$, $SD=4.2$) (see Figure 3). Sixty students (45.8%) scored less than our threshold score of 14 marks and 24 students (18.3%) scored less than 10 marks.

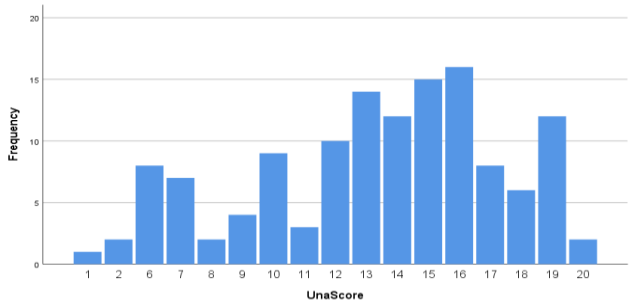

Figure 3. MATH151 distribution of students' scores (n = 131) on the 20 item UNA test

For students taking STAT115 *Introduction to Biostatistics*, the UNA numeracy assessment was completed by students in their own time in the first week of Semester 2 2019 and was unsupervised. However, students were encouraged to take the test to inform themselves of their numeracy needs and were given five marks towards their final grade in the course for taking the test. Of the 785 consenting students taking STAT115, 701 sat the UNA test, with the remaining 84 students opting not to do so, despite the inducements. Students scored between 0 and 20 on the 20-item test (M=14.9, SD=4.7) (see Figure 4). One hundred and eighty-eight students (26.8%) scored less than our threshold score of 14 marks and 90 students (12.8%) scored less than 10 marks.

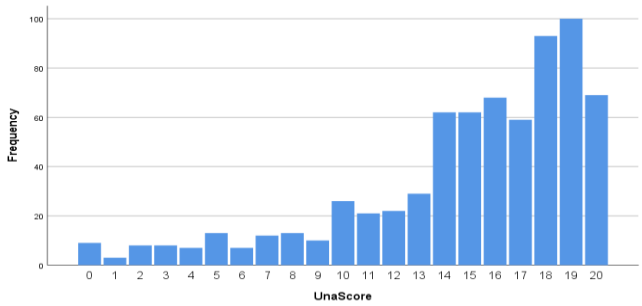

Figure 4. STAT115 distribution of students' scores (n = 701) on 20 item UNA test

As can be seen from Figures 3 and 4, the distribution of scores for STAT115 students sitting the test independently is rather different to that for MATH151 students sitting under exam conditions. Not only did a smaller proportion score less than our threshold score, but a much higher proportion scored 18 or more on the 20-item test. This difference could be accounted for by the variation in testing procedures rather than any differences between cohorts of students. The numeracy and attainment of the two cohorts is explored further in the next section.

Numeracy and Attainment

Overall, there was a strong and significant correlation of r=0.45 (p<0.001) between UNA numeracy score and the final mark of students in MATH151 and STAT115. Students who had a numeracy score less than our threshold of 14 marks had a 30.6% probability of failing their course, whereas students who had a numeracy score of at least our threshold had a probability of failing of only 8.0%. However, a much clearer picture is obtained by examining the attainment in MATH151 and STAT115 courses separately.

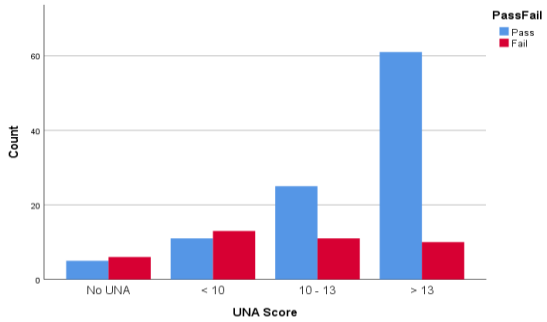

Figure 5. MATH151 students' attainment (n = 131) on course vs UNA score

For MATH151 there was a strong and significant correlation of r=0.41 (p<0.001) between UNA numeracy score and the final mark in the course. Of the students scoring less than 10 marks, 54% failed MATH151 (see Figure 5) with a mean score of 41% (M=41, SD=32). Similarly, 31% of students scoring 10 to 13 marks failed MATH151 with a mean score of 55% (M=55, SD=28). Only 14% of students scoring 14 or more marks failed MATH151 with a mean score of 71% (M=71, SD=26). It was interesting to note that the students who did not attend the tutorial and therefore did not sit the UNA test had a similar failure rate to those students who scored less than 10 marks. The failure rate (54%) for students scoring less than 10 marks or not sitting the UNA test was 3.9 times as high as the rate (14%) for students who achieved at least our threshold score of 14 marks.

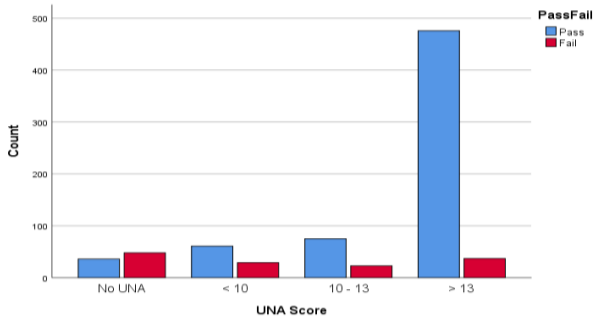

Figure 6. STAT115 students' attainment (n = 701) on course vs UNA score

For STAT115 there was a strong and significant correlation of r=0.46 (p<0.001) between UNA numeracy score and the final mark in the course. Of the students scoring less than 10 marks 32% failed STAT115 (see Figure 6) with a mean score of 56% (M=56, SD=17). Similarly, 24% of students scoring 10 to 13 marks failed STAT115 with a mean score of 62% (M=62, SD=20). Only 7% of students scoring 14 or more marks failed STAT115 with a mean score of 76% (M=76, SD=17). It was extremely interesting to note that the students who chose not to sit the UNA test had a failure rate even higher than those students who scored less than 10 marks. The failure rate (44%) for students scoring less than 10 marks or not sitting the UNA test was 6.3 times as high as the rate (7%) for students who achieved at least our threshold score of 14 marks.

Discussion and Conclusions

We used assessment items from UNA with students enrolled in EMAT198 to reliably calibrate using regression analysis against the LNAAT test to map a threshold score of 14

on UNA with the LNAAT adult progression at Step 6 and a score of 740. This score is higher than the 605 (Step 5) benchmark which corresponds to NCEA Level 1 numeracy assessment (Thomas et al., 2014) that is required for university entrance. Results from 832 students enrolled in mathematics and statistics courses within this study, using a UNA benchmark score of 14, indicate a significant correlation between UNA score and final examination result, demonstrating its suitability across a range of undergraduate courses with quantitative material. Furthermore, the cost and management of large-scale assessment (Brumwell et al., 2018; Hall & Zmood, 2019) can be mitigated by the provision of a well framed, 20 item assessment, which identifies a particular level of numeracy competence (Galligan & Hobohm, 2015) rather than a description of a learners' strategies, strengths, and knowledge (TEC, 2008) making it both time and financially advantageous. The importance of presenting questions in real-life contexts (Norton, 2006; Mason et al., 2009) is widely understood. Furthermore, UNA uses familiar adult contexts to assess the use of conceptual knowledge rather than procedural fluency (Hiebert & Carpenter, 1992).

In describing how the UNA was developed, we also demonstrated the efficacy of the UNA to identify whether students had a particular level of numeracy rather than measure what level of numeracy students had. This decision allows us to not only analyse the data but consider appropriate actions to take as a result (Blaich & Wise, 2011). The next steps are to examine how other disciplines, such as commerce, health sciences, and humanities, may use the UNA. Expanded use of the UNA may assist lecturers to question and examine their expectations about their students' mathematical competence and its alignment with numeracy entry levels (Parsons, 2010). Additionally, educators may find the UNA convenient for identifying the number of students who are likely to experience conceptual difficulties in their course. The UNA also provides an alternate source of numeracy feedback to educators that is consistent with other measures of adult numeracy such as the LNAAT. Educators may use results from the UNA to suggest that identified students seek numeracy support. To this end, students may be more likely to continue with the course and complete it successfully.

Further areas to address include: developing a larger bank of questions in the context of students' specific disciplines (e.g., nursing, pharmacy, business); and the process and potential issues (e.g., resources, time) in scaling up the use of UNA across an institution. We anticipate that educators will find the UNA useful for identifying if students have the prerequisite level of numeracy to enable them to be successful in their quantitative courses and that it will be a dependable assessment tool that is easy to administer, provides immediate feedback to students, and is consistent with other measures of adult numeracy.

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Why should we argue about the process if the outcome is the same? When communicational breaches remain unresolved

Joanne Knox
The University of Auckland
<j.knox@auckland.ac.nz>

This paper uses the commognitive framework to analyse how a group of four primary school students classify odd and even numbers. The findings show how students' reasoning is grounded in their personal uses of "odd" and "even". The students attend to different features of "oddness" and "evenness" and agree on *which* numbers are odd and even but disagree about *why*. The findings highlight the role that proving can play in signalling differences in reasoning within a group of students that may otherwise remain hidden. However, they also suggest students' awareness of the breach in communication may not be sufficient to engender a resolution, even when pedagogical moves toward this direction are made.

Mathematical proof is fundamental to the work of mathematicians, and many educators maintain that it should also be a fundamental part of school mathematics (CadwalladerOlsker, 2011). However, proving activity has been neglected in mathematics education (Stylianides, 2016), especially in the primary classroom. Accordingly, there have been recent calls recommending proving for all mathematical content areas and across the grades. For example, the PISA 2021 framework (OECD, 2018) highlights the centrality of mathematical reasoning and reforms in some countries' curriculum documents also now require proof and proving to be taught at all levels (e.g., Common Core State Standards Initiative [CCSSI], 2010; Department of Education [DfE], 2013; NCTM, 2000).

Although the fundamental purpose of a mathematical proof is to know whether a mathematical assertion or idea is true or false (CadwalladerOlsker, 2011), proving also has a more practical role in explaining and convincing others about our statements or theorems (Stylianides et al., 2017). It is through this practical role that proving has potential to support deep learning and sense-making. For instance, the NCTM's (2000) standards refer to proofs as offering "powerful ways of developing and expressing insights" through which "students should see and expect that mathematics makes sense." (p. 4).

What constitutes proof and proving at the primary level, however, is not entirely clear. Whilst it is unlikely that formal, deductive proofs expressed algebraically would be within reach of typical primary school students, Stylianides (2007) provided empirical accounts of how young students' informal arguments could be mapped onto corresponding formal proofs. These student arguments made use of manipulatives or diagrams to provide visual demonstrations of a generic example. Building from this research, Stylianides (2016) defined a proof as an argument, which is accepted by the classroom community and, uses and communicates reasoning in ways that are endorsable by the wider mathematical community but are also within reach of the classroom community. Nevertheless, even with a working definition of primary-level proofs, there is little research that explores *how* young students' arguments develop and become accepted within the classroom community. In this paper I utilise Sfard's (2008) *commognitive framework* to provide insights into how young students' arguments unfold as they *substantiate* (verify with evidence to prove why a reason is true) their classifications of numbers as even or odd.

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Theoretical Framework

Sfard (2008) defines mathematical discourse as a special form of communication, including self-communication (thinking), that is distinguishable via four interrelated characteristics: Its *keywords* (e.g., 'odd', 'even') and their use; its *visual mediators* (e.g., numerals, symbols, counters, pictures); its endorsed *narratives* (e.g., theorems, proofs, conjectures, definitions), and; its *routines* – discursive patterns, according to which mathematical tasks are being performed (e.g., the ways in which interlocutors substantiate oddness and evenness). Learning is seen as a lasting transformation in a learner's discourse, which is identifiable by changes in one or more of these four characteristics.

In terms of the keywords of interest to this study, 'odd' and 'even' are labels that function as nouns to denote discursive mathematical objects which may be *realized* in a multitude of ways; infinitely many numbers (e.g., 'odd' could be one, seventeen, one billion and one; 'even' could be two, forty-six, three million and eight) and each of these numbers could be realized as numerals (e.g., 1, 17; 2, 46), icons (e.g., an arrangements of dots) or symbolically as algebraic expressions (e.g., $2n+1$; $2n$). However, the illusory nature of mathematical objects (being products of our discourse as oppose to actual, tangible objects) entails that none of these *realizations* could be singled out as being 'the' object. During initial phases of learning, learners may have limited realizations of the signifiers 'even' and 'odd': Evidence of an expansion of realizations signals learning.

Another characteristic feature denoting the development of discourse is the level of *objectification*. Sfard (2008, p. 44) defines this as a process involving both *reification*– replacing talk about processes with talk about objects – and *alienation* – presenting phenomena impersonally, as if they were occurring independently from human participation. For example, when someone speaks of 'even' as "numbers that can be shared equally between two people", an activity (sharing) is indicated and the word 'even' acts as an adjective describing numbers. Whereas in the sentence "even plus even is even", 'even' has been objectified: The word is used as a noun that encapsulates all even numbers and realizations of even into one set, giving it separation from any activity and more permanence.

According to the commognitive framework, development occurs through the learner's exposure to, and participation in, the discourse he or she is supposed to individualise, and the support he or she receives from other participants. Encounters between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways, or perform the same mathematical tasks according to differing rules, have an indispensable role in this (Sfard, 2008, p. 162). Such encounters, termed *commognitive conflicts*, provide space for participants to consider new ways of talking, which is a prerequisite for experiencing a change in what they see. Sfard (2008, p. 258) maintains that resolving a commognitive conflict involves one of the interlocutors gradually accepting and adopting the incommensurable discourse and abandoning his or her own.

With regard to the group of students in focus, in this paper I ask, "What are the sources of commognitive conflict in the context of classifying odds and evens?" and "When and how can a commognitive conflict fail to give rise to a modification in students' discourse?"

Research Design

The present data is taken from a larger study aiming to investigate how students' arguments unfold and develop as they engage in proving activity. Year 4 students from two NZ schools were selected by their teachers to be withdrawn from their class to work in groups of four at a time with me (as a teacher-researcher) on three different tasks: (1) classifying numbers as

odd or even; (2) proving conjectures about the sums of odds and evens; and (3) proving conjectures about the products of odds and evens. As the unit of investigation in this study was discourse, teachers selected students and groups according to whom they considered would be willing and able to engage in dialogue.

The data presented in this paper is taken from one group of four 8-year-old students as they participated in the first task. The students took turns to classify numbers presented on cards as odd or even and, as each card was classified, they were asked to substantiate their classifications and were encouraged to consider one another's questions and thinking. The cards displayed increased in complexity, from single digit numbers (shown as Numicon tiles or numerals) to six-digit numbers. Numicon tiles are visual representations of numbers 1-10 presented as dots within a frameless 2 x 5 rectangle (see Figure 1).

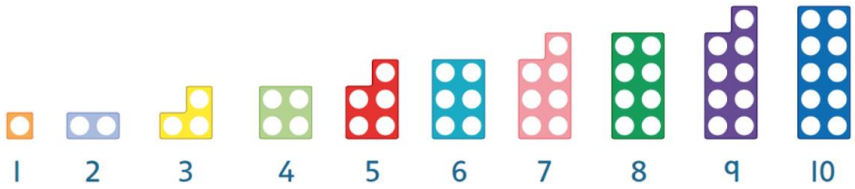

Figure 1. Numicon tiles.

The main aim of engaging students in this first task was to provide a baseline discourse for each student (i.e., what they already knew about 'odd' and 'even'), enabling me to track their learning (observable via the development of their discourse). The group sessions were audio and video-recorded, and their conversations were closely transcribed along with corresponding and relevant details about what the students did (e.g., gestures, facial expressions, actions, photos of their work). Here I conducted detailed discourse analysis utilising Sfard's (2008) commognitive framework to look for well-defined, repetitive patterns (routines) in students' discourses regarding their use of the words 'odd' and 'even' and their substantiating narratives about oddness and evenness. I also made use of *realization trees*. Whilst Sfard used "realization trees" (p. 165) to map personal realizations based on observations of the individual person implementing them, I constructed a *combined tree of realizations* for the group, mapping each interlocutor's observable realizations along specific branches, to help examine and clarify consistent and inconsistent uses of words within the group.

Findings and Discussion

Throughout the classification task all four students correctly classified the numbers and they agreed with the classifications made by their peers. Indeed, if the students had been asked to simply sort the cards into odd and even boxes with no justification, it would have been easy to assume that they held a common understanding about odd and even numbers. However, when I examined the routine ways the students substantiated their classifications, it became clear that their use of the words 'odd' and 'even' was different. Sfard (2008) states that interlocutors' word use (or what others might call 'word meaning') is important because "it is responsible for what the user is able to say about (and thus see) in the world." (p. 133).

Due to the scope of this paper, data that illustrate the students' word uses have been compressed in Table 1, rather than shown in their entirety. I include the turn number to provide the reader with an idea of turns elapsing and to enable me to refer to key turns within

my analysis. A combined tree of realizations (Figure 2), constructed from the student group's discourse, is provided to visually illustrate the students' word uses.

Table 1
Student Substantiations for Classifying Numbers as Odd or Even

Card shown and teacher's question	*Jane, Zara and Robert's responses*	*Danny's response*
Why is 'six' even? 	[31] Jane: Because three plus three equals six and that's even.	[32] What the heck! That's not right! [34] That's so not right… three plus three… how does the three come here?
Why is 'nine' odd? 	[43] Zara: It's got a one there. (Pointing to the single one at the top of the Numicon piece.).	[45] So, even are always first, 'cos zero, two, four, six, eight, ten. And then the odd numbers are starting from one—one, three, five, seven, nine, eleven. It goes like that. So that nine is odd.
Why is 'four' even? 	[55] Robert: Because, two and two. [59] Robert: Mm… er, because two plus two. [73] Zara: Erm also if, if you had two people then you'd be able, they'd both get two each.	[58] But how did you get the two? [60] How did you get the two?! Where's the two? How did you get the two? [70] It's just the sequence. Same as the Fibonacci sequence and the other sequences. [79] I know something. So, it's like always like even-odd, even-odd, even-odd a number. [81] No odd can, odd can still be a fair share 'cos you can split it up into decimals. Like with seven you can make three point five.
Why is 'five' odd? 	[90] Jane: Because a four and one. [92] Jane: Because the four is even, but five has like… [93] Zara: Instead of adding two on, you add on one and then it wouldn't be even. [94] Zara: So, two, two and one.	[91] But how did you get the four and one? [102] I know something. So, everything is like even-odd, even-odd, then even, then odd, so that's odd.
Why is 'twenty-two' even? 	[119] Jane: Because eleven and eleven.	[120] All you need to know is like if it ends with a… if… This is like a simple way- if it ends with a zero, two, four, six or eight it is an even number… [122] And if it's one, three, five, seven, nine it's odd.

Jane, Zara, and Robert's substantiations of evenness and oddness

Table 1 shows that Jane substantiates evenness as referring to numbers formed by adding the same, or an even amount to itself (a double) to make a number [31, 119]. Her realization of the signifier 'even' is characterised by Branch 2a on the combined tree of realizations. Similarly, Robert substantiates the evenness of four by attending to "two plus two" [55, 59] and so his realization is also aligned with Branch 2a. Jane substantiates numbers as being

'odd' because they are an even number "and one" [90] (shown on Branch 2b). Zara has two substantiation routines for evenness: When two people have equal shares [73] (shown on Branch 2a) and, numbers that are formed by "adding two on" [93] (shown on Branch 2c). Her substantiations show she realises oddness as being unlike evenness because the structure deviates from 'adding two', to having "a one there" [43, 93, 94] (shown on Branch 2d). Whilst Zara's substantiations of oddness and evenness are not completely identical to Jane's and Robert's, the student group share a common branch (Branch 2) because their realizations of the signifiers 'even' and 'odd' attend to the symmetrical or asymmetrical *structure* of such numbers. I refer to their discourses about odd and even as being *structure-based*.

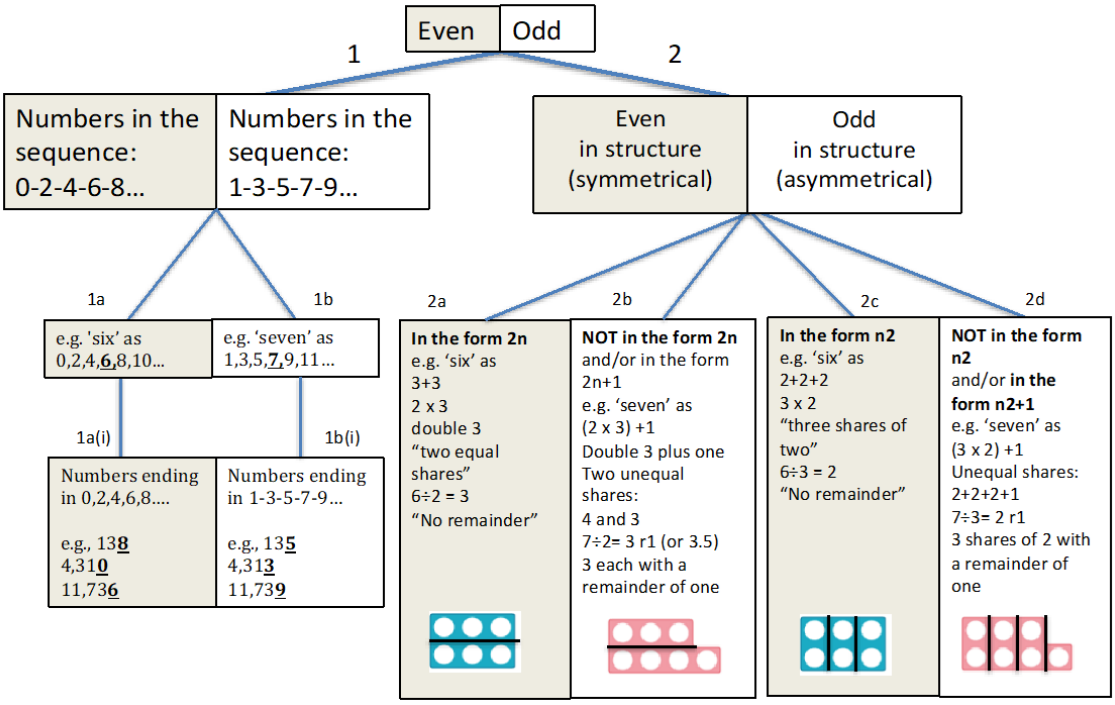

Figure 2. A combined tree of realizations showing the group's realizations of the signifiers 'odd' and 'even'.

Danny's substantiations of evenness and oddness

In contrast to the other students, Danny rejects (sometimes vehemently) structure-based substantiations of evenness and oddness [32, 34, 58, 60, 81, 91]. Even though the first three cards were presented as Numicon tiles, making the structure of these numbers visibly salient, his comments about Numicon 6 [32, 34] and Numicon 4 [58, 60] suggest he cannot even see the 'three plus three' in six nor see the 'double two' in four. However, Lavie and Sfard (2019) warn the researcher's tendency to look for things that children don't or can't do yet, means that they "remain oblivious to the possibility that the child's response to the [task situation]… may be about something else" (p. 423). Indeed, Danny's apparent bafflement and rejection of Jane's narrative is not necessarily because of an inability to see doubles but because, for him, his use of the word 'even' has nothing to do with a doubling criterion. Danny's routine uses of the words odd and even become apparent in his substantiation of nine as odd [45], and he repeats this substantiation for each subsequent number presented. For Danny, just as there is a Fibonacci sequence [69] and numbers within this set are '0,1,1,2,3,5,8…', the signifiers 'odd' and 'even' are sanctioned by sets of numbers in the

sequence of ‘even-odd-even-odd-...’. His substantiations are shown by Branches 1a (for even) and 1b (for odd). When the task changes to include numbers with more than one digit, Danny elaborates on his substantiation of oddness and evenness, adding a ‘check the last digit’ [120; 122] procedure to his ‘check for place in with sequence’ procedure (shown on Branches 1a(i) and 1b(i)). Accordingly, I refer to Danny’s discourse about odd and even as being *sequence-based*.

Level of objectification

Having illustrated how Danny’s realizations of the signifiers odd and even are different to those of the other students, I now point to a further point of difference between their discourses in terms of the degree of objectification. Jane’s, Robert’s and Zara’s substantiating routines refer to numbers on the cards as specific concrete objects that serve as realizations of ‘odd’ and ‘even’, and these routines require an action. According to their routines, one is required to check if the specific numbers can; be *made* by a double [31, 55, 59, 119]; *make* two fair shares [73], or; be *grouped* in twos to prove evenness [93]. Oddness is proven where an even result is not possible or in instances where a remainder of one or unequal shares are *created* [43, 90, 93, 94]. These students also tend to use the words ‘odd’ and ‘even’ as adjectives; for example, when Jane substantiates the evenness of six, she describes the *even quantity* of “three plus three” [31]. In contrast, Danny’s discourse replaces talk about processes on concrete objects with talk about ‘even’ and ‘odd’ as *mathematical objects* existing in their own right, each as a condensed set of numbers reified from his known sequencing procedures. And when Danny uses the words ‘odd’ [81] and ‘even’ [45] they serve as nouns rather than adjectives; for example, “odd can still be a fair share”. In short, for Danny ‘odd’ and ‘even’ is the sequence itself, just like the Fibonacci sequence [70], whereas for the other students, these keywords appear as describing features derived from actions on specific numbers. These characteristics all provide evidence to suggest that Jane, Robert, and Zara are in the process of discovering generalizable features of odd and even, and show Danny’s sequence-based discourse on odd and even to be more objectified (and thus more entrenched) than the structure-based discourse of the other students.

The (unresolved) commognitive conflict

The exchanges in Table 1 present an example of what Sfard (2008) calls “commognitive conflict” (p. 161): The students have realized the signifiers ‘odd’ and ‘even’ in different ways and so are classifying numbers as odd or even according to different rules. The different branches of realizations (Figure 2) illustrate the differences in the students’ substantiations of evenness and oddness and thereby expose the source of the breach in communication: Jane, Zara, and Robert have a shared *structure-based* branch of realization for odd and even and so are able to communicate their process of classifying odds and evens effectively with one another whilst Danny’s *sequence-based* branch of realization for odd and even is disconnected from the others’, meaning he is unable to communicate his process effectively with the other three group members.

To support the students to resolve the commognitive conflict, the teacher-researcher makes several attempts to scaffold their participation in one-another’s discourses. An example of this can be seen in Table 2, where the teacher-researcher has assumed that Danny cannot see the double structure of even numbers and the asymmetry of odds, and she attempts to scaffold his participation in this structure-based discourse.

Table 2
Why is 'Five' Odd?

Line	Speaker	What was said	What action occurred
103.	Teacher:	And you were a bit confused weren't you. When you said, "Where's the four and the one come from?" Is that right? When she [Jane] said, "Five is four and one".	Directed to Danny.
104.	Danny:	I know, I know. I know why. Cos five is like split and there's a four and there's a one but that doesn't make any sense in high-school maths, 'cos it goes like...	Points to the Numicon 5 piece.
105.	Jane:	We're not doing high school maths.	
106.	Danny:	I know but, but it doesn't make any sense to like a really good, really good mathematicians 'cos it has no sense.	

This exchange is interesting as it shows signs of Danny attuning (albeit very slightly) to the discourse he initially rejected. And yet even when he eventually begins to endorse the other students' structure-based substantiations, he positions them being substandard to his own routine when he says theirs' "doesn't make sense" in "high school" (a more authoritative setting) and by "really good mathematicians" (people who have higher mathematical status) [104, 106]. In other words, he elevates his sequence-based substantiation routine as one that *does 'make sense'* and *is endorsed* by people and places of mathematical authority. By doing so, he maintains incommensurability between the two discourses. Hence, the teacher's pedagogical move to encourage Danny to make sense of the other students' did not result in him endorsing their substantiations.

For group members to resolve commognitive conflict, a "gradual mutual adjusting of their discursive ways" is required (Sfard, 2008, p. 145). However, during the entire classification task there was little evidence to suggest this occurring. The failure to resolve the conflict can be attributed to two factors. Firstly, the conflict was not about the outcome of classifying numbers as odd or even, it was about how the students substantiate oddness and evenness. For the students, deciding which numbers are odd and even was the goal of the task so they had no reason to resolve it because, on this, they agreed. Secondly, although the teacher encouraged the students to share their thinking and participate in each other's discourses, Jane, Robert, and Zara's structure-based routines were supporting them to make sense of and explain generalizable properties of even and odd, where Danny's routine way of substantiating oddness and evenness was too objectified for this purpose. And Danny rejected the other students' structure-based substantiation routines because his sequence-based routines were more entrenched and, not only did they work and produce the same outcome, they also were more efficient than the alternatives. From David's perspective the structure-based substantiations required a process (checking for symmetry in one way or another) and so were time-consuming and unnecessary when, with his substantiations, the last digit, simply and instantly, confirmed a number's membership in the set of even or the set of odd numbers. Accordingly, even when he eventually endorsed the structure-based substantiations, he maintained incommensurability between these and his own by positioning his routines as superior ones that worked in more authoritative contexts and with people who had more authority.

Conclusion and Implications

For effective interpersonal communication within the group to occur, group members need to build on one another's ideas using "the same means as those endorsed by his or her

interlocutors" (Sfard, 2008, p. 173). This paper shows that whilst members of this group agreed on classifications of numbers as odd or even, they held different meanings of the keywords 'odd' and 'even' and accordingly substantiated oddness and evenness differently. Utilising the commognitive framework has helped highlight these distinctions in the students' reasoning that may otherwise be tacit. In terms of resolving the commognitive conflict, the students were unwilling to build on one another's ideas or reach a communicational agreement that rationalised these group decisions because they saw no incentive in doing so: They agreed on the classifications (which they interpreted as the goal of the task) and the alternative discourse did not serve them well with respect to this goal. Sfard (2007) notes that learners need good reason to change their routines, and I posit the classification task presented no such reason for any of the students to modify their substantiation routines.

The findings highlight the role proving activity can play in mathematics classrooms. In the absence of students' substantiations, a group consensus about an answer (in this case about which numbers are even and odd) may prematurely signal shared reasoning. Pressing students to publicly air their substantiations can bring differences in students' reasoning to the surface, which may otherwise be hidden. However, the findings also serve as a warning that common pedagogical moves to capitalise learning from mathematical disagreements by encouraging students to make sense of one-another's ideas may not necessarily result in students' adoption or even endorsement of them. Therefore, if a criterion for proof in the primary classroom is an argument accepted by the classroom community (Stylianides, 2016), the commognitive framework provides a useful lens to glean insights into barriers that a community may need to overcome in order to reach consensus.

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The metaphor of transition for introducing learners to new sets of numbers

Igor' Kontorovich
The University of Auckland
<i.kontorovich@auckland.ac.nz.>

Rina Zazkis
Simon Fraser University
<zazkis@sfu.ca>

John Mason
Open University
<john.mason@open.ac.uk>

Is the natural number 7 rational? Is it complex? We argue that the answers to these questions relate to the ways numbers are taught. Commonly, a new kind of numbers is presented as an expansion of a previously familiar kind of numbers, which results in a nested image of the relations between number sets. In this article, we introduce an alternative approach, in which one transitions between different numerical domains, some subsets of which are isomorphic.

Is the natural number 7 rational? Is it complex? Based on our experience with raising such questions to many students and teachers, we speculate that most members of the MERGA community will answer affirmatively. This might relate to a common way of teaching, where a new kind of numbers is presented as an *expansion* of a previously familiar kind, resulting in a nested image of number sets (see Figure 1). In this short theoretical discussion, we introduce an alternative perspective, in which one *transitions* between different numerical sets, some subsets of which are isomorphic.

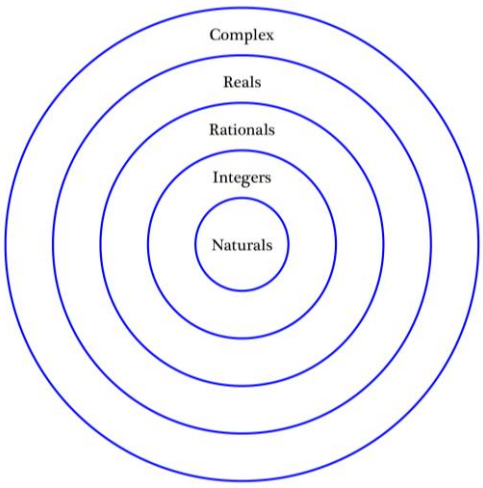

Figure 1. Nested image of number sets.

The Metaphor of Expansion

Many scholars argue that mathematics emerges from communication, which is replete with ubiquitous and often transparent metaphors (e.g., Lakoff & Núñez, 2000). Drawing on experiences that are expected to be common to the communicating actors, metaphors can open the door even to the most abstract mathematical ideas (e.g., Barton, 2008; Sfard, 2008). This feature turns metaphors into a powerful didactical tool that becomes handy when new

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 259-264. Singapore: MERGA.

numbers are introduced and when they are related to those numbers with which learners are already familiar.

In instructional settings, new kinds of numbers are often "grown" from an *expansion* of the concept of number: novel elements are introduced to a familiar number set yielding its expansion. For instance, González-Martín et al. (2013) maintain that,

> the learning of different sets of numbers can be seen as a progressive extension of the initial perception of numbers through the algebraic structure of nested number sets, from the primitive notion of counting, to the ideas of comparing, measuring and solving equations (p. 230)

At least three reasons can be offered for the didactical appeal of expanding learners' concept of number:

- Different number sets share many familiar number-symbols, words, related concepts, and properties (e.g., commutativity, associativity, identity). This allows teachers to develop new numbers out of the ones that students are already familiar with.
- The expansion epitomizes mathematics as a highly connected and coherent body of structural relationships. Given that numbers accompany students' learning all the way from kindergarten to university, every encounter with new numbers turns into an opportunity to perpetuate this image.
- This perspective aligns well with a common narrative, in which new numbers are positioned as a patch that resolves issues and inadequacies with numbers of the "old" kind. Naturals do not allow subtracting a larger number from a smaller one, hence the integers. Not all divisions of two natural numbers result in a natural number, hence the rationals. While bearing some resemblance to the development of numbers throughout mathematical history (e.g., Kline, 1972), an expansion of the familiar presents a sensible rationale for introducing new numbers.

As with any metaphor taken literally, expansion comes with its issues. For instance, it draws attention to the introduced add-ons, while glossing over the changes that they impose on the familiar structure. This might at least partially explain why students often assume that their previously held truths about numbers remain intact. At the elementary-school level, well-documented examples concern the notions of successor and density that children "carry over" from natural to rational numbers. For instance, pupils can claim that 2.4 is the next number after 2.3 and that 7.5 is the only number between 7.4 and 7.6 (e.g., Vamvakoussi & Vosniadou, 2010). Similar phenomena occur in a more advanced context. Kontorovich (2018a) showed that many tertiary students continue referring to complex numbers with a zero imaginary part as positive and negative. In fact, some of his participants even became irritated with the questionnaire specifying the number set for each question and lamented "Why do you always mention whether it's $\mathbb{R}$ or $\mathbb{C}$? 2 is positive no matter where!".

In research and practice, the exemplified ways of thinking are often stigmatized as products of students' "bias", "naivety", and "overgeneralization". However, we suggest that the metaphor of expansion may play a role in the robustness of these ways of thinking. Indeed, it seems more reasonable to expect expansion to enrich familiar concepts rather than transform them beyond recognition. Of course, a diligent teacher will emphasize the ways in which new numbers are different from the "old" ones. Yet, it is still not easy to keep track of what changes and what remains valid after the expansion. For instance, NCTM standards (2000) prescribe understanding complex numbers as solutions to quadratic equations that do not have real-number roots. Students are usually introduced to the quadratic formula in the system of real numbers. Accordingly, it seems to be taken for granted that the quadratic formula remains intact even after renouncing square-rooting negatives – one of the most prominent taboos of reals.

The Metaphor of Transition

The issues that we described in relation to the expansion metaphor appear serious enough to consider whether it is the only way to introduce new numbers. The alternative that we bring to the fore is the metaphor of *transition*. Within it, learners are not asked to mobilize familiar numbers to engender new ones but encouraged to depart from one numeric set to arrive at another. Transitions take place between distinct domains, situating the differences between them as an expected norm rather than an anomaly. For travellers, an appreciation of transition implies that the destination is foreign, and its mysteries are waiting to be discovered. It also means that the luggage carried from the port of departure should be selected carefully since not everything will continue to be useful. Overall, for the sake of a positive experience, transitioning students had better be attentive and alert to the rules and customs of the foreign terrain, as these are likely be different from the familiar. This is not to say that similarities between the new and the old will not be recognized. Such instances would be a pleasant surprise, enabling to leverage previously gained knowledge and experiences in new circumstances.

The transition metaphor may be viable for introducing new kinds of numbers. Specifically, it may offer a cohesive frame to attune learners' mindsets to the encounter with new number-names, symbols, and operations; to enhance their readiness to adjust and make sense of new number rules; and to explain why some familiar mathematical truths should be lost in transition. Transition also provides room to grow insights and appreciations of the familiar kind of numbers from the newly developed perspective.

To illustrate the metaphor of transition, let us consider an example where a somewhat extremal attempt is made to disconnect between real and complex numbers. Imagine a teacher who welcomes students to a new mathematical domain consisting of dots residing on a plane with one special dot O. "What can be done with them?", students ask. "Well, there is one operation we can do, let's call it "tāpiritanga" and "tāpiria" as its process." Then, the teacher shows how tāpiritanga of the dots z_1 and z_2 yields another dot z_3 via a so-called parallelogram law (see Figure 2). Through a guided investigation, students can find out that "tāpiritanga" is commutative (i.e., z_1 tāpiria z_2 is the same as z_2 tāpiria z_1), associative (i.e., z_1 tāpiria z_2 and then tāpiria z_3 is the same as z_2 tāpiria z_3 and then tāpiria z_1), and tāpiria of O to any dot leaves this dot intact. To impede students from carrying over "old" meanings of the concept, the teacher refrains from referring to dots as numbers. Instead, the teacher invites students to consider whether numerical domains with which students were familiar until now and the new world of dots have something in common. To support this process of discerning similarities, the teacher can reveal that "tāpiritanga" is "addition" in Māori (see Zazkis et al., 2021 for more illustrations of this sort).

We acknowledge that teaching with the metaphor of transition in mind is likely to come with issues. Supporting students in establishing productive relations between different kinds of numbers is probably among the first issues to emerge. Teaching experiments are needed to show what these issues can look like and how they can be handled. What we wonder about is whether students who transitioned between numerical sets will adhere to the abovementioned ways of thinking as students for whom the concept of number was expanded. Another point to consider is how the rules of new numbers can be harnessed to make students re-appreciate numbers of the familiar kind. For instance, will the students in our example enjoy the fact that a "flat" version of the parallelogram law works as the addition of reals on a number line?

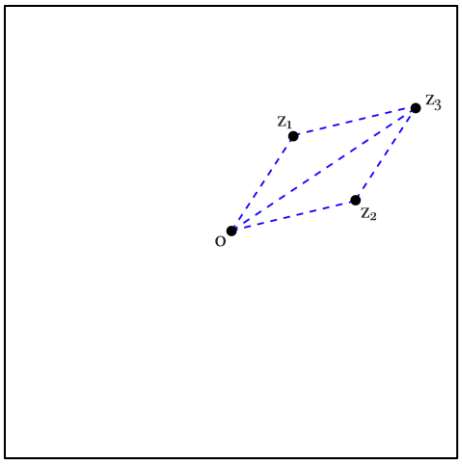

Figure 2. z_3 as a result of the 'tāpiritanga'-operation between z_1 and z_2.

Images Underpinning the Relations Between Number Sets

Herein we draw on the notion of subset to illuminate the mathematical grounds for the metaphors of expansion and transition. To recall, the set A is called a subset of the set B if every element in A is also an element in B. The expansion metaphor draws on the nested relationship among number sets, commonly visualized as presented in Figure 1: natural numbers are a subset of integers, which are a subset of rationals, which are a subset of reals, which in turn is a subset of complex numbers. To be explicit, we consider the subset relation of numbers as a mathematical stance rather than a deductively derivable result. Within this perspective, recognizing 7 as an element of natural numbers warrants its being an integer, rational, real, and complex number.

This recognition may become easier or harder depending on how numbers are represented. For instance, when numbers appear as dots, the dot entitled "7" remains fixed when the natural number line extends to the negative direction to become the integer line. The "7"-dot stays in place when the dotted line becomes dense with rationals and reals, and even when it expands to become the Argand plane. The situation is different when symbolic representation starts playing a more significant role, especially when different kinds of numbers are defined through symbols. For instance, complex numbers are often characterized by a real and an imaginary part. Then, $7 + 0i$ and 7 become different representations of the same mathematical object. In this sense, one could argue that $7 + 0i$ *is* 7, in more or less the same way that "seven" in English is "whitu" in Māori. This is opposed to a common students' claim that "the addition of zero i has no impact".

The transition metaphor draws on an image in which different number sets are isomorphic to some subset of each other. To recall, two sets are isomorphic if there exists a bijection between their elements that preserves a binary relationship, for instance addition and multiplication. Figure 3 depicts this relation with an example of real and complex numbers. From this standpoint, the natural 7 is different from the integer 7 (or $+7$), rational 7 (or $\frac{7}{1}$), and from the complex 7 (or $7 + 0i$). Yet, these numbers could be considered equivalent, if one wishes to identify them as such. Similarly, the relationship between natural and rational numbers is captured by considering naturals as isomorphic to a subset that, mathematically speaking, is perfectly embedded in rationals.

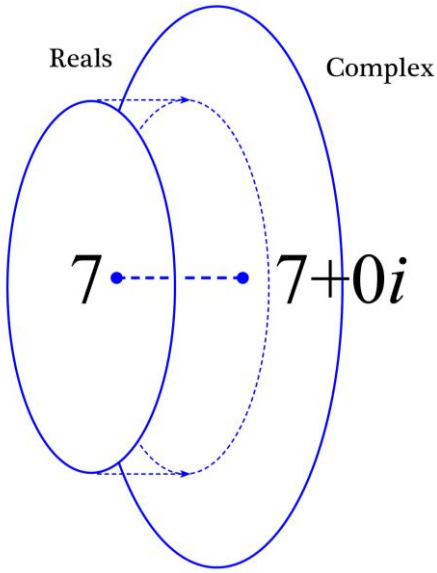

Figure 3. Isomorphic image of real and complex numbers.

An isomorphic image can help in resolving what may appear as an issue within the nested view on numbers. Zazkis (1998) discussed an incident, where her pre-service classroom was divided around the quotient in the division 12 by 5: some of the students argued for 2 with a whole-number quotient in mind, while others advocated for 2.4, implicitly assuming rational-number division. In a similar vein, Kontorovich (2018b) reported on a student who struggled to cope with the fact that $\sqrt{9}$ was 3 when approached as the (real) square root function, but the application of De Moivre formula on the complex 9 entailed 3 and -3. In both cases, the difference of the results is an issue within the nested number image but not necessarily with the isomorphic view. Through the latter lens, identically appearing words and symbols can be interpreted rather differently in different number sets.

Specific images of the relation between number sets underpin mathematical software. In MAPLE, the command $isprime$ tests for whether the input is a prime number. Working with an older version of MAPLE, we witnessed that it outputted "true" for $isprime(7)$ but "false" for $isprime\left(\frac{14}{2}\right)$, $isprime(7.0)$ and $isprime(3.5 \times 2)$. This was because the programmers intended for $isprime$ to operate with integer arguments. In MAPLE, the result of division was considered a rational number, and a rational $\frac{14}{2}$, and similarly 7.0 and 3.5×2, were not identified with an integer 7. Such programming may appear infelicitous to those adhering to the nested image: if all the four inputs point at the same number, how come that their outputs are not the same?! The devotees of the isomorphic image may be more accommodating since for them all these "7"s are different numbers a priori. Yet, we do appreciate that the current version of MAPLE explicates that the input of $isprime$ must be an integer.

Concluding Remark

We started with a question whether the natural 7 is also rational and complex, and suggested that the answer depends on the metaphoric lens through which one considers relations between number sets. We hope that the members of the MERGA community will share our curiosity in the metaphor of transition as a refreshing alternative to the hegemonic

metaphor of expansion. The nested and isomorphic images underpinning the metaphors may appear conflicting, but we consider them as complementary viewpoints – one from "above" and one from "aside" – on the same mathematical structure (see Figure 4). Furthermore, we believe that, for the learning of mathematics, it is useful for students and teachers to be able to flexibly switch between the two images.

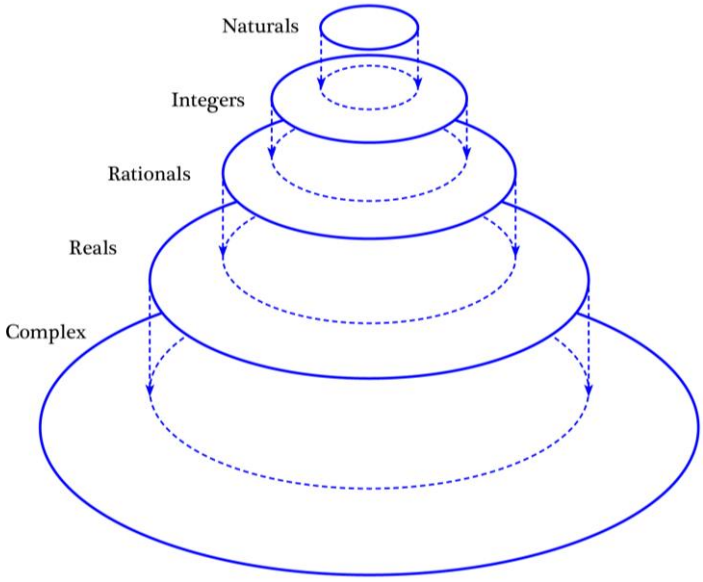

Figure 4. Visualization of relations between number sets.

Note. This paper is an amended version of Kontorovich et al. (2021).

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Engagement and outdoor learning in mathematics

Alexandra Laird
Griffith University
<alexandra.laird@griffithuni.edu.au>

Peter Grootenboer
Griffith University
<p.grootenboer@griffith.edu.au>

Kevin Larkin
Griffith University
<k.larkin@griffith.edu.au>

The study reported here was conceptualised using a theoretical framework that included three dimensions of engagement; emotional, behavioural, and cognitive, and these were used to structure the data collection and analysis vis-à-vis learning mathematics outdoors. This comparative case study involved 34 students from two Year 6 classes at a Queensland state primary school. The findings indicate that the students were engaged in their mathematics learning in the outdoor context. However, there was no compelling evidence that suggested the outdoor environment was any more emotionally, behaviourally, or cognitively engaging than the indoor context.

The concept of engagement has been a growing concern for researchers, particularly in mathematics education (Attard, 2012; Chan et al., 2015), where it has been seen as a key factor in ameliorating low levels of achievement and student boredom (Fredricks et al., 2004). For this reason, it is important that the concept of engagement be explored in mathematics education, as low levels of engagement can result in low participation and achievement (Attard, 2011). Consequently, this has the potential to affect Australia's perennial shortage of mathematically literate citizens (Attard, 2011). Engagement is a multifaceted concept that has been defined along three dimensions: emotional, behavioural, and cognitive (Fredricks et al., 2004). Researchers have suggested that utilising the outdoors in mathematics education helps to increase students' engagement (Fägerstam & Samuelson, 2014; Haji et al., 2017; Young & Marroquin, 2008). It seems there is a growing interest by researchers to evaluate and compare the efficacy of indoor and outdoor learning environments. However, it is seldom seen that the effectiveness of outdoor learning is holistically evaluated through the lens of the engagement dimensions. This study seeks to determine the effects that outdoor learning has on students' engagement in mathematics. To this end, this study will explore outdoor learning vis-à-vis the three dimensions of engagement: emotional (with aspects of affective engagement), cognitive, and behavioural, and investigate the engagement of students in relation to indoor and outdoor environments. In addition, this study will clarify distinctions among the three constructs of engagement and how they are both individually and holistically identified within the learning context.

Given the apparent gaps in the literature, this study sought to determine the effects that outdoor learning had on students' engagement in mathematics. The research question guiding this study were:

- *What sort of engagement (emotional, behavioural, and cognitive) in mathematics does an outdoor learning environment facilitate?*
- *In what ways, if any, does student engagement in mathematics differ according to the learning environment?*

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Theoretical Framework

Emotional engagement is defined as the positive and negative reactions that students have to their peers, teachers, academics, and school (Fredricks et al., 2004). Skilling (2014) suggests that when students are emotionally engaged, they demonstrate interest and enjoyment. Emotional engagement is also commonly labelled as affective engagement by mathematics education researchers (e.g., Attard, 2011; 2012; Grootenboer & Marshman, 2016), and these researchers frequently come from an educational background and explore the deeper internal state of engagement. Others, who label it as emotional engagement, typically come from a psychological background and look at student's reactions to school experiences and environments. *Behavioural engagement* is defined as an individual's active participation and involvement in academic and social activities (Attard, 2012). The concept of participation is inherent to the construct of behavioural engagement (Finn et al., 1995) with Skilling (2014) and Fredricks et al., (2004), suggesting that students who are behaviourally engaged actively participate, persist and concentrate, ask questions, and contribute to class discussions.

Cognitive engagement is defined as an individual's investment in, and acknowledgement of, the value of learning and their willingness to go above and beyond the minimum requirements of a task (Attard, 2012). It also refers to the ability to suppress distractions and to maintain and regulate efforts in sustaining cognitive engagement (Fredricks et al., 2004; Skilling, 2014). It is critical to acknowledge that these engagement constructs are not isolated processes occurring within the individual, but rather they are dynamically interrelated and a shift in one can dramatically influence the others. Therefore, in this article, the dimensions of engagement are considered holistically as a multifaceted phenomenon.

Attard (2012) suggests that effective mathematical engagement occurs when a student is enjoying the subject, can easily see the relevance that their work has to their own lives and future, and can make meaningful mathematical connections between their learning in the classroom and their learning beyond school environment. Also highlighted in her work is the significance of choice and creativity in the mathematical learning context, and the suggestion that, if students are engaged in activities that encourage creativity and that provide opportunities to make decisions about their learning, their engagement in mathematics will increase. Motivation concepts are suggested to have significant relevance and are often synonymous with engagement. Student motivation increases when they are able to make links between what they are learning, their knowledge, and their inside and outside classroom experiences (Opitz & Ford, 2014).

The literature frequently suggests that outdoor learning is an effective pedagogical approach to increase student engagement (Attard, 2012; Fägerstam & Samuelson, 2014; Haji et al., 2017) and consequently student learning. Outdoor learning can include activities that take place on the playground, the oval, or the garden, and it has been suggested that students perform significantly better in outdoor activities than in similar indoor classroom activities in mathematics (Fägerstam & Samuelson, 2014; Haji et al., 2017). Similarly, it is considered that exclusively learning mathematics inside the classroom hinders students from fully understanding mathematical concepts (Haji et al., 2017). There is a diversity of desirable learning features associated with outdoor learning that can be seen as prompting, and resulting from, increased levels of student engagement. Taking mathematical lessons outside adds a new dimension to the learning experience where opportunities for multi-sensory perceptions are increased (Fägerstam & Blom, 2013).

Linking conceptualisation of engagement to the effectiveness of outdoor learning

When reviewing the literature on engagement and outdoor learning, clear links can be made between the two. Table 1 outlines the links between emotional engagement and outdoor learning theories. Table 2 outlines the links between behavioural engagement and outdoor learning theories.

Table 1
Linking emotional engagement theories to outdoor learning theories

Emotional Engagement Theories	Outdoor Learning Theories
"The element of fun was identified as an element of "good" mathematics lesson" (p. 371) (Attard, 2011)	"The pupils in this study all described positive experiences regarding the outdoor lesson… all of them spontaneously uttered remarks such as 'it was fun'". (p. 68) (Fägerstam & Blom, 2013)
"When an individual is engaged with mathematics, he or she has been influenced by motivation" (p. 10) (Attard, 2012)	"Outdoor lessons in outdoor environments have positive impact on the pupils' interest and motivation" (p. 69) (Fägerstam & Blom, 2013)

Table 2
Linking behavioural engagement theories to outdoor learning theories

Behavioural Engagement Theories	Outdoor Learning Theories
It is emphasised that inherent to the construct of behavioural engagement is the concept of *participation*, which is a crucial component in achieving positive academic outcomes (Finn et al., 1995)	Students who are generally reluctant to *participate* in mathematics are more likely to engage in tasks when lesson are taken outside (Young & Marroquin, 2008)
Behavioural engagement is concerned with students' actions such as their "efforts, persistence, concentration, attention, asking questions, and *contributing to class* discussions" (Fredricks et al., 2004, p. 62)	Students are generally willing to take greater risks when mathematics is taken outside and are more likely to volunteer to *share their answers and justify their thinking* (Young & Marroquin, 2008)

Also outlined in the literature on engagement is the close connection that behavioural and cognitive engagement share (Fredricks et al., 2004). As many students are willing to take greater risks and persist when learning is outside the classroom, it is also probable that outdoor learning facilitates opportunities for cognitive engagement. A significant component regarding the effectiveness of outdoor learning in mathematics, labelled the 'novelty and variation dimension', is proposed in Fägerstam and Blom's (2013) study. It is suggested that since outdoor learning is often a new educational experience for students, this might have a high impact on students' positive engagement. In their study, students often regard indoor learning as boring and monotonous (Fägerstam & Blom, 2013). It can then be proposed that outdoor learning offers a valued variation to this type of learning.

Methodology

The methodology for this study was previously presented (see Laird & Grootenboer, 2018), so only a brief outline will be provided here. To establish what effect the mathematical learning site (outdoors and indoors) has on students' engagement, a comparative, collective, case study methodology was used. The study involves the comparison of two sets of two cases. Both cases were Year 6 classes undertaking mathematics lessons on the same topic and concept.

The first set of two cases involved the students initially participating in a mathematics lesson inside the classroom. Following this, they participated in a similar mathematical lesson outside the classroom (e.g., the playground, oval, or elsewhere on school grounds). The second set of cases involved the students participating in the same mathematical lesson, but in the reverse order where they participated in the outside lesson first and then the inside lesson second. The focus of the lesson, which was introducing students to the 'order of operations ("BODMAS"), was determined by the teachers to accord with their mathematics *scope and sequence* planning.

For this study, three methods of data collection were used: a simple survey, structured observations, and document analysis. They relate specifically to the three dimensions of the theoretical framework as is outlined in Table 3 below.

Table 3
Data collection

Dimensions of Engagement	Data Collection Method
Emotional engagement	A survey that the students completed at the conclusion of each lesson.
Behavioural engagement	Observations of students participating in the lessons using an observation framework.
Cognitive engagement	Student work samples* collected in each lesson

* The nature of these depended on the lesson focus that the classroom teachers chose

As there were no existing suitable instruments found in the literature, new instruments were developed using relevant theoretical literature on the nature and features of engagement in educational settings (see Laird & Grootenboer, 2018).

Findings

Emotional Engagement

The emotional engagement of the students in the study was measured through a post-lesson survey given to all the participants immediately following both their indoor and outdoor lessons. It is acknowledged that this instrument is limited in its capacity to measure emotional engagement; nevertheless, it provides some insights that are useful in considering engagement in mathematics learning. The first analysis was conducted to see if there were any statistically significant differences for the whole sample at the item level, and total score, between the indoor and outdoor lessons. The descriptive statistics are shown in Table 4 below.

Table 4
Descriptive statistics for emotional engagement indoor and outdoor

		N	Mean	Std. Deviation
1. I enjoyed the lesson	Inside	33	3.88	.927
	Outside	34	3.94	.814
2. I thought the lesson was interesting	Inside	33	3.61	.827
	Outside	34	3.59	.957
3. I had a lot of fun during the lesson.	Inside	33	3.42	1.146
	Outside	34	3.56	.894
4. I would like to do that lesson again.	Inside	33	4.06	1.059
	Outside	34	3.56	1.019
Total	Inside	33	14.97	3.359
	Outside	34	14.68	3.082

These results of the t-tests indicated that there were no statistically significant differences in the students' emotional engagement between the indoor and outdoor lessons as measured by the emotional engagement survey. Specifically, the students' post-lesson responses to individual items indicated that the outdoor lessons were not perceived as being more enjoyable, fun or interesting, and there was no distinction in their perception of whether they would like to do a similar lesson again.

An open question at the end of the survey provided the participants with an opportunity to express any other thoughts. 23 responded, and in their responses, they were generally positive about both the inside and outside lessons. Positive responses associated with emotional engagement (e.g., fun, liked, enjoy) for the indoor lesson were limited (n=7), whereas there were many more for the outdoor lesson, and several students (n=23) gave more than one positive response. The words used were often about particular features of the lesson including "being outside" and also being able to "move around", with, for example, one student stating, "I would like to do the lesson again because it was outside and I think we should do more outside tasks". Also, students often used positive emotional engagement terms in regard to being able to work in pairs/groups, the way their teacher taught in this context, and the lesson generally as a whole. By way of examples, one student responded, "It was much funner [sic] than the lessons in class and we got to work in pairs or in groups most of the time. We never get to do that in class", and another said, "I would like to do the lesson again because it was outside and I think we should do more outside tasks".

Behavioural Engagement

The behavioural engagement of students in the study was measured through a behavioural engagement checklist, which was completed by the lead author during both the indoor and outdoor lessons. Although it is acknowledged that this instrument is limited in its capacity to measure behavioural engagement, it does provide some insights that are useful in considering behavioural engagement in mathematics learning. The resulting data was multifaceted, nuanced, and intricate, but here only aggregated findings will be presented, and these will focus on the different phases (see below) of the lesson, and the two learning sites (i.e., indoor, and outdoor).

During the lesson observations, three distinctive lesson phases were identified and these were present in all observed lessons - both inside and outside. The three phases were listening to the teacher (LT), working as whole class (WC), and individual work (IW). The purpose of identifying these lesson phases is so that behavioural engagement can be compared vis-a-vis specific learning phases rather than just time phases. Student engagement was observed and recorded at 5 minutes intervals. The number recorded represented different levels of engagement from the students in the class: 1 = None, 2 = Some, 3 = Half, 4 = Most, 5 = All. The summarised data for behavioural engagement for both classes in both lesson sites is outlined in Table 5 below (note that LT was barely evident in the data so it is not included)

Table 5
Mean behavioural engagement ratings across lesson phases and lesson sites (n=34)

Lesson Phase	Site	Active participation	Ask questions	Contribute to class discussion	Persist with tasks	Display strong levels of concentration	Average
WC	Outside	4.5	2.3	2.5	4.5	4.3	3.62
	Inside	4.4	2.3	2.5	4.4	4.4	3.6
IW	Outside	4.15	2.75	2	4.2	3.65	3.35
	Inside	4.15	3	3	4.2	3.75	3.62
Average	Outside	4.325	2.525	2.25	4.35	3.975	*3.485*
	Inside	4.275	2.65	2.75	4.3	4.075	*3.61*

These results indicate that there were minor differences between the behavioural engagement levels of students in the outdoor and indoor setting. Overall, it seems that students were ‘actively participating’, ‘persisting with tasks’, and ‘displaying strong levels of concentration’ with similar or the same levels of engagement during outdoor and indoor lessons. The data for ‘asking questions’ and ‘contributing to class discussion’ showed some differences indicating that students were engaging with the teacher and the class more in the indoor setting. When looking at the ‘asking questions’ section of the checklist, there seemed to be minor differences between the two indicating that students were asking more questions in the indoor setting. These results indicate that there were minor differences between the behavioural engagement levels of students in the outdoor and indoor setting while they were working as a whole class and doing individual work.

Cognitive Engagement

Definitions of cognitive engagement relate it to an individual’s ability to persist when problem solving, endure in the face of failure, demonstrate highly strategic learning qualities, and adopt metacognitive strategies to arrange and assess cognition (Zimmerman, 1990). In this study these were ‘measured’ based on an interpretation of these features that could be identified in students work samples. The work samples collected provided some evidence of the students’ levels of cognitive engagement, albeit that it was difficult to clearly identify

certain features and to attribute them with any certainty to the particular site of the lesson. With this limitation in mind, in general the students demonstrated evidence of cognitive engagement when they were given the opportunity to immerse themselves in mathematics that required a high level of problem solving. This did not seem to occur in any particular environment - indoor or outdoor, but rather evidence of learning features that indicated high levels of cognitive engagement were observed only after some form of basic conceptual understanding was sound. For example, evidence of higher order thinking was found more in the students' second lesson work samples on order of operations because by this time they were able to participate in the more complex tasks. It is acknowledged that the data in this section is perhaps the least compelling, and in part, this is due to the 'internal' nature of cognitive engagement, meaning evidence often has to be inferred from behaviours and objects that can be observed. Also, there were some difficulties in even capturing the student work samples due to the activities of the lesson. Nevertheless, for the purpose of this small-scale study, cognitive engagement was examined using these qualities in an attempt to grasp some understanding of students' cognitive engagement levels.

Concluding Comments

This study focussed on student engagement in learning mathematics and, as is clear from this study and the relevant literature, this is a complex phenomenon. However, rather than being sidelined by the apparent difficulty in grasping the multifaceted and intricate nature of mathematical engagement, in this study the decision was made to accept the complexified quality of the phenomenon and then attempt to further investigate the topic. Unsurprisingly the findings are not conclusive; nevertheless, they are interesting and insightful. Put simply, rather than the indoor or the outdoor environment being more conducive to mathematical engagement per se, there is a need to appreciate all the pertinent factors (including the learning environment) when seeking to engage students in mathematical practices.

In general, the findings suggested that the students were engaged emotionally, behaviourally and cognitively in the outdoor learning environment. Although most of the data suggested that the outdoor learning environment was conducive to engendering all the dimensions of engagement, it was evident the dimension of emotional engagement was enhanced the most. However, although the outdoor environment was generally engaging for the students, it was not evident that they were any more or less engaged in their mathematics learning than in the indoor environment. This finding is noteworthy as it is somewhat at odds with the literature that indicated, albeit not conclusively, that an outdoor environment is likely to be more engaging.

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Teacher actions for consolidating learning in the early years

Sharyn Livy
Monash University
<sharyn.livy@monash.edu.>

Janette Bobis
The University of Sydney
<janette.bobis@sydney.edu.au>

Ann Downton
Monash University
<ann.downton@monash.edu >

Melody McCormick
Monash University
<melody.mccormick@monash.edu >

James Russo
Monash University
<james.russo@monash.edu >

Peter Sullivan
Monash University
<peter.sullivan@monash.edu >

All students should have access to learning experiences that help them make sense of important mathematical concepts. This study highlights teacher actions for consolidating student learning during teacher-lead discussion in the early years. We report on a case study of a Year 1 teacher involving a lesson observation. Highlights of the lesson include intended teacher actions that supported students to focus on the learning goals; use of work samples to make concepts clearer; fostering mathematical connections; and questioning strategies for promoting cognitive activation. Teacher actions such as questioning strategies and discussion of work samples may be key for helping students to achieve mathematical learning goals.

One of NCTM's (2014) guiding principles for school mathematics is that "effective teaching engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically" (p. 5). Many researchers would claim that effective teaching practices can be influenced by teacher actions. Teacher actions can include how teachers prepare for teaching; approaches for launching a lesson, how they promote student-centred learning; the types of questions they pose that guide learning; and how they help students to make mathematical connections, and develop reasoning and problem solving skills (ACARA, 2021; NCTM, 2014; Rowland et al., 2009; Smith et al., 2020; Sullivan et al., 2020b).

In our research project we aim to assist teachers to enhance the mathematical outcomes of Australian students by developing new understandings in ways mathematics is learnt by early years students' (5-8 year-olds). The project, Exploring Mathematics Sequences of Connected, Cumulative and Challenging tasks (EMC^3) provides teachers with sequences of lessons and new approaches to curriculum. Each lesson addresses a key mathematical concept and builds on students' mathematical learning from the previous lesson. To guide teachers' pedagogical actions, we have developed a student-centred Instructional Model for supporting teacher actions when facilitating lessons (Bobis et al., 2021). The Instruction Model extends the work of Sullivan et al., (2016) and the three phases of Launch, Explore, and Summarise. The revised framework includes an Anticipate Phase where the teacher identifies the learning goals of the lesson and considers ways the students might respond to the task; and a (Re)-Launch Phase, where the teacher can pose a further task that is the same in most respects, but different in terms of context, size of the numbers, or representation.

An outcome of the project is to report on ways teachers might use our sequences of lessons to inform their teaching and guide student learning. This paper reports on a case study of a Year 1 teacher and will inform further data collection as part of the larger project. The teacher was selected because she had been observed on several occasions throughout the year and was proficient at using the Instructional Model. Proficient teachers in the study

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 273-280. Singapore: MERGA.

were expected to follow the lesson structure of Anticipate, Launch, Explore and Summarise (Bobis et al., 2021). The research questions guiding the study were:

> What types of questions did a Year 1 teacher rely on when consolidating student learning and sharing work samples?
>
> How did a proficient Year 1 teacher rely on her teacher actions to guide her instructional decisions when discussing and sharing student work samples?

By observing teacher actions, we aim to identify how they increase opportunities for student learning and success through subsequent tasks. Findings will assist other teachers as they reflect on their actions when implementing the 10 sequences of lessons and suggestions.

Next is the review of literature and the theoretical model used to inform the study.

Literature Review

Effective mathematics teachers establish goals to focus and guide student learning (NCTM, 2014). Others suggest teaching approaches should be student-centred (Staples & King, 2017). Another attribute of quality teaching is to provide students with tasks that support cognitive activation by encouraging students to think in greater depth about problems (NFER, 2015). Such tasks may be open-ended, having more than one solution or have multiple approaches used for solving the task (open-middle) (Sullivan et al., 2020a). Other strategies intended to support cognitive activation include questioning techniques such as asking, “What if?” or “Might there be another way?” type of questions (NFER, 2015).

Effective mathematics teachers should provide opportunities for purposeful questions that promote reasoning (NCTM, 2014), guide learning, thinking and exchanging of ideas (Staples & King, 2017) and help students to make sense of solutions (Evans & Dawson, 2017). Sahin and Kulm (2008) described factual, probing and guiding question types in their review of literature. Factual questions require little cognitive challenge and are closed question types that usually require yes/no answers; probing questions help students to clarify, justify or explain; and guiding questions can assist students when responding to questions. When posing questions teachers must consider the types of questions they ask as well as the pattern of questions they use if they are to promote students’ reasoning skills (NCTM, 2014).

Theoretical Framework

An adapted version of Clark and Peterson’s conceptual framework (1986) was used to guide the study (Figure 1).

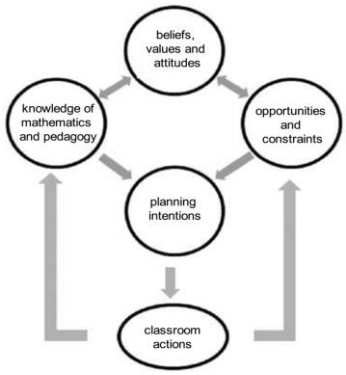

Figure 1. Framework adapted from Clark and Peterson (1986).

Clark and Peterson (1986) suggest teachers' classroom actions are informed by the relationship between knowledge of mathematics and pedagogy; dispositions including, beliefs, values and attitudes; opportunities and constraints they anticipate experiencing; and planning intentions. We anticipate that when teaching sequences of challenging tasks teacher actions guide their pedagogical decisions when posing questions, whilst sharing and discussing student work samples. Data analysis for the current study reports on one teacher's classroom actions when observed teaching a lesson.

Method

A qualitative study and case study were chosen to assist with providing an in-depth description of the circumstance (Yin, 2009). The study explored how Abby (pseudonym) approached her discussion with students when she was observed teaching a geometry lesson with Year 1 students. During the year, Abby first participated in a whole day of professional development to learn how to use the project resources; she attended six planning sessions with a member of the research team; was observed teaching on six occasions; and had trialled most of the ten sequences of lessons with her students.

Abby was observed teaching the first and second lesson of a shape sequence at the end of the year. The rationale for the shape sequence was to help students when classifying, making, naming and describing two dimensional shapes (polygons). The first lesson focused on students classifying groups of polygons, explaining similarities and differences. The second lesson (reported in the results) focused on students making and learning the names and properties of polygons. Students were asked:

> If you have 6 triangles all the same, what shapes can you make using all of the triangles; draw the new shapes you have made on dot paper [isometric] and name the shapes.

The next lesson in the sequence used trapeziums to make and name shapes and introduces the term chevron [and was not taught].

Proficient teachers in the study were expected to Anticipate students' solutions prior to teaching and launch each lesson without telling students how to respond to the task. Following the launch, students were expected to independently engage and attempt the task whilst the teacher observed and monitored their work as the lesson unfolded. The next phase of the lesson was the Summarise Phase [and occurred three times during the lesson reported in this study]. In this phase, students were selected to share their work samples. The teacher led a whole class discussion, similar to the framework for orchestrating mathematically productive discussion (Smith & Stein, 2018). Questions were posed by the teacher to the student(s) sharing their work sample or the whole class, helping students to clarify or explain their strategies, thinking, reasoning and/or problem solving skills.

Data collection and analysis

The launch and three Summarise Phases of the lesson were video recorded by the first author. Abby's lesson plan (including four anticipated student responses) and student work samples were collected. The lesson transcript was transcribed for coding and included the questions and student responses for each of the three Summarise Phases. The questions were coded as factual, probing, or guiding by two authors until a consensus was agreed (Table 1).

Table 1
Sample of coding teacher questions and explanation of coding

Coding	Illustrative question	Explanation of coding
Factual	Should I call this shape a trapezium?	Yes/no answer closed question
Probing	Why can't I call this one a trapezium?	Asking students to justify or explain their thinking
Probing	Who can tell me why?	Seeking clarification
Guiding	What do you notice about the edges?	Prompting students to focus on the edges when answering

Two authors partitioned the transcript into eight segments. Each segment included discussion of a key mathematical concept and/or student work sample. The segments assisted with identifying teacher actions Abby modelled during the lesson. Highlights are reported and discussed next.

Results and Discussion

The length of each summarise phase increased throughout the lesson and the lesson took 90 minutes to complete. During the lesson, five student work samples were shared. Table 2 reports the number and type of questions Abby posed for each Summarise Phase of the lesson and number of segments within each phase.

Table 2
Number of Summarise Phases of the lesson, segments and types of questions

Phase and Segments	Factual	Guiding	Probing	Total
Summarise 1(3 minutes) 1 Segment	10	3	0	13
Summarise 2 (8 minutes) 3 Segments	20	4	21	45
Summarise 3 (19 minutes) 4 Segments	48	15	25	88
Total	78 (53%)	22 (15%)	46 (32%)	146 (100%)

Effective teachers use a variety of questioning types as part of their teacher actions (NCTM, 2014). The results in Table 2 show Abby relied on different question types. Half of Abby's questions were factual (closed) questions and less than a quarter were guiding questions. As the topic of shape relied on naming shapes and their properties this may be a reason more questions were closed question types as Abby posed closed questions to help students develop geometric language. One third of the questions were probing questions. Probing questions are important for helping students to clarify, justify and explain their thinking (Sahin & Kulm, 2008), assisting students to make sense of mathematical ideas and demonstrate reasoning.

Next a selection of Abby's teacher actions is reported focusing on the summarise phases.

Selecting work samples

For each Summarise Phase Abby selected one to three student work samples to discuss with the whole class. The work sample was projected onto a white board. The students sat together on the floor and the students sharing their work stood next to the teacher. The first student work sample that Abby chose showed four polygons, including two quadrilaterals (Figure 2).

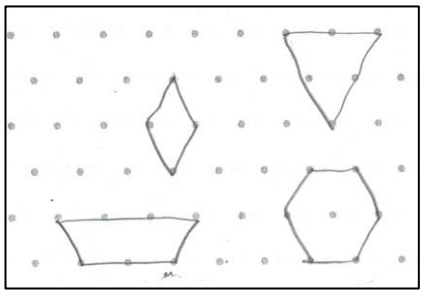

Figure 2. Four responses recorded on dot paper.

As a number of students had not attempted to record their solutions the work sample (Figure 2) provided an opportunity for all students to see how to use the dot paper, which they had not used previously. Arguably there are different reasons the teacher may select a student work sample first; the most commonly used strategy; an incorrect solution; misconception; or example of concrete to abstract (Smith & Stein, 2018). The first work sample Abby shared helped clarify how to record solutions and guided the students to focus on the learning goal, using triangles to make, name and record polygons. When discussing the work sample Abby also made connections to the previous lesson. A student named the first shape a diamond and Abby replied, "Yesterday we decided not to call these shapes a diamond … yes a quadrilateral."

This discussion demonstrated how Abby's classroom actions were influenced by her own mathematical content knowledge of how to name polygons.

Questioning strategies

When asking students to discuss their work samples, Evans and Dawson (2017) noted that teachers usually prompt students by first posing an open-ended question such as, "How did you solve this problem?"

Abby asked the following closed questions at the beginning of each of the three summarise (S) phases:

S1: I want you to tell me out of these four shapes, which one do you think meets the problems keywords?
S2: This is a fun one isn't it?
S3: Should I call it a trapezium?

Interestingly Abby chose to ask closed questions when commencing each Summarise Phase. There may be any number of reasons Abby chose closed questions, such as wanting to help the students to re-engage with the task, providing a warmup question, or because she considered that an open question to begin with may cause students to encounter challenges

and disengagement. Another conjecture might be that Abby was assessing or reviewing student understanding before moving on, helping her to think in the moment and therefore guide her follow-up questions.

Supporting cognitive activation

Cognitive activation occurs when students think more deeply about facts or concepts (NFER, 2015). An important observation was how Abby used different question types to cognitively activate students as she engaged them in mathematical discourse and consolidated their learning. Segment 3 provided an example of Abby's teacher actions when demonstrating her questioning strategies for supporting cognitive activation. The students were discussing an irregular hexagon (one student named an apple core) and a regular hexagon, both constructed with six triangles.

Abby: Hands up if you don't think it is a shape [pointing to the irregular hexagon]? (factual)

Abby: Why don't you think it is a shape? (probing)

Some students thought it was not a shape because they had never seen the shape before, one that goes in and out like that.

Abby: What name did you give this shape? (probing because there is more than one answer)

Student: An irregular hexagon.

Abby: Why is it an irregular hexagon? (probing).

Abby: How many sides does your shape have? (factual)

The use of a factual question was followed up with probing questions demonstrating how Abby posed questions to help make the mathematical concepts clearer for the students. Abby's actions show skilful use of a factual question (typically) having a lower level of cognitive demand, followed by probing questions (typically) having a higher level of cognitive demand, encouraging students to justify and explain the properties of regular and irregular hexagons. In other words, the factual question required students to engage in the discussion by choosing a yes or no response, focus their thinking, ready for the following probing questions that supported cognitive activation.

Fostering mathematical connections

Fostering mathematical connections for students was another action Abby modelled to help make concepts clearer. The focus in Segment 2 was to clarify the properties of 'real shapes.' A student named the bottom figure a candy-bar (see Figure 3).

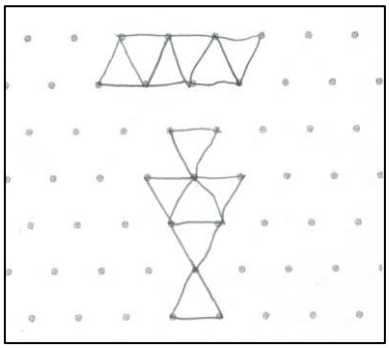

Figure 3. A quadrilateral and a candy bar

During this discussion Abby used 'why' and 'what' type questions such as "Why do you think it's not a shape?" and "What do you think?" Some students considered it was a shape and others disagreed, but the students were not really sure. The nature of these questions provides another example of Abby engaging students in cognitive activation. The combination of why, what and probing questions encouraged students to reason, clarify and justify their thinking about the 'candy bar' and experience a light bulb moment. In particular, Abby asked questions designed to support students to notice that the edges of the triangles did not always overlap ("What are those bits called?"), and that the corners or vertices need to overlap to make a (real) shape. Other teachers in the project also reported experiencing light bulb moments with their students when important mathematical connections were highlighted for, and by, the students (Russo et al., 2020).

In terms of the study's conceptual framework (Clark & Peterson, 1986), Abby's beliefs and values influenced her choice and ordering of work samples during the lesson. Abby valued the importance of the Anticipate Phase, particularly anticipating her student work samples prior to teaching as she considered how they might respond to the task prior to teaching. Doing so allowed Abby to increase the level of cognitive activation because she was familiar with different solutions and therefore could focus on discourse for consolidating student learning. Further evidence of Abby's beliefs could be gained from an interview after the lesson, which did not occur.

Conclusions

During the Summarise Phase of the lesson Abby relied on a combination of factual, probing and guiding questions to make connections among important mathematics concepts/ideas and student work samples that ultimately helped to consolidate their learning. Abby modelled a well-developed understanding of the different terms used to describe the properties and names of different polygons when questioning students as part of her teacher action and knowledge of mathematics and pedagogy. Without such knowledge this would have impacted on her classroom actions especially selecting and discussing examples to make concepts clearer when guiding students to the learning goal of the lesson and sequence.

When helping students to make connections in the elementary classroom, Smith et al., (2020) state the importance of the role of the teacher for helping students "see connections between the solutions that are shared and the goals of the lesson (p.141)". Specifically, Abby was able to help students make connections with the goal of the lesson by asking students to explain their thinking related to the names and properties of the different polygons, and to make, name and describe two dimensional shapes (polygons). This lesson approach is different to that described by Smith & Stein (2018) in that our research-based teaching suggestions, Instructional Model and teacher actions support students to make connections with the mathematical goal of a sequence of lessons. Such student-centred pedagogical approaches aim at consolidating student learning in greater depth.

Further lesson observations and assessment of student knowledge prior to and after a sequence will assist with extending understanding of the strengths and weaknesses of how students learn during a sequence and as a consequence of teacher actions. We note the limitations of reporting on one case study and a single lesson but anticipate the findings from this small study will help teachers to reflect on their questioning approach for deepening the learning during the Summarise Phase of lessons.

In terms of more general research directions suggested by this study, it is notable that Abby used different question types in complementary ways. In particular, we discussed how a factual question was often followed by a probing question. We commented that the purpose

of asking the factual question appeared to be to engage students, whilst the follow-up probing question served to activate cognition. Future research could consider whether this strategy was idiosyncratic to Abby, particular to a lesson exploring properties of shapes, or whether using the different question types in this manner characterises effective teachers more generally when teaching primary mathematics.

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Teaching towards Big Ideas: A review from the horizon

Yi Fong Loh
School of Physical & Mathematical Sciences, Nanyang Technological University
<yloh027@e.ntu.edu.sg>

Ban Heng Choy
National Institute of Education, Nanyang Technological University
<banheng.choy@nie.edu.sg>

To understand what teachers need to teach towards big ideas in the classroom, there is a need to systematically interface different conceptions of big ideas in mathematics with models of teacher knowledge. We conducted a literature review on horizon knowledge and big ideas to clarify both constructs and their relationships. Twenty-one journal articles were initially shortlisted, with within-case and cross-case analysis finally performed on four articles after inclusion/exclusion criteria. While it is clear that more work needs to be done, we tentatively conclude that to teach towards big ideas is to emphasise disciplinary ways of thinking that are empirically demonstrable to be fruitful for the learning of mathematics.

Teaching towards big ideas is a key shift in Singapore's most recent mathematics curriculum revision, implemented in 2020 (Toh et al., 2019; Choy, 2019). Teaching towards big ideas may present a huge pedagogical challenge for teachers. Firstly, there is a lack of clarity about what big ideas are. Although Charles (2005) defines a big idea as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10), different conceptions of big ideas continue to abound both in literature and in the practice of teaching. For example, the big idea of equivalence has a 'bigness' that can range from an understanding of the equal sign to the logical equivalence underlying every step in a series of algebraic manipulations to the equivalence relations that appear much beyond the domain of mathematics. Secondly, it is not clear what is meant by teaching towards big ideas. Some researchers highlight the importance of making explicit both "big content ideas" and "big process ideas" during lessons (Hurst 2015a). Others highlight the importance of reflecting on "issues of student learning and engagement as well as the domain", allowing mathematically worthwhile learning experiences to emerge from the connection of numerous smaller ideas (Mitchell et al., 2017). Such conceptions of big ideas may even seem not too different from existing understandings of expert teaching (Choy, 2019).

In a crowded curriculum, teachers may be tempted to force-fit the teaching of big ideas directly rather than teaching towards big ideas. How teachers can understand and appropriate the new notion of teaching towards big ideas, and yet, maintain the coherence and connection with their current pedagogical practices will depend on their mathematical knowledge for teaching (Ball et al., 2008). Ball et al. (2008)'s conceptions about Mathematical Knowledge for Teaching (MKT) make a distinction between Pedagogical Content Knowledge (PCK) and Subject Matter Knowledge (SMK). In particular, the notion of Horizon Content Knowledge (HCK), a component of SMK, resonates with 'teaching towards big ideas' since it isolates those aspects of mathematical knowledge which constitute an "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). The idea of seeing connections and coherence within and between mathematical topics may provide a way for teachers to navigate the challenges of teaching towards big ideas. However, like the notion of big ideas, 'horizon knowledge' has been defined differently and utilised in various ways (e.g., Jakobsen et al., 2014). Teaching towards big ideas requires teachers to present mathematics as a "coherent and connected enterprise" (NCTM, 2000, p. 17). To do this, there is a need to have some clarity regarding

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 281-288. Singapore: MERGA.

the kind of knowledge needed. This begs the following research question: *How does the construct of 'horizon content knowledge' explicate how teachers can teach towards big ideas in the mathematics classroom?*

To answer this question, we adopted a systematic approach towards reviewing the literature discussing both 'horizon knowledge' and big ideas in conjunction. In particular, this research question is framed by addressing how the constructs of 'big ideas in mathematics' and 'horizon content knowledge' are respectively conceptualised in the mathematics education literature.

Method

Taking into account the best practices for conducting a systematic review (Alexander 2020; Siddaway et al. 2019, 2019), we took a systematic approach towards conducting a literature review by searching through four databases: EBSCO's Academic Search Complete, British Education Index, Education Source, and ERIC. Using a search term for 'big idea' or big ideas in "All Text", a total of 929 articles were initially obtained. A further refinement for texts that also contain 'horizon knowledge', 'horizon content knowledge', or 'mathematical horizon' yielded a total of 21 articles using Boolean search. We applied our inclusion/exclusion criteria to obtain four articles for our focus, as summarised in the chart (see Figure 1).

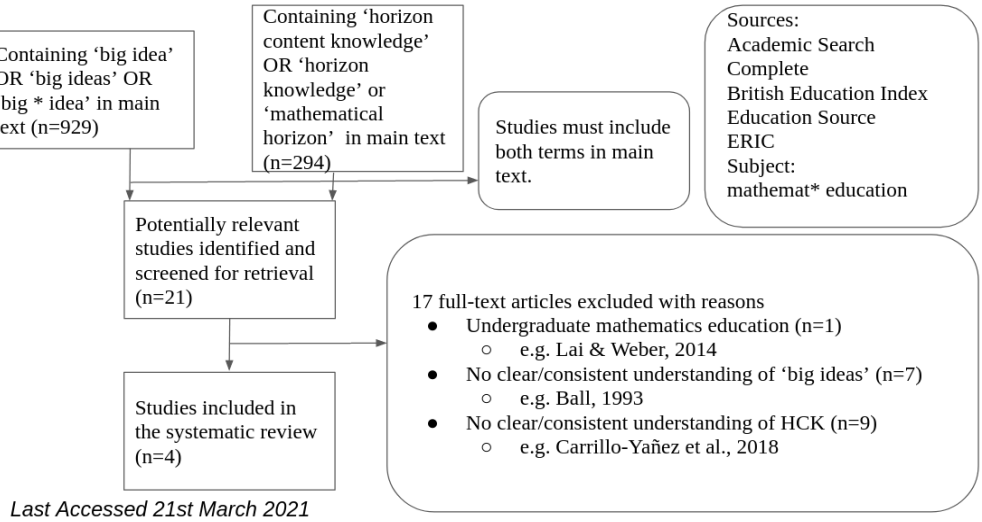

Figure 1. Systematic inclusion and exclusion

An example of a journal article with no clear understanding of HCK offered is Carrilo-Yanez et al. (2018). The article proposes a new model of mathematical knowledge. On one hand, the Mathematics Teacher's Specialised Knowledge (MTSK) has a component Knowledge of the structure of mathematics (KSM) that could be coded for big ideas with its distinction between 'intra-conceptual and inter-conceptual connections' (Carrilo-Yanez et al.,2018, p.8); on the other hand, its lack of explicit reference to HCK or ideas of the horizon outside of its literature review prevents an independent coding for any implicit concept of HCK which it could hold. This inhibits any conclusion that could be drawn from the comparison of the two constructs big ideas and HCK. An example of an article rejected for no clear understanding of big ideas is Ball (1993). While there is a singular occurrence of the phrase 'big ideas' in viewing "students as capable of thinking about big and complicated

ideas" (Ball 1993, p. 384), the notion was not explicitly discussed. After obtaining the resulting set of articles (see Tables 1 and 2), we conducted a vertical or within-case analysis followed by a horizontal or cross-case analysis (Miles et al. 2014) for each of the concepts 'big ideas' and 'horizon content knowledge' respectively.

Table 1
Conceptualisations of big ideas in review

Author	Conceptions of "big ideas"		
	Connectivity of knowledge	Disciplinary practices	Setting up powerful teaching moments
Hurst (2015)	See mathematics as 'coherent set of ideas'. Encourage deep understanding of math: enhance transfer, promote memory, reduce amount to be remembered, how topics are connected across years. Knowing 'about' the link rather than knowing a particular link (p. 2).	Problem solving skills and other 'big process ideas' need to be at the heart of teaching and learning. Includes deciding how to tackle problems, gather and organise data, represent and communicate findings. Could be reflected in practices documented in syllabi, e.g. ACARA (p. 9).	
Hurst (2017)	Grants an ability to shift between 'inner' and 'outer' horizons, which respectively denote objects' properties and connections to larger mathematical structures (p. 117).		"It is the 'enabler' that allows teachers to set up, the contingent moments that are the essence of powerful teaching" (p. 117)
Seaman and Szydlik (2007)		Mathematical sophistication: beliefs about nature of mathematical behaviour, values concerning what it means to know mathematics, and particularly in avenues of experiencing mathematics objects and in distinctions about language Explicitly identified some fundamental norms of the community of mathematicians and demonstrated how these norms can help to understand why many preservice teachers find mathematics difficult (p. 170)	
Quebec Fuentes and Ma (2018)		Norms of discussions specific to the field of mathematics, sociomathematical norms, including what makes various explanations mathematically different, sophisticated, efficient and/or acceptable (p. 11)	

Table 2
Conceptualisations of 'horizon content knowledge' in review

Author	Conceptions of *horizon content knowledge*		
	Connected content knowledge based on big ideas	Mathematical knowledge to situated *mathematical horizons* of students in terms of their understanding	Awareness of the affordances of mathematical competencies to highlight mathematical connections
Hurst (2015b)		Teachers with well-developed *horizon content knowledge* (HCK) are able to look both forwards and backwards from a particular point of mathematical understanding and consider how to help a child to develop new knowledge or to see what understanding might be lacking in order to correct a misconception (p. 8).	
Hurst (2017)	Consists of connected content knowledge based on big ideas and also a sensibility about mathematical proficiencies (p. 120).	Teachers need: knowledge of students' mathematical horizons, situated in terms of his/her mathematical understanding (p. 115).	A sensibility about mathematical proficiencies and processes that can be invoked to help children reach their mathematics horizons and move beyond them (p. 120).
Seaman and Szydlik (2007)	The teacher must understand the rich connections among mathematical ideas (p. 168).	Be better able to identify specific mathematics needs to help children in particular situations (p. 168).	
Quebec Fuentes and Ma (2018)		Having connections to mathematical concepts, as presented in various ways, and requiring metacognition. (p. 15)	Developing an understanding of the specific ways of communication and representation centred on developing particular mathematical ideas as well as constituting the disciplinary discourse of mathematics. (p. 11)

Results and Discussion

Three main strands of big ideas and HCK emerged from our analysis. For both big ideas and HCK, two of three strands were respectively grouped around content knowledge and characterisations of mathematical thinking. In the third strand, HCK is explicitly defined in terms of big ideas or vice-versa, and thus could not be coded independently of the other construct.

Conceptualisations of big ideas

Firstly, big ideas are a "coherent set of ideas" (Hurst 2015b, p. 2) which allow for the connectivity of mathematical understanding. This was coded in two of the four studies. Hurst (2015b), quoting from Charles (2005), states that connectivity is important to "encourage a deep understanding of mathematics, enhance transfer, promote memory and reduce the amount to be remembered" (p. 2). Interestingly, big ideas are distinctive in allowing for cognitive improvements such as improved transfer learning and memory performance through a reduction of cognitive load. In consequence, future research could underpin this conception of big ideas based on empirical work. Further, Hurst notes importantly that "[t]here is not necessarily any one particular way in which content ideas can be linked around big ideas", as the big ideas can be linked together in different ways (Hurst 2015b, p. 2). For example, consider the big idea of 'proportionality'. Depending on the lesson objective, students could be led to the "inner horizon" (Hurst 2017, p. 116) to understand why 3/15 is equal to 1/5, or to the "outer horizon" (Hurst 2017, p. 116) to understand why fractions, decimals, percentages (Hurst 2015b, p. 6) are ultimately different representations of the same mathematical object. Ultimately, this first strand of big ideas emphasises the connectivity of mathematical content knowledge, and could be elaborated in specific versions such as in Charles (2005), Ma (2010), and Clarke et al. (2012).

Secondly, big ideas are disciplinary norms and beliefs about the nature of mathematics, which can be shown to affect mathematical understanding. This strand was evident in three of the four studies reviewed. Seaman and Szydlik (2007) most clearly show this through an inability of preservice elementary teachers to re-construct a correct mathematical understanding of the greatest common divisor, even when given mathematical definitions in a teaching resource. Instead, some preservice elementary teachers cling to a procedural approach to mathematics. This differing view on the nature of mathematics prevented them from even attempting to making sense of the relevant definitions. This was a lack of "mathematical sophistication" (Seaman & Szydlik, 2007, p. 169) on the pre-service elementary teachers' part, as Seaman and Szydlik observe amongst other deviations from a non-exhaustive list of nine disciplinary norms. Further empirical work may strengthen this claim to show how disciplinary norms such as problem-solving habits can improve general mathematical performance. Moreover, a closer look at the three studies indicates different understandings of what constitutes mathematics as a discipline. While Seaman and Szydlik (2007) and Quebec Fuentes and Ma (2018) refer explicitly to university mathematicians, Hurst (2015b) does not explicitly address the possibility that the mathematics education community and, what we loosely call the 'university mathematics' community, could have a differing set of norms. Any conception of big ideas based on disciplinary processes must clarify its definition of 'mathematical discipline' before it can be further demonstrated how such big ideas improve mathematical understanding and learning.

Thirdly, big ideas enable better teaching by setting up "contingent moments that are the essence of powerful teaching" (Hurst, 2017, p. 117). Of the four articles, this was explicated

only in Hurst (2017). This conception of big ideas has a clear link to the context of teaching. The resemblance to HCK is not an accident as in his view, "HCK and big ideas are inextricably linked, or even could be considered as one and the same" (Hurst 2017, p. 114). Whether HCK and big ideas are separate constructs need to be further evaluated. This evaluation could happen on the conceptual front as evidenced by further systematic reviews, or on the empirical front by investigating the separability of big ideas and HCK as constructs.

Conceptualisations of Horizon Content Knowledge

First, one concept of HCK is that it is the knowledge required to situate the "mathematical horizons" of student mathematical understanding. This view was found in all four studies. An example of this is provided by Quebec Fuentes and Ma (2018, p. 19), where an open-ended question is posed about a 'yellow square'. Students were to debate if the square is both a polygon and a quadrilateral, and the teacher needs to work with students' definitions of squares and rectangles in order to convince them that a square is a special kind of rectangle. That is, the teacher's content knowledge about mathematical definitions at different curricular levels are required for teachers to look "both forwards and backwards" (Hurst 2015b, p.8) so that the visual understanding of rectangles is connected with an inclusive definition of rectangles. This strand of HCK is thus characterised by open-ended engagement with students' ideas that does not fall into the other categories of Ball et al.'s (2008) categories in SMK or PCK.

A second related concept is that HCK is a sensibility for mathematical horizons, as understood in the previous sense. HCK consists of ways of representing and communicating mathematical ideas that can help children reach beyond their current mathematical horizon. This view was shared by Quebec Fuentes and Ma (2018) and Hurst (2017). Using the same example from Quebec Fuentes and Ma (2018, p. 19), the crux of this conceptualisation of HCK is in the sensitivity of the teacher to student's open-ended answers about squares. The teacher needs to apply "mathematical proficiencies and processes such as reasoning, justifying, hypothesising and problem-solving" (Hurst , 2017, p. 115) to transform student's answers into precise mathematical language, so that the students can understand that "a square is a special kind of rectangle". Like the difference between the first and second concepts of big ideas, the difference between HCK of the first and second kind is in the focus on the teacher's cognitive processes in navigating mathematics, in contrast to the content knowledge invoked for the same purpose. This parallel distinction will be significant in our later conclusion.

The third concept of HCK is that it consists of "connections and links within and between big ideas" (Seamand & Sydzlik, p. 8). This was also found in Hurst (2017), which held a view that HCK and big ideas might be the same (Hurst 2017, p.114). Again, the link between HCK and big ideas ought to be evaluated empirically as well as theoretically. We attempt to evaluate the latter in the next section.

How can teachers teach towards big ideas in the classroom?

Our review of papers discussing both HCK and big ideas simultaneously reveals that the two concepts are related but ultimately distinct. While both concepts are sometimes used to refer to both the content knowledge and the processes underlying it, our review shows that the focus of the two terms are different. For the big ideas of mathematics, the crux of the concept refers to knowledge and mathematical processes that unify the discipline, whereas, for HCK, the corresponding focus is within the context of teaching: HCK is the knowledge consisting of content knowledge and processes required to diagnose a student's current

mathematical horizon and advance it. The difference that characterises big ideas, in contrast to HCK, is an intentional usage of knowledge and practices from mathematics as a discipline. This makes sense given that HCK was originally a component of MKT. Whilst terminological ambiguity could have diluted its meaning (Jakobsen et al., 2014), HCK cannot be dissociated from teaching contexts. In contrast the construct of 'big ideas' runs in the opposite direction with the mathematical discipline influencing teaching.

The distinction of HCK from big ideas helps explicate how teachers can teach towards big ideas in the classroom. We propose that to teach towards big ideas is to embody the epistemic norms of the mathematical community in the classroom. By epistemic norms, we mean habits of the mathematical community that are demonstrably productive towards the generation of mathematical knowledge and the improvement of learning outcomes. This is supported, firstly, by the non-uniqueness of the mathematical connections between content knowledge, as discussed in the first strand of big ideas. That these connections need not be unique suggests greater importance for the habit of deepening one's mathematical content knowledge. Secondly, the second strand of big ideas emphasises strategies of knowing in mathematics that can be meaningfully brought into the classroom. The embodiment of epistemic norms requires the possession of both these strands of big ideas.

We suggest, then, a two-step characterisation for preparing teachers to 'teach towards big ideas'. First, 'teaching towards big ideas' involves an understanding of how mathematicians think. Whether big ideas are conceived as connective content knowledge (Hurst 2015b) or mathematical habits, characterised as "mathematical sophistication" in Seaman and Szydlik (2007), the teacher has to reflect in order to improve her classroom practice. This involves noticing differences from the mathematical discipline in how they conceptualise mathematical connections and in how they think mathematically. Secondly, 'teaching towards big ideas' needs to be evaluated on empirical metrics such as improved classroom practice and/or student learning outcomes. 'Teaching towards big ideas' takes time for its efficacy to be evaluated, and this evaluation could be incorporated into existing frameworks of professional development. The educative curricular approach of Quebec Fuentes and Ma (2018) is but one example of other approaches to professional development, such as lesson study, that can be pursued and investigated.

Finally, a comparison across the four studies suggests that the two distinctions between HCK and big ideas, and between content and process, are valuable for unifying the mathematics education literature. In our review, the studies that surfaced ran across the literature's breadth. The empirical work of Quebec Fuentes and Ma (2018) and Seaman and Szydlik (2007), and the theoretical works of Hurst, form a closed loop, that can benefit from more work that clarifies both constructs in theory and practice.

Conclusion

In conclusion, we suggest that to teach towards big ideas is to emphasise disciplinary ways of thinking that are empirically demonstrable to be fruitful for the learning of mathematics. Our review was limited by three factors. First, the current range of databases could be extended. Second, literature beyond journals should be considered in a more thorough review. We excluded grey literature including books and dissertations for practicality. There is reason to believe that grey literature may be useful for our research question due to its practitioner-oriented focus. Further, given a relatively new focus on big ideas in the literature, new ideas could be expected to be articulated outside of journal publications. Third, whilst teaching towards big ideas may seem to be a new trend in English language mathematics education journals, a systematic review across multiple languages

may reveal greater insights. In German-language literature, for instance, there has long been a tradition of established mathematicians interacting with their mathematics educators along the lines of "fundamental ideas" in mathematics (Vohns, 2016).

Given that mathematics as a discipline needs to be understood both universally as well as contextually, especially in connection to teaching, 'teaching towards big ideas' may benefit from a closer look at the existing interdisciplinary study of mathematical practice. Historical, sociological, and philosophical standpoints can have meaningful contact with the mathematics education literature in creating an empirically-grounded study of successful and diverse mathematical practices (Hamami & Morris, 2020; Kerkhove & Bendegem, 2007). Disciplinary features such as mathematicians' judgements about the elegance of a proof, explanation and understanding, the visualisation of mathematical objects, and the differences between informal and formal proofs are just some of the topics investigated in this burgeoning focus of interdisciplinary inquiry (Hamami & Morris, 2020), of which their contact with 'teaching towards big ideas' is not coincidental. To advance the 'teaching towards big ideas' successfully, it is suggested for further research to integrate knowledge across disciplinary divides, where meaningful.

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The Tattslotto question: Exploring PCK in the senior secondary mathematics classroom

Nicole Maher
University of Tasmania
<Nicole.Maher@utas.edu.au>

Helen Chick
University of Tasmania
<Helen.Chick@utas.edu.au>

Tracey Muir
University of Tasmania
<Tracey.Muir@utas.edu.au>

Pedagogical Content Knowledge (PCK) is a powerful construct for examining the complexity of teacher knowledge. Together with teachers' moment-by-moment choices of action, it provides insight into teachers' knowledge and its influence on student learning. This paper investigates the PCK experienced by a senior secondary mathematics class during a lesson on probability. Data were gathered through observation, and student and teacher interviews. Multiple aspects of PCK were evident and were used in complex and dynamic ways.

This study is from a wider investigation of *pedagogical content knowledge* (PCK) at the senior secondary mathematics level. PCK has become a powerful construct for examining the complex relationship between content and teaching (e.g., Ball et al., 2008). PCK is an intricate blend of content and pedagogy, described by Shulman (1986) as knowledge that embodies those qualities of the content "most germane to its teachability" (p. 9). There has been little research into PCK at the senior secondary mathematics level, with only a few small studies (e.g., Maher et al., 2015; Maher et al., 2016) exploring the complexity of teachers' PCK and its relationships with broader contextual factors. The present study builds on this research by examining the moment-by-moment enactment of a senior secondary mathematics teachers' PCK during part of a lesson, and how this is perceived by the students. Data were collected from a lesson observation, a post-lesson interview with the teacher, and from students' perspectives on their teacher's knowledge and actions. This paper will explore the following research questions: *What aspects of PCK does a teacher of senior secondary mathematics display while demonstrating a worked solution? What do multiple sources of evidence of PCK reveal about teaching and learning during a teacher's worked solution to an item?*

Review of Literature

In the past 35 years, PCK has received considerable attention in the mathematics education research community (e.g., Hill et al., 2008). The appeal of PCK may be attributed, in part, to its potential to more precisely describe teacher knowledge in action (Gess-Newsome, 2015). While "teacher knowledge in action" refers to important practices such as the preparation of meaningful explanations in predictably challenging content areas, it does not necessarily concentrate on what it means to "teach effectively moment by moment" (Mason & Davis, 2013, p.186). It is posited that teachers' moment-by-moment pedagogical choices of action are potentially the most influential source of insight into mathematics teacher knowledge (Mason & Davis, 2013; Mason & Spence, 1999). Mason and Davis (2013) pinpoint the vital role of the "connective tissue" between *mathematical awareness* (e.g., noticing an absence in understanding from a learner) and *in-the-moment pedagogy* (e.g., having an appropriate pedagogical action come to mind when needed).

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Several frameworks have been developed to identify aspects of teacher knowledge including PCK (e.g., Chick et al., 2006; Hill et al., 2008). The Chick et al. PCK framework (2006) used in this study provides a detailed inventory identifying key elements of PCK, designed for observing teacher knowledge in action in the classroom. The framework has been applied to the work of mathematics teacher educators (e.g., Chick & Beswick, 2017), and within the context of secondary mathematics teaching. Vale and her colleagues (e.g., Vale, 2010) have used it to examine the mathematical knowledge of out-of-area mathematics teachers.

The elements of the framework offer a set of filters through which to explore teaching in action. The framework reflects the complexity of PCK, by identifying its components under three broad categories: "clearly PCK", "content knowledge in a pedagogical context", and "pedagogical knowledge in a content context". These categories represent the varying degrees to which content and pedagogy are intertwined rather than specifying sharply defined boundaries. Space prevents the inclusion of the entire framework but brief descriptions of selected PCK elements specific to this study are given in Table 1.

Table 1
Excerpts from a Framework for Pedagogical Content Knowledge (from Chick et al., 2006)

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill
Student Thinking	Discusses or addresses typical/likely student thinking about mathematics concepts (either generally or with reference to specific students).
Cognitive Demands	Identifies aspects of the task that affects its complexity.
Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Explanations	Explains a topic, concept or procedure
Knowledge of Examples	Uses an example that highlights a concept or procedure
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Structure and Connections	Makes connections between mathematical concepts and topics, including interdependence of concepts
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning (e.g., justifies an activity as developing understanding of long-term probability).

Methodology

This paper uses data from a wider investigation into PCK at the senior secondary mathematics level and explores aspects of PCK from the perspectives of a teacher, his

students, and the researcher, by examining an episode from one lesson. A Year 11/12 Mathematics Methods class from a Tasmanian independent school participated. Mathematics Methods is the main pre-requisite mathematics course offered for students who intend to pursue tertiary studies in areas such as science and engineering.

The participants in the present study were the teacher, Mr McLaren, who had taught Mathematics Methods for 12 years, and the nine 16-18-year-old students in his class, five of whom provided data. Teacher and student names are pseudonyms. Data were generated during part of a lesson where Mr McLaren provided a worked solution to an item involving the practical application of a concept the students were studying. The episode was observed, video-recorded, and transcribed in full. After the lesson, the participating students completed a short questionnaire that asked them: (a) What did you find to be the most helpful explanation, example, or strategy that your teacher used in today's lesson? And (b) What did it help you learn? At the end of the lesson, the students participated in a 15-minute semi-structured focus group interview during which they commented on aspects of the lesson that they perceived were particularly useful. Mr McLaren also participated in a 20-minute interview after the lesson, discussing his actions during the lesson episode discussed in this paper. Both interviews were recorded and transcribed in full.

Teacher actions were identified by the authors and aligned to relevant PCK elements of the Chick et al. (2006) PCK framework. The teacher and student interview and questionnaire responses were also analysed for further insight into teacher PCK.

Results and Discussion

This section begins with a description of the lesson scenario involving a number of aspects of teacher knowledge. Insights into Mr McLaren's PCK—as illuminated by multiple sources of data—are then discussed. The scenario focuses on Mr McLaren's demonstration of the solution to the Tattslotto problem in Figure 1.

In Tattslotto, your chance of winning first division is $\frac{1}{8145060}$. Find the number of games you would need to play if you wanted to ensure a more than 50% chance of winning first division at least once.

Figure 1. The Tattslotto problem (condensed from Hodgson, 2013)

The Tattslotto problem scenario

During the lesson, Mr McLaren had introduced his students to applications of the binomial distribution and chose to demonstrate the solution to the Tattslotto problem as an example of a problem where the probability is known and the number of trials (n) is unknown. He guided the students through the process of setting up the inequality to model the problem. He identified that winning first division Tattslotto involved a binomially distributed random variable X with n trials and probability of success $\frac{1}{8145060}$, that is, $X \sim \text{Bi}(n, \frac{1}{8145060})$. With a mix of focused questions and explicit direct teaching, he helped the students to recognize that, given the item stipulates winning first division *at least* once, the situation can be expressed as "one minus the probability of *not* winning first division [in n trials]". He thus established the inequality $1 - \Pr(X = 0) > 0.5$ using the fact that $\Pr(X \geq 1) = 1 - \Pr(X = 0)$. Mr McLaren then guided the class through the development of the inequality shown in Figure 2, by using the formula for the probability distribution of a binomial random

variable X, given by $\Pr(X = x) = {}^{n}C_x\, p^x q^{n-x}$ where ${}^{n}C_x$ represents the number of ways that x different outcomes can be obtained from n trials, p is the probability of success (in this case winning first division), and q the probability of failure (not winning first division; equal to $1 - p$). The students had been introduced to this formula during the previous lesson.

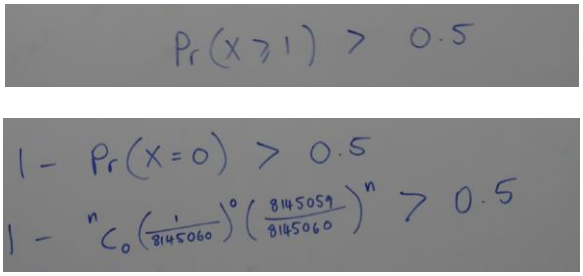

Figure 2. Initial stages in determining how many games are required in order to have a 50% chance of winning Tattslotto

After some procedural manipulation, which included dividing both sides by negative one and changing the sign of the inequality as a result, the inequality shown in Figure 2 was expressed as: $n \log_e \left(\frac{8145059}{8145060}\right) < \log_e 0.5$. Mr McLaren pointed to the inequality and asked the class, "What do we do now?" David suggested dividing both sides of the inequality by $\log_e 0.5$, so Mr McLaren reiterated that "we are trying to get n on its own, so we need to divide by the log of all that [points to $\log_e \left(\frac{8145059}{8145060}\right)$]. Now, is there anything else we need to know about?" There was a pause before Toby tentatively suggested that "The [inequality] sign changes". Kale quickly retorted "No it doesn't. I thought you said it only changes when you divide by a negative?" "That's right, so why would the inequality change?" asked Mr McLaren. "It doesn't" Kale persisted, looking puzzled. Mr McLaren assured them "It does change, but why?" Someone suggested, "because it's a log" to which the teacher responded, "Yes, well, in a way because it is a log, but why?" David offered "Because there is a rule on our formula sheet?" Mr McLaren shook his head with a smile "No, there is no rule on your formula sheet". He paused for a short while and then said, "OK, let's have a look". Mr McLaren began to write something on the white board but then quickly rubbed it off and changed tack. "OK, let's think of any log. Now remember the log graph, this is the easiest way to look at it". He sketched the graph of $y = \log_a x$ as shown in Figure 3.x

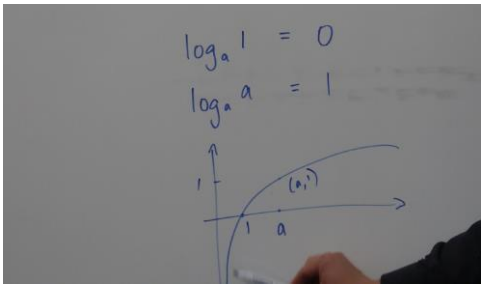

Figure 3. Mr McLaren's sketch of the graph of $y = \log_a x$ used to show when $\log_a x$ is negative.

Mr McLaren highlighted the point at x=1 and Toby suddenly called out, "Oh, so that's below one, so it's a negative, so that's why you change it around!" Mr McLaren nodded "Good, yes, any value of x less than one, or between zero and one, is negative". He pointed

to the region of the graph between x=0 and x=1 and reiterated that the logarithm to any base of any value for x between zero and one, in this case $\frac{8145059}{8145060}$, is negative. This was used to explain that when both sides of the inequality are divided by $\log_e\left(\frac{8145059}{8145060}\right)$, the inequality sign changes. "And it's a good thing too," Mr McLaren commented as he rubbed the board down, "otherwise we would find that we need to buy less tickets than we would actually need to buy. So, it's a good thing to look at what you're actually doing rather than just performing the calculations. OK, can someone evaluate that for me please?" [points to the right hand side of the inequality shown in Figure 4].

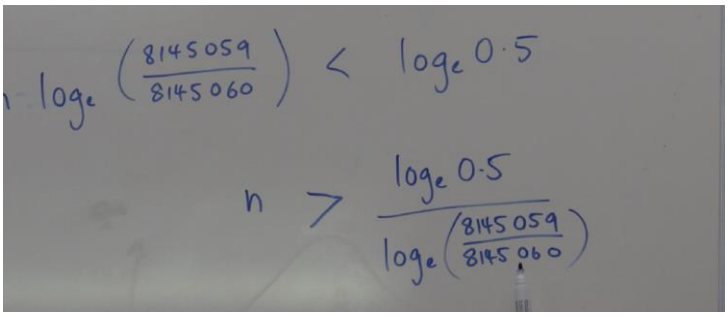

Figure 4. Final stages of the calculation of the inequality to determine the number of games.

Jonti performed the calculation, yielding 5645727.4. Mr McLaren asked "Can you buy 0.4 of a ticket? [The students shook their heads.] You would still write it to one decimal place, but for your final answer you would round up. You have to round up because if you go less than the 0.4 then you won't have greater than 50% chance of winning". Kale exclaimed, "So you'd need to buy that many tickets?!" Someone else added, "Just to have a 50% chance of winning once! What?" Mr McLaren smiled, "Yes, so you need to buy a lot."

Discussion of PCK

Multiple elements of PCK from the Chick et al. framework provided filters through which to examine the teacher's PCK in action in this episode. *Knowledge of explanations* was evident throughout Mr McLaren's worked solution. A combination of *knowledge of student thinking* and *knowledge of the cognitive demand of the task* were apparent, in that Mr McLaren was aware that students may not make the necessary connections with their previous work on logarithms to recognize that $\log_e\left(\frac{8145059}{8145060}\right)$ is negative. These aspects of PCK were intertwined with *knowledge of teaching strategies,* evident when Mr McLaren posed strategic questions to encourage the students to make the connection between the value of the logarithm and the reversal of the inequality sign. As the students did not appear to make this connection by themselves, Mr McLaren sketched the graph of $y = \log_a x$, where "a" represents any base, to assist them to recognise that the value of $\log_e\left(\frac{8145059}{8145060}\right)$ is negative. Mr McLaren's decision to sketch the graph appeared to be made in-the-moment, in that it seemed not to be something that was planned in advance, which highlights the complex and dynamic nature of teacher knowledge. During this in-the-moment event, Mr McLaren drew upon his own mathematical content knowledge and demonstrated several aspects of PCK including *representation of concepts, knowledge of mathematical structure and connections,* and *knowledge of the curriculum* (evident because the teacher drew upon

his knowledge of where logarithmic graphs were placed within the course). Further evidence of Mr McLaren's PCK was provided in the post-lesson interview, as seen below:

Researcher: The reversal of the inequality sign generated a lot of interest. How did you come to decide on how to show them why the sign changes?

Mr McLaren: … It's a hard one to remember because it [$\log_e\left(\frac{8145059}{8145060}\right)$] doesn't look like a negative number but umm I suppose it strengthens their understanding of logarithms. They were not understanding; well, they hadn't made any connections at that point.

While Mr McLaren's comment provides further evidence of *knowledge of mathematical structure and connections* and *knowledge of student thinking* it does not offer specific insight into his in-the-moment decision to use the log graph to show why $\log_e\left(\frac{8145059}{8145060}\right)$ is negative. On reflection, it may have been valuable if the researcher had phrased her question more carefully to probe for specific details about Mr McLaren's in-the-moment decision to draw the graph. Nevertheless, it is apparent that Mr McLaren's content and curriculum knowledge were sufficient for him to bring to mind (a) the reason for the change in sign and (b) a representation that would help students see why the value of the logarithm is negative.

Several students commented on the usefulness of the way Mr McLaren unpacked the solution to the Tattslotto problem, as indicated in their responses to the researcher's question about the teacher's useful explanations, examples, or demonstrations.

Jonti: The log one … [Toby concurs with "The Tattslotto one"]. It was good he kind of like decided on that Tattslotto question because it sort of recaps other things that we knew already so you go through it and refresh your mind on log laws and add the new layer of um technicality to it … Umm I don't know, it's just, well, it doesn't look that hard but then the way you've got to go around it with the logs and switching the inequality sign as you go through as well.

Researcher: Did you find anything in the explanation useful in helping you to piece it all together?

Jonti: Yeah umm I liked how he went through each step not like skipping over any one of them assuming you would know it.

Carl: Yeah umm just I kind of understood the thing except for getting tripped up when you've got to remember your log laws and like, and it was funny that you even had to draw a graph so go right back to the start to show us like if it's below like why you have to switch.

Jonti: The graph made it a lot clearer as to why you change the sign.

Kale also recorded that "the log explanation was the most useful because it explained and refreshed things for me like the log laws and changing < and >" (post-lesson questionnaire). These responses suggest that the students appreciated Mr McLaren's approach to solving the problem and that the graph had assisted them to realise why the inequality sign changed. Here the teacher's fluency across topics is a key part of his knowledge, and something that he wants to convey to students.

Mr McLaren also discussed the reversal of the inequality sign within the context of the problem, highlighting that it "makes sense because otherwise we would find that we would need to buy less tickets than we would actually need to buy". This aspect of Mr McLaren's PCK was identified as *purpose of content knowledge* because he alluded to the way the mathematics content may be used within the context of the Tattslotto scenario. This connection between the mathematics itself and the context of the problem resonated with Carl in the student focus-group interview:

> Carl: Yeah, because I was sitting there and I was like why did you switch it because it wasn't dividing by a minus but then it's like no because if you think about it, it's common-sense you're not going to have to only buy a small number of tickets.

During his post-lesson interview, Mr McLaren identified his reasons for selecting the Tattslotto problem.

> Mr McLaren: Umm probably more so from a non-maths kind of perspective to sort of demonstrate the futility of umm Tattslotto and the chances of winning umm that's probably the main reason why I chose that particular question. It wasn't so much a maths choice in that respect.

The teacher's response provides evidence of both *choice of examples* and *goals for learning* since he justified choosing the Tattslotto problem because it illuminates the very low probability associated with winning first division. One student, David, commented on this aspect of Mr McLaren's approach in the following written response:

> The Tattslotto question was the most useful. It helped me to find the number of games needed for a 50% chance of winning the game and how stupid gambling is. (David, post-lesson questionnaire)

It is noteworthy that teacher and student responses in relation to the Tattslotto problem focused on the involvement of logarithms during the solution process rather than on probability concepts per se. For example, although Mr McLaren solved the problem by setting up the inequality as 1 – Pr (not winning in *n* games), thus demonstrating that content knowledge, the significance of this probability technique was not evident in the other data sources in that neither teacher nor students mentioned this as a key learning outcome. This might mean that students were familiar with the technique, and that Mr McLaren had *knowledge of student thinking* to be confident that they could use it fluently, or, alternatively, this was not identified as a key learning point, which may reflect some shortcomings in PCK.

Conclusions

While this study is limited to one account of a lesson episode, it provides a detailed snapshot of the nature of a senior secondary mathematics teacher's PCK in action, and its influence on students from their perspectives. The level of detail and specificity afforded by the Chick et al. (2006) framework rendered it useful for examining the moment-by-moment teaching and learning interactions between Mr McLaren and his students. As such, the study contributes to the field of research into the complexity of PCK at the senior secondary level from multiple perspectives, including the researcher, the teacher, and his students.

Multiple elements of PCK were evident in the scenario, particularly those from the "clearly PCK" section of the framework, including *knowledge of student thinking, knowledge of the cognitive demand of the task*, *knowledge of teaching strategies*, and *representations*. *Mathematical structure and connections* from the "content knowledge in a pedagogical context" section of the framework, and *Goals for learning* from "pedagogical knowledge in a content context" section were also evident in the data.

The combination of PCK elements from across the framework provided insight into the nature of the interactions between Mr McLaren, his students, and the mathematics itself, highlighting the complex and dynamic nature of PCK. For example, Mr McLaren called upon his own content knowledge to decide on which action to take in order to make visible for the students why the value of the logarithm was negative. There were also interactions with the broader teaching and learning context, with Mr McLaren's expressed reason for choosing the Tattslotto problem being to illuminate the "futility of gambling" rather than the

mathematics per se. On the other hand, some key probability ideas, such as the role of the complement, may have been underemphasised because of this focus.

Other post-lesson teacher interview data supported the researcher's observations of *knowledge of student thinking* and *mathematical structure and connections* but was limited in terms of providing insight into Mr McLaren's specific choices of action. The Chick et al. (2006) framework for analysing PCK, however, offers a level of detail and specificity that is potentially useful for examining what comes to a teacher's mind in moment-by-moment teaching and learning interactions.

The students' perceptions of Mr McLaren's actions gave useful insights into PCK and supported evidence from the other data sources. For example, Mr McLaren's representation of the log graph representing the relationship between the value of x and its logarithm was particularly noticed and appreciated by the students. Similarly, the connection Mr McLaren made between the mathematics involved in the reversal of the inequality sign and the reality of the number of games that would need to be played, was valued by some students.

. Further studies that investigate PCK in different senior secondary mathematics contexts with a particular focus on the moment-by-moment pedagogical choices of action would also add to the limited research in this area.

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Capitalising on student mathematical data: An impetus for changing mathematics teaching approaches

Tracey Muir
University of Tasmania
<Tracey.Muir@utas.edu.au>

National testing and reform agendas, with their focus on school improvement, has led to increased collection and scrutiny of student data. The analysis of these data usually occurs at a school level, often by school leaders. What is less common is the opportunity for students to scrutinise their individual data and take ownership over the results and subsequent learning experiences. This paper reports on a study whereby students and teachers collaboratively interpreted mental computation test results and identify future teaching and learning directions. The findings showed positive outcomes for students led to changes in teacher growth and approaches to their teaching of mathematics.

A key finding from a study aimed at developing an evidence base for best practice in mathematics education (Smith et al., 2018) identified that data can be used to monitor student outcomes and progress in mathematics. Purposeful use of data was a characteristic of the successful schools in the study, with the report recommending that sharing best practice models for using data would benefit all schools. Direct measures of student outcomes, and the collection and analysis of data, have also been identified as essential contributors to school improvement (ACER, 2019). 'Analysis and discussion of data' is one of the eight domains identified in the Teaching and Learning School Improvement Framework (Masters, 2010) whereby outstanding schools are characterised by having established and implemented a systematic plan for the collection, analysis and use of student achievement data. Furthermore, data are used throughout the school to identify gaps in student learning, monitor improvement over time, and monitor growth across the years of schooling (Masters, 2010). The Grattan Institute also recommended the use of data to inform teaching through the provision of a checklist of effective uses of data such as a shared sense of responsibility for students' learning; developing a common language across the school; and in-house professional learning (Goss et al., 2015).

This paper reports on a study which was part of an ARC research project which aimed to improve students' learning and wellbeing through a focus on personalised learning and team teaching in six different Australian schools. Each school identified a curriculum focus, which in this case was mathematics, and for the purpose of this paper, the topic of mental computation. Clarke and Hollingsworth's (2002) model was used to inform the professional learning and subsequent professional growth experienced by the teachers in the study, with student data collected pertaining to students' performance in mental computation. These data were subject to subsequent analysis, and formed the basis for future teaching and learning experiences. The following research questions were addressed for this paper:

1. *In what ways can student data on mental computation performance inform subsequent mathematical teaching experiences?*
2. *In what ways does a shared responsibility for teaching mental computation contribute to teacher growth?*

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 297-304. Singapore: MERGA.

Theoretical Background

Schoenfeld (2014) identified five key dimensions which characterise quality learning in mathematics: curricular coherence of the subject, cognitive demand of tasks, student access to mathematical content, opportunities for student agency, authority, and identity, and effective use of assessment. Cox et al. (2015) documented a case study whereby students experienced personalised learning in mathematics that met Schoenfeld's dimensions. Curriculum coherence, for example, was provided through a focus on student learning intentions, and individualised mathematics programs, and students worked in groups at the same level. Of particular relevance to this paper is the consideration of effective assessment, whereby consideration is given to the monitoring of student understanding and timely planning that addresses students' needs and offers ways to progress in performance (Black & Wiliam, 2009). The schools in Cox et al.'s (2015) study used student data collected through NAPLAN and diagnostic tests to design individualised programs for their secondary classes. The researchers found that test results, despite being trenchantly criticised elsewhere for their reductive effects on curricular content and methods, actually allowed teachers to tailor curricular experiences and progressions to meet the developmental needs of individuals in mathematics.

Teacher change and professional growth

According to Guskey (1986), teacher change is likely to occur only after changes in student learning outcomes are evident. Guskey and others (e.g., Fullan, 2015) highlight the limitations of one-off professional learning opportunities and advocate the situating of professional development within realistic contexts. For teachers to make significant changes to their practice, multiple opportunities are required to learn new information, trial new approaches and evaluate the impact of these approaches (Timperley, 2008). In addition, collegial interaction and expertise are required to challenge existing assumptions and develop new knowledge and skills associated with positive outcomes for students (Timperley, 2008). Change is more likely to occur if teachers are seen as learners and schools as learning communities (Clarke & Hollingsworth, 2002), and more likely to be sustained if there is evidence of student learning success. With this in mind, Clarke and Hollingsworth (2002) developed an Interconnected Model of Teacher Professional Growth which identifies the mediating processes of reflection and enactment as the mechanisms by which change in one domain leads to change in another. As shown in Figure 1, four domains are identified, with the type of change reflecting the specific domain. For example, using a new teaching approach is relevant to the domain of practice and a changed perception of salient outcomes related to classroom practice would reside in the domain of consequence. Through the use of the model, Clarke and Hollingsworth (2002) found that having a community of colleagues with whom consequences of experimentation were shared facilitated documented changes in teachers' practice. These findings are consistent with other research that endorses collaboration, where practices are deprivatised (Vescio, Ross, & Adams, 2008), enabling teachers to engage in meaningful reflection alongside colleagues working in similar contexts (Buysse, Sparkman, & Wesley, 2003). Clarke and Hollingsworth's (2002) model has been applied in a range of contexts to identify growth in teachers' learning (e.g., Downton, et al., 2019), and was used to interpret the changes in teachers' practice reported in this paper.

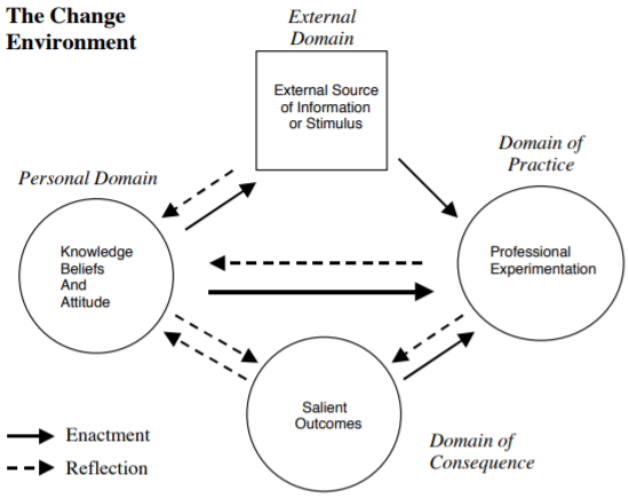

Figure 1. Interconnected model of Teacher Professional Growth (Clarke & Hollingsworth, 2002, p. 951).

Mental computation

Mental computation and the explicit teaching of strategies was selected as a focus by the teachers in this study as they consistently found that students in their Year 5 and Year 6 classes were relying on written methods, rather than mental methods, to solve basic number fact problems. Mental computation is emphasised in the Australian Curriculum: Mathematics (ACARA, 2018). By Years 5 and 6, students are expected to solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental strategies (ACMNA100), use efficient mental strategies to solve problems (ACMNA291), including those involving all four operations with whole numbers (ACMNA123). While specific mental computation strategies are not identified, McIntosh and Dole (2004) identified a number of strategies including bridging 10, commutativity, and doubles.

Methodology

The study reported in this paper was part of a wider collaborative project which aimed to improve regional low SES students' learning and wellbeing. Involving six different schools from two Australian states, each school developed their own projects which included individualised approaches to supporting learning and wellbeing, in response to the interests and needs prominent in each site. The project reported here involved teachers capitalising on mathematics test results to personalise students' mathematics learning.

The project used a longitudinal multi-phased mixed methods design study (Creswell, 2003) to examine the effects of the proposed strategies as they were enacted in each school site. Each project site entailed an interpretative cycle whereby observations and teacher insights and practices, gleaned from interviews and meetings, progressively fed into the findings and forward planning.

Context and participants

The site which is the subject of this paper was Epping Primary School (pseudonym). Epping Primary School is a semi-rural school with a total student population of

approximately 500. The participants for the study were four Year 5 and Year 6 teachers and their classes which totalled approximately 120 students. Following ethical approval, full consent was given by the teachers and the participating students' parents.

Data collection and procedure

The researcher's role was partly observer, participant-observer, and an external source of information or stimulus (Clarke & Hollingsworth, 2002). Beginning in 2017, the researcher met each term with the Year 5/6 teachers and school leaders to identify the mathematical focus or topic. The researcher and teachers worked collaboratively to either develop or adapt a pre-test on the topic (e.g., mental computation) which was administered to all students. The teachers marked the tests and then organised students into four similar ability groups based on the results. They also conducted interviews with the students to share individual test results and have students write their personal goals for mathematics learning. With the support of the researcher, the teachers adopted a shared responsibility through collaboratively planning for and then teaching, the selected mathematical topic to the whole cohort of 120 students.

In addition to 'regular' mathematics classes, 2-3 sessions were planned weekly whereby all the students gathered in the Performing Arts Centre (PAC) space. PAC maths (as it came to be called) involved a 15-20 minute session which was planned for and led by one of the teachers. For mental computation, the sessions would involve familiarising students with different mental computation strategies, providing them with problems to calculate mentally and then whole group sharing of selected students' strategies. Students used individual whiteboards to record their thinking when required. Following the whole group session, students were then split into four groups and moved to their allocated teacher's classroom. Each teacher was responsible for adapting instruction on a previously agreed strategy for their particular group. The emphasis was on increasing students' range of strategies and teachers typically made use of games and activities to develop the strategies. The experiences for each group were similar, but tended to differ in terms of the magnitude of the numbers involved. The teaching of mathematics continued in this way for 4-6 weeks, and then students were given a post-test. Results were again discussed between the teachers and the students, and a new focus was identified. The data reported on in this paper relates to a fourth cycle undertaken on mental computation in Term 3, 2018.

Data analysis

The data discussed in this paper include pre- and post-test results, student data interviews and a teacher focus group interview. The pre- and post-tests were co-designed with the teachers and the researcher and contained 73 items. Essentially the items were the same for both tests with some variation in the numbers given. There were 50 items that required instant recall, but most items required application of strategies as illustrated in Figure 2. The strategy items were marked according to a rubric designed by the teachers and used a rating scale of 0 (no or incorrect response), 1 (partially correct response) and 2 (complete correct response). The response shown in Figure 2 scored a 2. It was possible to score a total of 116 in both the pre-test and post-tests. Interviews were semi-structured, audio-recorded, of approximately 15 minutes in duration and fully transcribed. Student data interview scripts were open coded, with mathematical language and goals being examples of two codes applied. The teacher's focus group interview was analysed to look for evidence in changes

in practice using codes related to the four domains of Clarke and Hollingsworth's (2002) model. The next section presents the results of the study.

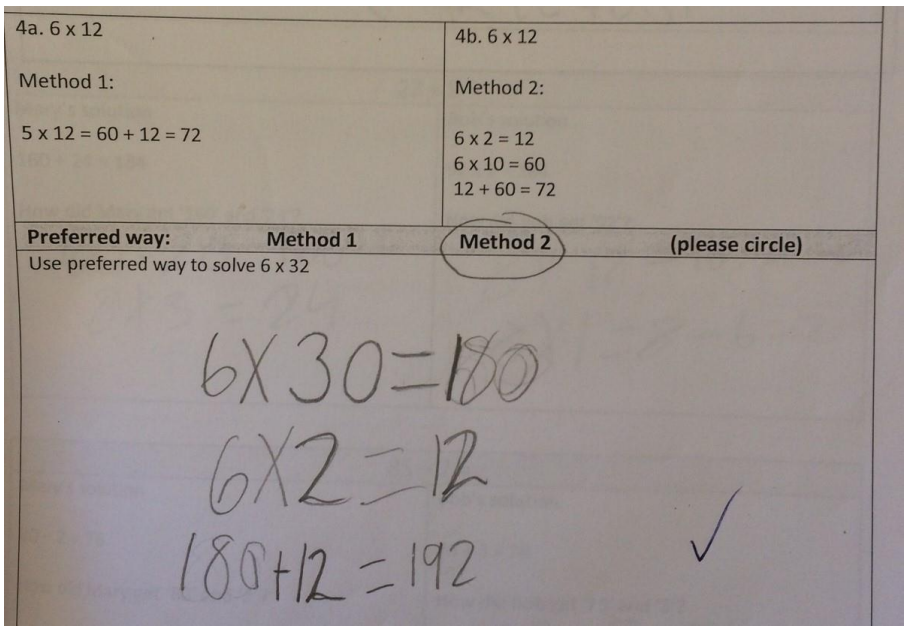

Figure 2. Example of test item and student's response

Results

Prior to participating in the project, each of the Year 5/6 teachers were responsible for individually planning, developing and teaching mathematics to their own classes. As a result of the project, the teachers assumed a sense of shared responsibility for students' learning, and changed their approach to collaboratively plan for the whole cohort of Year 5/6 students, based on the results of a pre-test. Individual data interviews were held with students, 15-minute introductory sessions were conducted with the whole cohort, and students were organised into fluid groupings for targeted instruction, based on the results of the test. Students' and teachers' experiences of this approach are detailed in the next sections.

Capitalising on student pre-test data

In order to capture how students experienced PAC maths, two students, John and Tina (pseudonyms), have been selected to illustrate how the approach worked in practice. John scored 56 in the pre-test and was particularly confident with instant recall. He indicated in his interview that:

> I reckon I did pretty good. I like times tables, so I'm pretty good with times tables. I think I did pretty good … I liked part two where you had to choose your method, then you had to tell in your answer why it was preferred.

He was less confident with the items that required him to interpret the work sample responses and left most of Part 5 blank which required the use of specific mental computation strategies to solve the problems ("it was a bit hard for me"). John initially identified that his goal would be to "work on harder questions". His teacher helped him to refine the goal in the following way:

> We've got to have a look at your goal and see if it's very specific or not. So, work on harder questions. We might not know exactly what that looks like, so is there another way that we can be a bit more specific about that? Questions that involve what?
>
> John: Involve maybe harder times tables, like 23 times 200 or something.

In the interviews each student was also encouraged to set a mathematical behaviour goal. After some discussion, John identified that this goal would be to have a go at questions he was not sure about, rather than leaving them blank.

Tina scored 92 in her pre-test and made an attempt to answer every question. While she could identify two different ways to solve problems when asked, her responses showed a preference for the formal algorithm because "it is quicker and easier for me". She was able to articulate areas in which she was confident with and others which she found challenging:

> I did well in the timed questions, then I sort of went downhill through the rest of the test but I still did my best. It was hard for me to say how I did it because most of it was in my head. Division was more challenging because most of the division questions I get always have sevens and eights in them and I can't really divide with seven and eights.

Salient outcomes: Students' perspectives

Along with the whole cohort of Year 5/6 students, John and Tina participated in 2-3 whole group 15-minute sessions in the PAC, followed by 45 minutes of targeted group instruction. According to John's teacher, he was placed in the 'second top' group, where specific mental computation strategies such as doubling and bridging 10 were taught. In his post-interview, John indicated that he thought he had achieved his goal and learned about strategies such as bridging 10 and doubling and halving. He scored 92 in his post-test (an improvement of 31%) and indicated that he liked the PAC maths approach:

> You get to work in groups where you can interact with other people, plus they help you out if you don't know a sum, like, they can teach you how to do it.

Tina was placed in the 'top' group, for her targeted mathematics instruction and scored 105 in her post-test. In her post-test interview she acknowledged that the teaching approach had helped her towards achieving her goal:

> [We learned about] split and divide and split and multiply and friendly and fix … where you make one of the numbers up to ten, instead of having a unit in it and you add that back on later … so 49 plus 20, and you make it into 50. 50 plus 20 is 70, then you minus the one that you added on, so that's 49.

Tina also expressed a liking for the PAC maths approach:

> There's other people in the room … and I like watching what answers they get, once I've got my answer and I'm holding it up. I like seeing how other people have thought, that's my favourite thing about PAC maths … and I like the groups because in [regular] class I have people that are lower than me, so we have to teach them stuff I already know.

Salient outcomes: Teachers' perspectives

The focus group interview provided teachers with the opportunity to discuss the benefits and challenges of the PAC maths approach. They found the pre-test was useful in terms of identifying that while many students did well on the multiplication items, many students found division challenging and did not see the connection between the two operations. The collaborative planning and whole group teaching sessions enabled "all teachers to be using the same language which is good" and the targeted instruction groups provided for differentiation with a smaller range than typically experienced in a regular class grouping:

> Maths is hard to teach in our [regular] class … the range is just so huge … all across the 5/6 cohort there's a big lot of D kids and because you've got Ds and you've got your As and a few Es, it's really hard to plan at everyone's level. [Jane]

> I think that's the key to helping them go forward because you don't have to worry about those ones - you've just got that core of the kids. You know very explicitly what they can and can't do and how you can just push them to move that little bit further because it's just targeted at them. [Tim]

Other than logistical issues early on with factoring in planning time and booking the PAC space, the teachers all agreed that PAC maths was not only beneficial for the students, but also for their own teaching practice:

> I think it's been good in the sessions that we do have together that they [students] realise that sometimes we can be so isolated in our rooms, "Oh, we're all learning this." That's quite a powerful thing … it's pushed us out of our areas as well. It's been really powerful for the kids to see that we all teach – I mean, I've gone from taking the top group to the bottom group and that has been really powerful for the kids to see … it's just been good … everyone's been happy. [Jane]

Discussion

There is evidence that Schoenfeld's (2014) five dimensions of quality learning were enacted through the PAC maths approach. Collaborative planning provided for curriculum coherence across the Year 5/6 cohort. The whole cohort grouping at the beginning of each session ensured that all students received the same core content, experienced different teaching styles, developed a common vocabulary, and were exposed to a wide range of different students' thinking strategies. Like the students in Cox et al.'s (2015) study, these students worked in similar ability groupings, with the establishment of personal learning goals fostering the development of individualised learning.

Clarke and Hollingsworth's model (2002) provided a useful lens for understanding teachers' growth and commitment to sustain the practice. Through the researcher and their involvement with other schools in the project, the teachers were exposed to an external source of information or stimulus. Site visits to other participating schools allowed teachers to observe different enactments of personalised learning and they were particularly impressed with the shared practice of capitalising on the use of student data. The teachers then engaged in professional experimentation through their use of pre- and posttests, whole cohort PAC sessions, which deprivatised their practice (Vescio, et al., 2008) and grouping for instruction. They were motivated to continue with cycles involving different topics when students' results improved from pre- to posttests (salient outcomes) and they experienced satisfaction from their teaching approaches.

In terms of improving students' mental computation skills and knowledge, the interviews showed that students were able to identify learned strategies that were helpful and efficient and helped them to achieve their personalised mathematical goals. The test results allowed teachers to identify gaps in students' learning (Masters, 2010), while the interviews allowed teachers to tailor curriculum experiences to meet the individual needs of students (Cox et al., 2015).

Conclusions and Implications

While the collection of student data is becoming increasingly prevalent in our schools, more could be done to capitalise on this valuable source of information. Analysis and discussion of data has been identified as an essential component of school improvement (ACER, 2019), yet examples of effective ways of how this might be done is limited in the literature. The approach detailed in this paper provides such an example, which could be

adapted by schools in similar contexts. It is likely that within any school, that is typically organised in year cohorts, students are exposed to different mathematical experiences depending upon their teacher's interpretation of the curriculum. PAC maths made provisions for teachers to develop a shared responsibility for the Year 5/6 cohort and students and teachers benefited from being exposed to different teaching, deprivatisation of practice (Vescio, et al., 2018) and interaction with different students. The project provided teachers with the opportunity to engage in professional experimentation (Clarke & Hollingsworth, 2002), and salient student outcomes provided an impetus for professional growth to occur. It is hoped that the project detailed in this paper has provided teachers and school leaders with an insight into how rethinking current teaching approaches can lead to improved mathematical outcomes and experiences for students.

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The development of predictive reasoning in Grades 3 through 4

Gabrielle Ruth Oslington
Macquarie University
<gabrielle.oslington@mq.edu.au>

Joanne Mulligan
Macquarie University
<joanne.mulligan@mq.edu.au>

Penny Van Bergen
Macquarie University
<penny.vanbergen@mq.edu.au>

This longitudinal study aimed to determine changes in students' predictive reasoning across one year. Forty-four Australian students predicted future temperatures from a table of maximum monthly temperatures, explained their predictive strategies, and represented the data at two time points: Grade 3 and 4. Responses were analysed using a hierarchical framework of structural statistical features. Students were more likely in Grade 4 than Grade 3 to make reasonable predictions (87% vs 54%), to demonstrate data transnumeration in their representations (71% vs 19%), and to describe data prediction strategies based on extraction, clustering, aggregation of data, and observations of measures of central tendency.

Predictive reasoning is an everyday process where the decision-making process is informed by chance events placed in a context of underlying causal variation (Makar & Rubin, 2018). Research with primary school students on probability and prediction often focuses upon deterministic experiments using devices such as random draws; for example, balls or lollies (Reading & Shaughnessy, 2004). These tasks are helpful for investigating and promoting student reasoning because they reduce the number of sources of variation under consideration. However, at the heart of statistical analysis lies an understanding of the relationship between variables (Biehler et al., 2018). To answer many "real world" statistical questions of interest, students must also be capable of making distinctions between authentic correlations and genuine randomness (Bryant & Nunes, 2012). Designing predictive tasks, which include both causal and random variation, can therefore have exceptional potential for exploring some of the big ideas about probability and variation, such as distribution, expectation and randomness, and inference and sampling.

Appreciating distributional relationships and identifying authentic data patterns while predicting can be challenging for young students. Watson and Moritz (2001) described predictive strategies used by young students which included seeking missing or unused numbers, as also reported in Oslington et al., (2020). This may be linked to students' perception of "fairness" whereby variation is controlled by even allocation across groups (Reading & Shaughnessy, 2004). When describing the data predictions of Grade 2 students, for example, Ben-Zvi and Sharett-Amir (2005) noted that students in their sample appeared to conceive the data as flat distribution with all values equally likely. However, Grade 3 students with an aggregate view of data were generally able to make reasonable predictions (Oslington et al., 2020) suggesting a developmental progression towards an understanding of data as a distribution containing a central signal with random variability around the signal (Konold & Pollatsek, 2002).

Conceptual framework

Students' development of predictive reasoning competencies can be supported in multiple ways, including via opportunities to create and analyse data (English, 2012; Makar,

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2016), opportunities to represent data by drawing, describing and graphing (English, 2012; Mulligan, 2015), and via other low-stakes learning experiences in which predictive reasoning language, context, and content is scaffolded (Kazak et al., 2015; Makar, 2016). To measure these competencies, researchers have drawn upon frameworks that observe what students can *do*—such as the Awareness of Mathematical Pattern and Structure (AMPS) (Mulligan & Mitchelmore, 2009) or the Structure of Observed Learning Outcomes (SOLO) model (Watson et al., 2017)—as well as what students *notice and describe* (English, 2012; Konold et al., 2015; Mulligan, 2015).

Mulligan and Mitchelmore's (2009) five-level AMPS conceptual model demonstrated that some young students spontaneously sought patterns, structures, and relationships by noticing similarities and differences between mathematical quantities, objects or relationships. In this process, students showed an understanding of emerging generality when they identified and applied common structural features and noticed regularities of spatial structures. These students could also think relationally. Based on these findings, AMPS describes two interdependent components: one cognitive—a knowledge of structure, and one meta-cognitive—a tendency to seek and analyse patterns. When applied to the development of statistical reasoning, it is implied that pattern and structure refer to the general properties within the data set, which can be expressed through relationships between the elements or subsets of the data set (Mason et al., 2009).

Our recent study (Oslington et al., 2020) found individual differences in reasoning among Grade 3 students. For example, students who viewed the data represented in a table as a single dataset also tended to use that same table as a resource for making predictions; integrating data with personal experiences and general knowledge. However, other third graders used the table in an inconsistent and idiosyncratic way. The current study, which included the same cohort at the beginning of Grade 4, focused on the changes in structural features of student predictions, reasoning, and representations between Grades 3 and 4. The AMPS framework was extended to capture the structural features of statistical development, drawing on research which suggests that the development of complex cognitive processes underpinning predictive reasoning occurs over time and from an early age. These studies imply that students who engage in pattern-seeking behaviours such as seeking similarities and differences are likely to understand the mathematical and statistical structures behind these patterns, while those who do not notice the patterns are likely to focus upon idiosyncratic or non-mathematical features. Thus, the developmental process of predictive reasoning might be interpreted through observing the pattern-seeking behaviour of students while engaged in a predictive reasoning task.

The Design Study

An earlier report (Oslington et al., 2020) described the first of three iterations of a design study on predictive reasoning conducted with a single student cohort (Oslington, 2020). This report describes the second of the three iterations and two research questions were addressed:

(1) Which structural features of the data were identified by students when predicting from, reasoning about, and representing a data table?

(2) How do the predictions, reasoning, and representations of Grade 3 students compare with those same students when in Grade 4?

Participants

Students attended a K-12 independent school in metropolitan Sydney. The school was relatively advantaged with an ICSEA score of 1080 and 75% of students were in the top half of Australian students. The report here covers two data points: the first week of Term 1 for Grade 3 students and the first week of Grade 4 with the same students. Forty-four students were available at both time data points representing 96% of the year group. Ethical permission was provided for all students. Data collection on both occasions was by the first author as teacher-researcher.

Table 1
Coding for student predictions, reasoning and representations (AMPS level)

AMPS level	No. of reasonable[a] predictions	Explanations	Representations
Advanced structural	12	Identified and described relationships between variables across the table	Relevant data transnumeration demonstrating association between variables of time and temperature (e.g., dot and line graphs)
Structural	10-11	Patterns observed across the row and column structure	Relevant data transnumeration through reorganisation of variables of time and temperature (e.g., bar graphs, tables extracting highest values from years or months)
Partial	5-9	Patterns observed using column structure	Attempted transnumeration, such as changing the orientation of the table without relevant data extraction (e.g., reorienting the data table so time series is on the horizontal axis)
Emergent	3-4	Patterns described not related to data context or content	Reproduction of data set without transnumeration (i.e., copy of table)
Pre-structural	≥ 2	Systematic pattern seeking not employed	No interaction with the numbers in the data table observed (e.g., weather pictures, empty grids or invented data)

[a] Predictions were considered reasonable if falling within the 5th and 95th percentiles of temperatures historically recorded for the month at Observatory Hill meteorological station, Sydney.

Lesson design and data collection

As previously described (Oslington et al., 2020), students were withdrawn from the classroom in convenience groups of 9–12 students. Each student independently: (1) predicted future maximum monthly temperatures for Sydney using a table of past maximum temperatures, (2) constructed a representation of the temperature data, and (3) explained their predictive strategies via a videoed interview. The data collection process was identical in both Grades 3 and 4, with the exception that the temperature table provided to Grade 4 students (Figure 1) contained one extra year (2017) of temperatures. Data consisting of

student predictions, video interviews, and representations for students at each time point were coded using a five level AMPS scaffold (Table 1).

Highest daily temperature in the month												
Max Temp	Jan	Feb	March	April	May	June	July	August	Sept	Oct	Nov	Dec
2010	41	38	31	31	27	21	21	25	27	29	30	31
2011	35	42	34	27	23	21	21	26	33	34	37	27
2012	30	29	27	26	23	20	20	21	24	26	27	29
2013	46	30	34	29	26	22	24	25	32	37	34	36
2014	37	32	30	30	27	23	25	23	34	34	37	32
2015	36	32	36	32	28	23	21	28	30	37	41	35
2016	39	33	31	34	28	22	26	25	25	34	33	38
2017	39	38	32	29	26	20	27	26	34	35	28	38
2018												

Figure 1. Maximum temperatures table for Sydney provided to Grade 4 students.

Results

As shown in Table 2, evidence of structural and advanced structural awareness underpinning temperature predictions was more widespread in the Grade 4 cohort relative to Grade 3. In Grade 4, 87% (n=38) of students predicted sequences of temperatures with at least 10 reasonable predictions. In contrast, just 54% (n=24) of the students had reached this same milestone in Grade 3. Almost half the Grade 4 cohort (48%) made reasonable predictions for all 12 monthly values, compared with 27% of Grade 3 students. The percentage of students making emergent or pre-structural predictions declined from 22% of Grade 3 students to only 9% of Grade 4 students.

Table 2
AMPS levels of student predictions, representations and explanations in Grades 3 and 4

	Predictions (%)		Explanations (%)		Representations (%)	
	Grade 3	Grade 4	Grade 3	Grade 4	Grade 3	Grade 4
Adv structural	27	48	2	7	5	12
Structural	27	39	15	36	7	45
Partial	23	5	34	36	7	14
Emergent	11	7	25	14	36	20
Pre-structural	11	2	23	7	45	9

Reasonable predictions relied on identification of patterns in the data, which some students could articulate at interview. Strategies unrelated to the data table (pre-structural), or that erroneously applied personal knowledge and experience in isolation from the data (emergent), were reported by 48% (n=21) of students in Grade 3 but just 20% (n=9) of the same students in Grade 4. By Grade 4,79 % (n=21) of the cohort described some relevant aspects of the data table when explaining their predictive strategies (i.e., a partial, structural, or advanced structural response) relative to 22 (50%) in Grade 3. Students' strategies that

focused upon a single aspect of the data were categorised as partial. These explanations often reflected awareness of the similarities or patterns in the vertical column structure of the table only, but were not coordinated with the seasonality implicit in the horizontal row structure. Other strategies included matching the tens' digits or looking at the highest or lowest value in each column. Approximately one third of each of the Grade 3 and 4 cohorts' (34% and 36% respectively) elicited explanations that were coded at the partial level. Structural responses successfully integrated seasonal knowledge and changes in temperature with the yearly trends of the data table by coordination of vertical and horizontal data entries. Such responses were more common in Grade 4 (36%) than Grade 3 (15%). Grade 4 student Luca, for example, first described observing the range of values in columns, and noticed that some values were more frequent than others, implying an expectation of a clumped distribution. He then explained that the previous two July temperatures were the hottest July values, predicting a similarly hot value for 2018.

Finally, and in contrast to the temperature prediction data, verbal explanations at the advanced structural level were quite uncommon, observed in only one Grade 3 student (2%) and three Grade 4 students (7%). These students described the data table holistically, often observing multiple interrelated components (e.g., noticing a higher level of variability with increasing temperatures). They differed from those with structural explanations by describing how they had sought central tendencies in the data, or by selecting an average or representative figure. In Grade 3, for example, Joseph observed differences in monthly data range and benchmarked values by intentionally making adjacent months a few degrees higher or lower than adjacent ones. Grade 4 student, Rhys, explained he sought the "approximate average temperatures", describing these as middle values, closely aligned to the median. Grade 4 student, Stuart, started his predicting with winter temperatures where the variability was least, and explicitly linked the higher range of temperatures in the summer months to the amplified impact of climate change on warmer seasons relative to cooler ones.

Students' representations of predictions in Grade 3 were predominately pre-structural, or emergent (Table 2). Pre-structural representations did not include data from the original temperature table, instead consisting of weather pictures, empty grids, and tables with invented data (pre-structural, Figure 2). Students with emergent responses appeared to recognise the importance of the original temperature data by copying the table, but provided no interpretation or additional manipulation to illustrate understanding (emergent, Figure 2). In both Grade 3 (7%) and Grade 4 (14%), a relatively small number of students also constructed representations at the partial structural level, using aspects of the table data in a non-systematic way. For example, the transnumeration labelled partial in Figure 2 was constructed as a time sequence on the *x*-axis prior to stacking the temperature data values on the *y*-axis above.

Finally, by Grade 4 more than half of the students created representations that were categorised at the structural (45%) or advanced structural levels (12%). These students were able to organise their data in a new way, distinct from the original data table: either by sorting (e.g., listing years as hottest to coldest), grouping the original data to create a new variable (e.g., determining hottest temperature for each month), or by focusing on a specific statistical exemplar, such as the median value. While some students listed these as abridged tables or lists, students in Grade 4 (39%) were often able to coordinate two sets of variables and create bar graphs (23%) or line graphs and scatterplots (16%). Only 7% of students in Grade 3 were able to do the same. The examples of structural and advanced structural representations in Figure 2 include structural elements such as approximate equal spacing, the range on the *y*-axis starting above zero, correct sequencing of months, and coordination of bivariate data.

Discussion

The aim of the study was to track the development of students' predictive reasoning capacities across one year. Students demonstrated important gains in development by Grade 4, evident through the reasonableness of their predictions, and examples of transnumeration and coordinate graphing in their representations. In order to accurately make predictions, students required an understanding of the variability in data, the capacity to reason logically about random events, and to appreciate associations between events (Bryant & Nunes, 2012). Consistent with Watson and Moritz (2001), who found increases over time in students' ability to represent, predict, and interpret pictograms, students' predictions in the current study progressed markedly between grade levels. By Grade 4, many students' explanations of their predictions reflected an understanding of multiple forms of variability (see Shaughnessy, 2007), for example, noticing extremes and outliers in temperatures, discussing changes over time, noticing variability in the table or monthly range; variability associated with seasonality, and awareness of distributions.

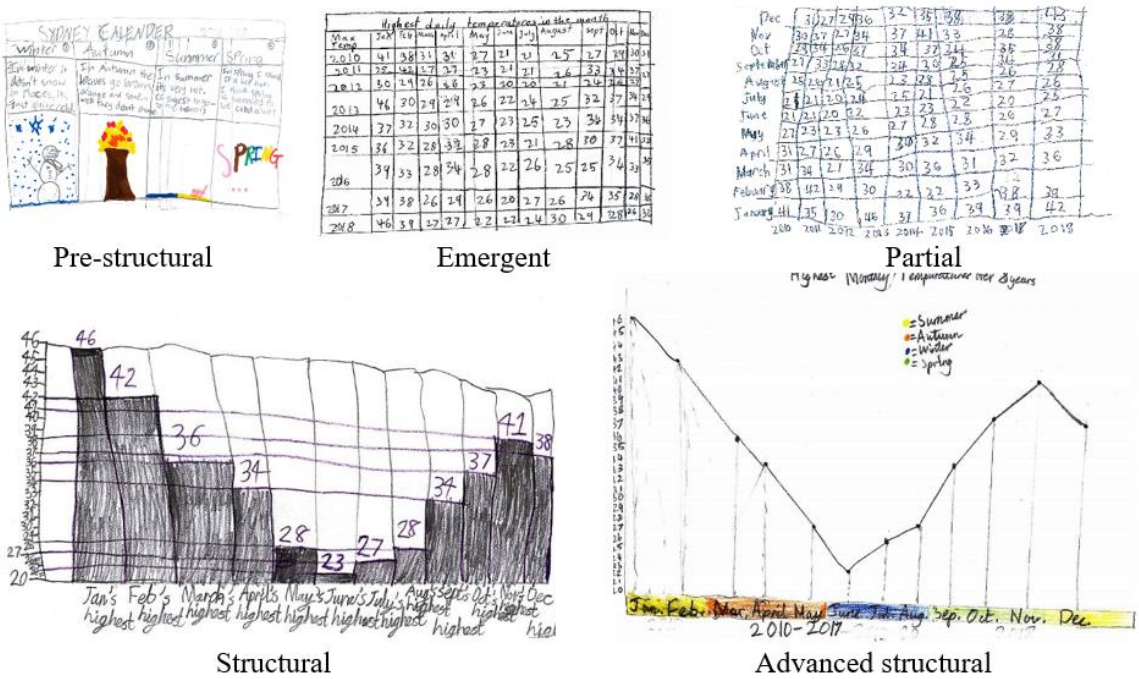

Figure 2. Student representations of Sydney maximum temperatures at five AMPS levels.

Coding with the AMPS conceptual framework highlighted the degree to which students identified statistically-meaningful patterns within the data. While previous research has shown that students with a high awareness of patterns are also likely to develop coherent mathematical concepts and relationships (Mulligan et al., 2020) and representations (Mulligan, 2015; Oslington et al., 2020), this longitudinal study is the first to apply the AMPS framework to statistical reasoning (also see Cycle 1 findings in Oslington et al., 2020). Awareness of mathematical pattern and structure was fundamental to students' perception and representation of the data set, enabling an understanding of how the variables of time and temperature could be related and organised, how students coordinated bivariate lent data, and an understanding of collinearity. For example, students in the cohort were

frequently able to identify the base-ten structure evident in the monthly columns, note the repeated data values, and identify variations and similarities in data range. Students who had structural or advanced structural AMPS levels demonstrated conceptual understanding of central tendency, variability and could abstract and generalise about relationships within the data, which are both crucial components of AMPS (Mulligan & Mitchelmore, 2009).

The data set chosen in this study reflected underlying causal variation due to seasonal change, which was apparent to students' reasoning at the structural and advanced structural levels. In each case, students were able to identify this causal pattern and use it in their own predictions and representations. However, the variables of time and temperature could also be organized in several ways, each giving a different grasp of the distribution (Gattuso & Ottaviani, 2011). If a student selected the temperature as the dependent variable, for example, the independent variable could be either month or year. Selection of month as the independent variable emphasised the relationship between months and temperatures, provided they were organised in calendar order. This made it easier for students to read beyond the data (Makar & Rubin, 2018) and make inferences about other years. For some students, drawing their own (structural) predictions in a figure or graph clarified this relationship in a way that simply viewing the historical data table could not. This became apparent at interview. Students who selected the year as the independent variable still demonstrated coordination of bivariate data, but their resulting graph lacked the relational component apparent when months were selected. This is because, at least on the timescale used, the data showed no clear trend across years and temperatures. This type of representation provided less opportunity for inference, as interpretation was limited to noticing (for example) the hottest or coldest year out of the limited range.

Limitations and directions for further study

This study illustrated development in the predictions, explanations, and representations of students at Grade 3 and 4. The study was nonetheless limited by focusing on a single aspect of predictive reasoning with one cohort. It also involved a repeated task and the impact of task familiarity upon the achievement of the students has not been explored. Caution is therefore required before such findings are generalised to other contexts. Notwithstanding these limitations, there are also recommendations for future research.

First, future research should investigate the interplay of students' prior knowledge, autobiographical memory, and data interpretation skills when making predictions. In the current study, students' explanations revealed gaps in their assumed semantic knowledge of relevant concepts such as the timing of winter, the importance of the values in a data table, and the repeated pattern of months. This was particularly true in Grade 3. For example, few students appreciated that the two dimensions of the data table actually represented one continuous data sequence, which formed a pattern of highs and lows repeated every 12 units. In contrast, references to students' prior experiences were common. The Grade 4 data collection occurred in February 2019, after an Australian summer widely promoted as having record-setting maximum temperatures. While predictive strategies described by the Grade 4 students were typically based upon either the data values, the seasons, or an integration of both season and data table contents, the discussion of temperature within the media and community may still have helped students to better understand what might count as a realistic temperature. Indeed, practice at making predictions in other contexts—such as literacy studies, problem-solving in mathematics, social science lessons on climate change and geography—may also have contributed to the growth in predictive reasoning capacity between the two grades.

Secondly, for more advanced and older students, further research should consider the optimal timing for introduction of formal statistical concepts. Several of the students already appear to be independently seeking central tendencies in the data. Research on the use of exploratory graphing as a tool for promoting a relational understanding of data sets would also provide guidance regarding appropriate statistical learning sequences.

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Comparative judgement and affect: A case study

Jennifer Palisse
The University of Melbourne
<jennifer.palisse@student.unimelb.edu.au>

Deborah King
The University of Melbourne
<dmking@unimelb.edu.au>

Mark MacLean
The University of British Columbia
<maclean@math.ubc.ca>

Comparative judgement is a relatively new way of facilitating peer-assessment where students are shown pairs of other students' work and judge which of the two is *better*. Literature on example-based learning suggests that students should be able to learn from comparative judgement. We present the case of one student, Josie, whose understanding of rational inequalities did not improve while assessing other students work comparatively. We argue that her self-explanation attempts were limited because comparative judgement created an environment that threatened her goal of understanding. Knowing the correct answer may have helped alleviate some of these issues.

Comparative judgement is increasingly being used as a digital peer-assessment tool where students evaluate, provide feedback, and rank pairs of other students' work. It involves showing pieces of work as pairs, side-by-side, where students select which of the two they feel is *better*. This relies on the fact that humans are more reliable at comparing two objects against a given quality, for example, which object is heaviest, than they are at making an absolute judgement, such as stating how heavy an object is in kilograms (Thurstone, 1927).

In the context of mathematics education, comparative judgement has been used successfully to assess problem solving (Jones et al., 2014), students' understanding of fractions (Jones & Wheadon, 2015), and proof comprehensions (Davies et al., 2020), as well as understanding of calculus (Jones et al., 2019, 2013). Each of these studies focused on establishing reliability, that is, to demonstrate that the rankings produced by comparative judgement agree with those that would have been produced using traditional marking methods. High inter-rater reliability has consistently been demonstrated.

Other studies have explored what type of construct comparative judgement might be measuring. For example, Jones et al. (2013) advocate for the potential of comparative judgement to measure what they term as conceptual understanding. In their study, teachers were asked to give an estimate of their middle school students' mathematical ability. Students were then given a task where they were asked to order a set of fractions from smallest to largest and explain their solution. Teachers then used comparative judgement to rank students' responses. Jones et al. found that teachers' estimates of mathematical ability were a better predictor of final rankings than task accuracy, and argue that comparative judgement measures something other than procedural understanding.

Given the large number of studies already available exploring issues of reliability and validity, this paper instead focuses on recent calls to explore the potential use of comparative judgement as a learning tool rather than an assessment tool (e.g., Strimel et al., 2020). As of yet, there has been minimal research looking at the role comparative judgement might play pedagogically, particularly in the context of mathematics. From the few studies that do exist, it seems that students both in the secondary school setting (Jones & Alcock, 2013) and university setting (Potter et al., 2017) find pairwise peer-assessing valuable and worthwhile.

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In the context of design, Bartholomew et al. (2019) have shown that comparative judgement can improve student outcomes, where students who participated in pairwise peer-assessment outperformed students who assessed their peers' work one at a time. This suggests that comparative judgement influences learning beyond the process of just giving and receiving feedback.

In this paper, we add to the limited knowledge of comparative judgement as a learning tool by presenting the case of one student, Josie, who did not find comparative judgement helpful. We present an analysis of Josie's experience to better understand what aspects of the comparative judgement process were helpful or otherwise. We suggest that the way in which we organised our comparative judgement activity may have limited Josie's willingness to engage in deep reflection, reducing the potential for learning to occur.

Before reviewing the case of Josie, we draw on the framework of learning from worked examples to provide some justification for why comparative judgement might be useful for students' learning.

Learning from Worked Examples

Learning from worked examples involves providing students with the following: the problem itself, the steps taken to reach a solution, and the final solution (Renkl, 2014). In a seminal study, Sweller and Cooper (1985) compared learning through worked examples and learning through problem solving in mathematics and found that learning through worked examples required less time to process, problems were solved faster, and students made less mathematical errors. Since then, several studies have replicated these findings (for an overview, see Renkl, 2014).

The reasoning behind why example-based learning is effective draws on *Cognitive Load Theory* (Sweller et al., 1998). Learning by practicing problems without the guidance of a worked example places heavy demands on working memory. To solve a given problem, learners must work on finding a set of steps that can lead them to the desired goal. Because such an approach imposes a heavy load on working memory it generally does not lead to learning. Studying worked examples reduces cognitive load. Learners no longer need to find a set of required steps themselves as these are included within the worked examples. Learners can then avoid searching their own prior knowledge for solution methods. This thereby reduces working memory leaving cognitive resources available for self-explanation (Sweller et al., 1998). This is important as only when self-explanation is encouraged, that is, when students explain the reasoning behind a solution to themselves, do worked examples appear to be effective (Chi et al., 1989; Renkl, 2014).

We hypothesise that comparative judgement, as with worked examples, may also facilitate learning, where the solutions students compare are analogous with worked examples used during example-based learning. One key difference is the emphasis on *comparing*, however, this too has supportive evidence for learning in the literature on example-based learning. Empirical studies have demonstrated that presenting worked examples side-by-side, rather than one at a time, consistently results in greater learning gains (Star & Rittle-Johnson, 2009). By having two worked examples available simultaneously, the learner is no longer required to hold a representation of the previous worked example active in their working memory in order to compare with the next example. The effect is to reduce cognitive load making learning more likely. To put it simply, in the words of Rittle-Johnson and Star (2009, p. 529) "Experts agree; comparison is good."

Lastly, comparative judgement includes both correct and incorrect examples. The framework of example-based learning suggests that providing students with both correct and

incorrect examples is more beneficial than providing correct solutions only since incorrect solutions help students recognise incorrect strategy choices by drawing attention to the feature of the problem that makes the strategy inappropriate (Booth et al., 2013).

Research Design

The results reported here form part of a larger pilot-study. As such, we limit our explanation of our research design and detail only aspects needed to understand this single case study.

Participants included first-year undergraduate students studying an entry level calculus course. Eight students participated in an individual problem-based interview. Students completed a pre-task that asked them to solve the rational inequality $\frac{x+1}{x-7} > 3$. No feedback was provided, and they were not told whether they had answered the problem correctly or incorrectly. The students were then shown six pairs of other students' work on the same problem. For each pair, students selected which of the two they thought was the better solution. What *better* meant was up to students to decide. Students decided based on a variety of factors including the choice of method, whether any mistakes had been made, and even the neatness of the handwriting. Students were not provided with a rubric, marking scheme, or correct answer to help inform their decisions. While assessing each solution, students were asked to think-aloud. The think-aloud method involves verbalising one's thoughts when first noticed and is seen as a valid way to capture individuals' working memory (Ericsson & Simon, 1993). Once students completed their pairwise judgements, they then completed a similar problem and solved $\frac{5x-2}{x+5} > 6$. This allowed for comparison between students' pre- and post-tasks to analyse any changes students might have made between tasks. A short semi-structured interview followed where we asked students to explain any changes they made to their solution technique between their pre- and post-tasks.

Analysis

From the literature, we anticipated that comparative judgement should be useful in improving students' understanding of rational inequalities. To measure this, we intended to compare students' pre- and post-tasks to see if students who had made errors in their pre-task had rectified their mistakes in their post-task. Surprisingly, for students who held misunderstandings during the pre-task, we found little evidence for improved understanding in the post-task.

Since we were unsure why there had not been any improvement our next steps were exploratory. We found students' think-aloud comments during comparative judgement enlightening and analysed students' comments using thematic analysis (Braun & Clarke, 2019). This involved a progressive process of systematically comparing and grouping segments of think-aloud data firstly into meaningful smaller groups, and later into broader themes. These themes were developed inductively from the data. We present two themes here: the frustrations caused by not knowing the correct answer, and the lack of willingness to interpret other students' solutions if knowledge from external, rather than internal, sources was expected.

Below is our analysis of one student, Josie, who had not found comparative judgement helpful. Josie represented an atypical case since she was able to complete both the pre- and post-tasks but was still unable to understand the different approaches used in the solutions presented to her. As such, Josie was the only student for whom there was no evidence to

suggest any improvement in understanding had resulted from the comparative judgement task. As Josie was forthcoming with her opinion, her case was useful in illustrating why comparative judgement may not be helpful for all students.

Results and Discussion: The Case of Josie

Background

Josie completed the pre-task by rearranging $\frac{x+1}{x-7} > 3$ as $\frac{x-11}{x-7} < 0$ and solving for when the numerator was positive and denominator negative, and vice versa. Unfortunately, she made a minor numerical error, leading to an incorrect final answer. This proved to be significant as her answer differed to most of the answers in solutions provided for the comparative judgement task. Josie noted while completing the pre-task that she had never understood how to solve rational inequalities and did not know why the same techniques used for solving equations could not be used to solve inequations. For example, Josie did not understand why you could not multiply both sides by $x - 7$ and why there would be "two answers". We refer to this type of approach as a two-cases approach.

Josie's goal of understanding and not knowing the correct answer

Josie was shown sample solutions from other students, presented in pairs, from which she was asked to choose which of the two solutions she felt was *better*. For Josie, the better solution tended to be the one with the correct answer.

> The thing I mostly go off is if the answer is right. If one was right, one was wrong, most of the time I picked the one that was right.

Not knowing for sure the correct final answer proved to be a major source of frustration for Josie as her strategy for choosing between the two solutions relied heavily on which solution was correct. This meant she was at risk of not making what she felt to be the correct choice between solutions. As a result, Josie did not trust her pairwise judgements.

> But if I actually knew the right answer, then I'd be like well that one's better because it's the right answer.

Having been denied access to the correct answer after multiple requests to the interviewer, Josie felt frustrated and as such, perceived it to be the interviewer's fault, not her own, that she was unable to make a correct pairwise choice. The above statement was said with a tone of irritation suggesting she felt a sense of pointlessness to the whole activity. From Josie's point of view, there was no point in engaging with the task if her pairwise choices were going to be wrong anyway. Without a means of resolving her uncertainties, the task had little perceived value.

This sense of pointlessness was again felt as Josie began working on the post-task. She believed her answer to the new problem to be incorrect and attributed this again to not having been provided with the correct answer to begin with:

> Now see if you'd told me how to solve this [pre-task] then I could probably solve this. So, I'm just going to be doing the wrong thing. [Huff] Is that right?

Josie continued working with a 'huff'. These statements carried a strong sense of exasperation and her final comment "Is that right?" was said sarcastically. By this late point in the activity, Josie was frustrated.

Josie's inability to resolve her uncertainties appeared to have affected her efforts in understanding the relevant mathematical concepts. In one example, Josie had tried to understand why the student had solved for two-cases but ended her reflection shortly:

> I'm just like sort of trying to tell myself what they're doing and why they're doing it. But I don't know if that's the right answer… [Moves onto next solution]

Here Josie made a sincere attempt to understand what the student had written but quickly gave up. Her comment about not knowing the right answer was said with a slight antagonistic tone directed towards the interviewer. From Josie's perspective, there was no point spending time interpreting the student's work if there was no means of checking the correctness of her interpretation.

In a second example, Josie tried to understand why a particular student's solution included two cases, noting two columns, one labelled $x - 7 > 0$ and $x - 7 < 0$. She quickly gave up, this time with self-deprecating comments:

> I'm just trying to work this out for my own benefit here and think, why are they doing the
>
> $x - 7 > 0$ [Points at two column headings]? [Silence] Honestly, I'm probably just too dense to get this. [Quickly starts evaluating next solution]

Josie had been silent for some time, suggesting she had been deep in thought and genuinely tried to work out why the student was solving for a second case. Additionally, her self-criticism was not uttered with her usual air of exasperation, but instead said with a more sombre tone, suggesting a sense of hopelessness or even sadness. We interpret this to mean that Josie really had tried to understand the need for a second case but felt unable to do so. Furthermore, the negative emotional response suggests Josie may have been experiencing fear as she perceived her goal of understanding to be at risk. As such, Josie may have felt her own reasoning was not a safe strategy (Sumpter, 2013) and rather than risk investing more time into resolving her confusion, she avoided this situation by quickly moving on to evaluating the next solution.

Together these instances paint a picture of a student who found comparative judgement frustrating primarily because she had no internal means available to resolve her uncertainties. This subsequently created an environment that repeatedly placed Josie's goals of wanting to understand the problem under threat. As a coping strategy, Josie often appealed to not having the correct answer as the cause of not being able to understand. By placing blame on external factors, that is, not having the answer, Josie was able to increase her own feelings of safety rather than risk relying on her own reasoning. As a result, Josie seemed to have viewed comparative judgement as somewhat pointless for learning. From Josie's perspective, what was the point of investing time and effort into understanding someone else's solution if her reasoning of what the student had written could not be verified? This provides some explanation into why comparative judgement was not helpful in this case: if there was no point spending time interpreting each worked solution, then self-explanations were unlikely to be generated. With no self-explanations, learning from worked examples was unlikely to be successful (Chi et al., 1989). As such, if comparative judgement is to be used with the educational goal of improving mathematical understanding, we recommend providing students with, at minimum, the correct answer before completing any pairwise judgements. We suspect had Josie known the correct answer with certainty, she might have been willing to spend more time reflecting and interpreting what other students had written, making learning more likely.

Josie's views on the role of the expert and usefulness of other students' solutions

In this section we discuss who Josie felt was responsible for explaining mathematical ideas so that she could understand them, and whether she felt comparative judgement was effective in improving her understanding.

Ultimately, Josie felt the responsibility for her understanding of the mathematics content lay with her lecturers – an external source of expertise:

> I guess I don't have as good an understanding of inequalities as I should. I guess you can't multiply across like that [multiply by $x - 7$ without considering both $x - 7 > 0$ and $x - 7 < 0$]. I don't know why you can't. That never really got explained to me.

Furthermore, she felt frustrated with her mathematics lecturers, who she felt were not providing adequate explanations:

> Cause there's a lot of stuff I don't understand, and the lecturers aren't really thorough with explaining it.

From Josie's point of view, the responsibility for generating understanding came from her lecturers. It was their role to provide a thorough enough explanation to generate understanding and her role to receive such knowledge. We suggest that by placing expectations of understanding on an external source, it increased Josie's feelings of psychological safety by removing her ownership of her own understanding. By claiming her lecturers had failed in providing her with suitable explanations, it may have resolved her experience of doubt by surrendering control to an external authority (Bendixen, 2002). From Josie's point of view, it was not her fault if she could not understand when the responsibility for understanding lay with her lecturers.

These sentiments were echoed as Josie explained why she felt comparative judgement had not been helpful for her understanding. Firstly, Josie acknowledged that comparative judgement had been useful in pointing out that her initial assignment had been flawed. However, it had not been useful for understanding why they were flawed:

> But yeah it definitely helped me figure out oh yeah, I need another answer. Stuff like that. But it didn't help me understand why.

Here Josie was referring to when she had completed a comparative judgement activity as part of a class assignment earlier in the semester. For her assignment, she had multiplied both sides of the inequality by the denominator but failed to consider when the denominator was either positive or negative, yielding the partial solution $x < 11$. By looking at the solutions of her peers, she realised that there was "another answer".

Josie continued explaining why comparative judgement might not have helped her understanding:

> Well there's either really smart kids that just go bang, bang, bang. That's the answer and they don't really explain it. I don't understand what you're doing [the smart kids]. Or they're just not getting the right answer in general.

Given it was likely Josie believed the role of her lecturers was to provide her with an explanation, it may be that as she read these solutions, she similarly felt that it was the responsibility of the student (as an external source of knowledge) to provide all necessary explanations. That is, it was not her role to spend time interpreting the solutions but rather the solutions should have been presented in a way that did the thinking for her. Because of the value Josie placed on figures of authority such as her lecturers, it was surprising to see that Josie did not value the work of the "really smart kids". This speaks to the interplay between Josie's views on the role of the expert and her goal of understanding. While Josie

viewed her role as the receiver of knowledge from those more knowledgeable than her, she simultaneously expected the knowledge giver to provide such knowledge in a way that was easy to understand. This meant that for Josie, comparative judgement failed to provide an environment conducive to learning as the solutions were not written in a clear and easy to understand manner.

Lastly, linking back with our previous theme, Josie implied that incorrect solutions were not useful in helping her mathematical understanding. Because of the emphasis she placed on knowing the correct final answer, it was unlikely that Josie thought she could learn from incorrect examples. However, even the correct worked solutions seemed equally unvalued as they did not include enough detail or explanation to help resolve Josie's knowledge gaps ("they don't really explain it"). This is true - none of the worked solutions in this study included much detail or written explanation as to exactly why two cases needed to be considered. We found this surprising given that research in the area of learning from worked examples suggests that learning from incorrect examples is often more beneficial to learning than correct examples only (Booth et al., 2013). Results here suggest that this may not carry into the context of comparative judgement. This might be because the comparative judgement solutions we used were not labelled as either correct or incorrect, whereas studies exploring incorrect worked examples typically label such solutions as incorrect. One avenue of future research could be to explore whether marking comparative judgement solutions as either correct or incorrect has any influence on how students interact with the written solutions.

Final Comments

For Josie, comparative judgement was useful in helping her notice that she had been incorrect, but not useful in improving her understanding of solving rational inequalities. We noted two reasons for this. Firstly, it seemed that comparative judgement threatened Josie's goals of understanding – Josie did not feel her own reasoning was a safe strategy and did not feel she had a way of resolving her uncertainties. Josie's coping strategy was to blame the interviewer for not providing her with the answer. As Hannula (2006, p. 169) states "students may decide not to pursue learning goals when they feel that one or more of their psychological needs are thwarted."

Secondly, learning from other students' solutions did not appear to align with Josie's expectations of where knowledge comes from. From Josie's perspective, it was the role of the worked solution to provide her with a clear and easy to understand explanation and her role to receive such knowledge. In short, she expected to understand each solution without needing to think and the solutions we included in this study lacked the type of explanation Josie expected.

What ties these themes together is the interplay between Josie's desire to understand but not wanting to take ownership of generating this understanding. This seems to have impacted her ability to act strategically, limiting the amount of time she spent reflecting on the underlying mathematics, making learning from the worked solutions less likely. As such, the way in which we set up our comparative judgement activity may not be appropriate for the purposes of improving understanding for students like Josie.

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A primary education mathematics initiative in an Indigenous community school

Bronwyn Reid O'Connor
The University of Sydney
<bronwyn.reidoconnor@sydney.edu.au>

This study investigated the implementation of a primary mathematics initiative in an Australian Indigenous community school designed to improve students' mathematical proficiency. Throughout the 7-month initiative, four classes ranging from Year 2 to Year 6 participated in the initiative. Findings indicated an increase in mathematics achievement as measured by the Progressive Achievement Test – Mathematics (PAT-M) throughout the initiative. As the findings from the Year 3/4 class displayed the largest growth in mathematical achievement throughout the initiative (equivalent to 1 year and 2 months), the specific pedagogies and practices enacted in this class that were found to influence students' achievement will be considered.

The gap in achievement between Indigenous and non-Indigenous students in Australia is regularly referred to in literature, and there has been little change in this gap over time despite several initiatives, both government and non-government, attempting to address these reported gaps (e.g., Dreise & Thomson, 2014; Thomson et al., 2016). Though the Council of Australian Governments (COAG) goal to halve gaps in numeracy achievement for Indigenous students by 2018 has not been realised (Department of the Prime Minister and Cabinet, 2018), this paper will discuss a case where the gap between Indigenous students' and non-Indigenous students, as measured by PAT-M, had been successfully reduced in a sample within one Australian Indigenous community school.

Literature Review

A substantial body of research in Indigenous education focused on achievement has been established in recent decades and continues to be of importance due to equity implications (Hunter & Schwab, 2003). Beyond conforming ideals of creating capable citizens, whereby mathematical knowledge enables students to understand how the world works, those who develop mathematical knowledge potentially have the capacity to also create the world in a new way (Atweh & Brady, 2009). Due to equity concerns and the importance of mathematical knowledge, research exploring effective education practices in Indigenous education is important as "future Indigenous education policy decisions must be based upon real research findings, and where these findings necessitate policy action, those actions must be taken" (Mellor & Corrigan, 2004, p. iv). These sentiments have also been re-iterated in reports on Indigenous primary school achievement by the Productivity Commission (2016).

Despite several programs focused on Indigenous mathematics education currently operating across Australia, no consistent improvement has been realised according to current large-scale, standardised measures of achievement (e.g., the National Assessment Program – Literacy and Numeracy [NAPLAN], Programme for International Student Assessment [PISA], and Trends in International Mathematics and Science Study [TIMSS]). Therefore, it is important to continue investigating the impact of specific educational practices to further understand underlying reasons for these gaps and to focus on identifying and supporting teachers in implementing successful practices. This study is significant for its contribution to our understanding of these issues, and the findings reported focus on the practices enacted

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by teachers, supported by the researcher in the role of a mentor, that resulted in significant positive change in students' mathematics achievement as measured by the PAT-M.

A summary of seminal literature regarding effective teaching of mathematics to disadvantaged students is outlined in Table 1. These practices were also supported by common findings from research concerning effective numeracy development for Indigenous learners through the What Works program developed by the Australian Government, Department of Education, Science and Training (DEST, 2010). This literature guided the conceptualisation of the mathematics teaching initiative in the initial stages of development. The intention of the review in the development of the initiative was to establish a mathematics program that was structured in a manner that provided appropriate opportunities to thoughtfully develop foundational mathematics concepts.

Table 1
Effective pedagogies informed by literature

Initiative elements	Components of effective mathematics education from literature
Developing the initiative	High levels of teacher collaboration and a shared, school-wide approach (Boaler & Staples, 2008; Jorgensen, 2018).
Lesson elements	Explicit teaching of new mathematical concepts (Baker et al., 2002; Good & Grouws, 1979; Hattie, 2009; Jorgensen, 2018; Pegg & Graham, 2013).
	The central role of feedback, & providing feedback data to students (Baker et al., 2002; Hattie & Clarke, 2019).
	A mastery learning cycle (Hattie, 2009; Jorgensen, 2018; Kulik et al., 1990; Pegg & Graham, 2013).
	High mathematical expectations (Jorgensen, 2018).
	Focus on number (place value and operations) as a priority (Jorgensen, 2018).
Teaching number facts	Explicit strategy instruction in conjunction with timed tasks (Cumming & Elkins, 1999; Pegg & Graham, 2013).
Teaching computations/algorithms	CRA teaching sequence, and consistent language and methods for teaching algorithms (Mancl et al., 2012).
Teaching problem-solving	Utilisation of Polya's problem solving heuristics (Ozsoy & Ataman, 2017)

One of the dominant practices focused on in this initiative was explicit instruction to conceptually develop students' deep understanding of key mathematics concepts. Explicit instruction is "a systematic method of teaching with emphasis on proceeding in small steps, checking for student understanding, and achieving active and successful participation by all students" (Rosenshine, 1987, p. 34). Supported by evidence from meta-analyses and other studies (Hattie, 2009), explicit instruction practices have been shown to significantly increase students' achievement. Often explicit instruction (sometimes referred to as direct instruction) is criticised due to its presumed association with a didactic teacher instructional model. To clarify, Hattie (2009) provided a list of seven elements of effective explicit instruction which consisted of clear lesson goals (learning intentions) and success criteria, student engagement, lesson structures designed to accommodate modelling and checking for understanding, guided practice, independent practice, and effective lesson closures. Many of these practices align with key practices noted in other projects involving Indigenous students (e.g., Jorgensen, 2018).

Two research questions were answered in this study by observing how teachers employed the suggested practices and by tracking changes in students' achievement on the PAT-M. The first research question was: *How did students' mathematical achievement, as*

measured by the PAT-M, change as a result of the initiative? And *how did teachers implement a mathematics initiative in an Indigenous community school?*

Method

The Sample: Study context

The students in this study were of Aboriginal and Torres Strait Islander heritage, which encompasses a great diversity of people. The local context of these students was an urban community (city) in Australia. The context of the study school is a F-12 community operated school (under the banner of an Australian Independent school, which is government funded but run by a community identified school board) for Indigenous students. This study involved four composite age classes, comprising of students from Year 2 to 6 (Year 2/3, Year 3/4, Year 4/5, and Year 5/6), four individual classroom teachers, and 57 students. The findings from 11 students in the Year 3/4 cohort (out of a total class number of ~20-25 students) are reported. The final sample is smaller than the total population of the class due to students leaving the school throughout the initiative resulting in incomplete data for some students, and movement of students between classes meaning that achievement could not be tracked and attributed to the teaching and learning occurring in a single classroom. The total class numbers fluctuated in the outlined range throughout the school year due to student movement in and out of the school, and within the school. The teacher for Year 3/4, Diane (a pseudonym), was a practising teacher for approximately 15 years who had been teaching at the sample school for 12 years.

Implementation of the initiative

The mathematics initiative in this study was conducted over 7-months from March to October of the school year. The initiative was implemented through professional development sessions delivered by the researcher. The first of these sessions was run at the beginning of the initiative and involved dissemination of the summary of literature findings to teachers. The researcher then maintained an interactive role throughout the initiative by providing support to teachers when planning and implementing their mathematics programs. The interactive nature of the research was facilitated by the researchers established role as a teacher in the school prior to the initiative. The prolonged engagement of the researcher in the school helped to increase the credibility of findings, which worked to ensure the validity of conclusions from findings (Guba & Lincoln, 1985). Also, due to the researcher's pre-established role in the school, the context and culture of the school were well understood, and rapport was established with students and staff, increasing the credibility of findings by providing cultural sensitivity (Gay et al., 2006). Further professional development sessions were facilitated part-way through the initiative during school hours to provide interim analysis of class data; this was conducted in the interest of ensuring the collected data supported the school's mathematics programs in a meaningful way.

Overview of Methodology

The research design for this study was mixed methods, typified by the collection and interpretation of quantitative data, followed by qualitative data to provide breadth and depth of data to explain a complex phenomenon (Cohen et al., 2011). To address the first research question, changes to students' proficiency throughout the initiative was measured using a standardised mathematics test (PAT-M). To address the second research question, rich qualitative data in the form of classroom observations were collected throughout the initiative. The quantitative findings relating to changes in students' achievement were then

connected to and explained by the qualitative classroom observations. Interpretations on how teaching practices impacted on students' mathematics achievement were then made, following a sequential mixed methods design (Creswell & Plano Clark, 2018). Though these findings are not reported within the scope of this paper, this study was situated within a larger research study which involved the collection of further data on students' mathematical proficiency (Reid O'Connor, 2020; Reid O'Connor & Norton, 2020). The use of multiple data sources and triangulation of data in this mixed methods study helped work towards increasing the trustworthiness and confirmability of findings (Guba & Lincoln, 1985).

Data analysis: Progressive Achievement Test – Mathematics

Changes to students' mathematical proficiency were measured by administering the standardised PAT-M at the beginning and conclusion of the initiative. The PAT-M consists of 30-40 multiple choice items across number, algebra, geometry, measurement, statistics, and probability. The PAT-M provides a measure of students' skill and understanding of school mathematics (Stephanou & Lindsey, 2013). The intended use of these tests aligned with this study, which aimed to assess students' current achievement, monitor the impact of an initiative over time, and to inform the development of the initiative. Utilising a standardised test also provided a method for comparing class mean scores to the national norming sample of over 500,000 students. Cohen effect sizes were also calculated to compare the initiative to the national norming sample. Effect sizes are an empirical measure that answers the question "how does the effect of an initiative compare to a typical year of growth for a given target population of students" (Hill et al., 2008, p. 173). Cohen (1988) classified effect sizes of 0.20 as small, 0.50 as medium, and 0.80 as large; other literature has found 0.40 to be the average effect size expected during a school year (Hattie, 2009). For more meaningful comparisons, the effect sizes from the norming sample data were also calculated.

Data analysis: Classroom observations

Classroom observations were carried out throughout the initiative and provided important data to identify how and why classroom practices were influencing students' achievement. Classroom observations focused on qualitatively recording: (1) lesson structures, (2) pedagogical approaches, (3) frequency and type of teacher interaction with students, and (4) student time on task. Approximately two mathematics lessons were observed in each class during each week of the initiative. Observations were recorded by the researcher as field notes in the form of diary entries throughout the initiative. The intent on such a method was to be able to tell the story of what happened in classrooms to allow for teaching practices to be described in detail and linked to student achievement. Observations of the classes during mathematics lessons also allowed for identification of other relevant occurrences in students' learning environments that may have impacted on teaching and learning. Summaries of teaching practices were member checked with teachers at the conclusion of the initiative to help ensure the authenticity and dependability of the data and analysis, increasing the validity of findings (Guba & Lincoln, 1985).

These qualitative observations were analysed by the creation of case reports for each class, and Wellington's (2015) stages for interpreting qualitative data were utilised. These stages consisted of immersion in the data and reflection, analysing and taking apart the data, and recombining and synthesising the data. Patterns in teaching practices across observations were identified relating to the four observed elements in lessons. These trends were then reduced by searching for overarching themes; for example, patterns relating to

classroom routines and daily activities that fostered student's independence were grouped under an overarching theme of *consistency*. These themes were then located with reference to the literature (Wellington, 2015), to propose links between teaching practice and student achievement.

Results: PAT-M

The mean PAT-M score and effect sizes for the Year 3/4 class comparative to the national norming sample is outlined in Table 2. The comparative mean scores reported for the norming sample are what would be typically expected of a Year 3 student in March and October respectively.

At the beginning of the initiative, the mean score reported by the Year 3/4 class was below the norming sample mean. The mean score of 96 reported by Year 3/4 was equivalent to an achievement standard of an early Year 2 student when compared to the norming sample. By the end of the initiative, the Year 3/4 class reported positive gains in mean score, and closed the gap in achievement comparative to the norming sample. The Year 3/4 post-initiative mean score of 109 was indicative of an achievement standard similar to that of a mid-Year 3 student. Comparing the effect size found for Year 3/4 to Hattie's (2009) classification, the large effect size of 1.36 was above the zone of desired effects for an educational intervention.

Table 2
Results from PAT-M pre- and post-initiative for Year 3/4

	M (Mean Rasch Scaled Score)		Effect size
	Pre-initiative: March	Post-initiative: October	
Year 3/4, n=11	96.57	109.88	1.36
PAT-M norming sample	106.29 (Yr3)	110.90 (Yr3)	0.32
Gap in achievement	-9.72	-1.02	

Overall, the gains in achievement reported by the Year 3/4 cohort throughout the initiative equated to an improvement of approximately 1 year and 2 months within the 7-month initiative, twice the expected gain. Figure 1 graphically outlines the change in scores throughout the initiative.

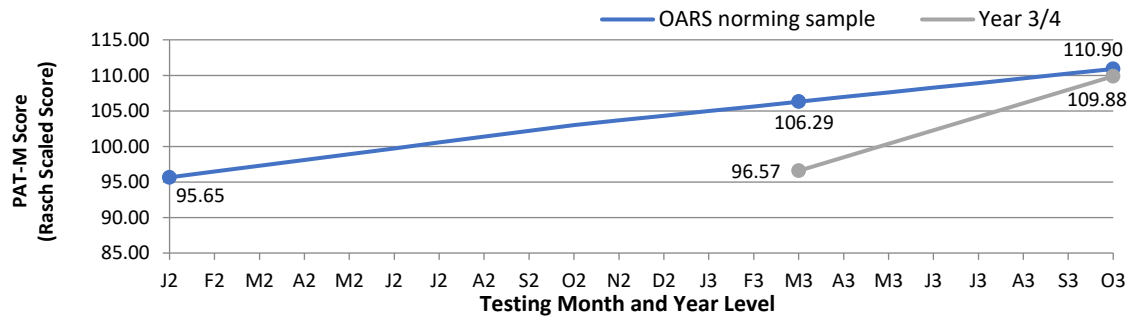

Figure 1. Graph of the growth in PAT-M achievement for Year 3/4 compared to the norming sample, n=11.

The Year 3/4 cohort experienced a more substantial improvement in a shorter timeframe than the national norming sample. The significant gains in achievement, and subsequent closing of the achievement gap, and high effect size indicated that the initiative was highly effective in advancing students' learning in this class. Exploring the pedagogical practices enacted by the classroom teacher assists in understanding why the initiative was effective in advancing students' mathematics achievement in this case.

Results: Classroom Observations

The teaching approach of the Year 3/4 teacher, Diane, involved mathematics lessons that followed a consistent daily structure, with the intention that students in the class would also learn the structure and become self-sufficient with lesson routines (i.e., students knew what activities were coming next and could get appropriately prepared without added teacher instruction). When asked to comment on her specific strength in teaching mathematics, Diane self-identified that her strength was implementing consistent routines by noting "I stick to routine quite stringently…I think that is a strength because the children know what the expectation is for the day, for the week".

In addition, Diane focused on maintaining high behavioural standards during classes as a priority, and subsequently was observed to consistently minimise behavioural interruptions that may have occurred without her supervision and guidance; this was achieved by Diane establishing a trusting and respectful rapport with students by maintaining firm, consistent, and reasonable expectations regarding classroom behaviour.

High academic standards were also a major feature of Diane's teaching approach. On a daily basis Diane constantly consulted with students one-on-one during mathematics lessons by marking students' work and giving feedback, with the expectation that students would then immediately return to independently fix any errors. This consultation and feedback process formed a continuous cycle. Feedback included supporting the students in identifying their error/s, followed by discussion and explanation of correct techniques or strategies. What Diane was observed to have implemented was a method of maintaining high academic standards through short diagnostic teaching cycles where a small number of tasks (questions or problems) were set and completed by students then checked by a teacher, and where accuracy of answers was required. This cycle is outlined in Figure 2.

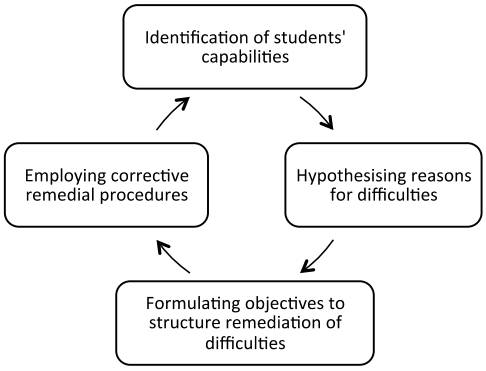

Figure 2. Description of diagnostic teaching cycle employed in Year 3/4 (developed from Reisman, 1982).

The way in which Diane influenced students' achievement in mathematics through the combination of consistent lesson structures, high behavioural standards, and high academic standards was observed to result in high levels of time on task for students in Year 3/4. The continual checking of student work and continuous feedback cycle was observed to foster a learning environment where most students strived for accuracy.

Diane's pedagogic approach focused on explicit individual and whole class instruction, and group work was not a feature of instruction. In terms of the quantity of work set, Diane set small quantities of work (i.e., questions) for students to complete as part of her diagnostic teaching cycle. Diane followed recommendations concerning the need to conceptually develop ideas at the introductory phase of learning by following the recommended, consistent pedagogy and language for teaching algorithms, however Diane focused on clear

strategy discussion in mathematics. On this, Diane noted that "I always try and teach my kids other ways of doing things. There's not just one way".

Discussion

Results from this study indicated that, as measured by the PAT-M, students in the reported cohort substantially increased in their mathematical proficiency over the course of the initiative as illustrated by high effect sizes and the closing of the gap between the Indigenous students and the norming sample comparisons. Pedagogical practices supporting consistency (predictable lesson structures fostering students' self-sufficiency), feedback (in the form of a diagnostic teaching cycle), and high expectations (relating to both behaviour and academic expectations) were critical features of the teaching approach in mathematics in Year 3/4.

One potential explanation for practices associated with consistency and high expectations underpinning positive gains in students' achievement is the outcome of increased academic learning time. Research in Indigenous settings, including the Success in Remote Indigenous Contexts project (Jorgensen, 2018), has proposed that consistent lesson structures reduce student confusion, which subsequently enables students to focus on the tasks rather than guessing teacher or classroom expectations. This finding was also supported in this study as a known and consistent lesson structure fostered students' independence in their learning and reduced many classroom or behaviour-related disruptions related to students being off-task or needing to ask what was required of them. The result is that the total time students spend on task is increased.

The diagnostic teaching cycle was a central element in successfully implementing and maintaining high academic expectations within the classroom. The continuous feedback cycle fosters a mastery teaching approach which is supported by empirical research (e.g., Good & Grouws, 1979; Hattie, 2009; Hattie & Clarke, 2019; Jorgensen, 2018; Kulik et al., 1990; Pegg & Graham, 2013). This diagnostic, mastery approach increased students' experiences with success on mathematical tasks in the observed class. Feedback through the diagnostic cycle achieves a meaningful and practical way for the teacher to ascertain what students understand and what mistakes they are making, and through this process teaching and instruction can be accurately tailored. By doing this, teaching is never "missing the mark" of where students are at in their learning process, and instruction can be tailored to ensure it is highly relevant and appropriate to students' ability and learning needs.

Overall, the enacted practices in this class that were supported by the initial literature review including consistency, mastery teaching approaches, feedback, and high expectations were successful in raising Indigenous students' mathematical achievement throughout this initiative.

Conclusions

The positive findings from this study relating to Indigenous students' achievement is an important contribution to literature, particularly due to the wealth of deficit-based findings that are currently reported in the field related to large-scale standardised testing. Whilst the findings of the larger study from this initiative indicated that the factors relating to Indigenous students' mathematical proficiency in this setting were complex and inter-related, with several affective factors playing a critical role (Reid O'Connor, 2020), the conclusion from the initiative's findings were that positively influencing Indigenous students' achievement is a worthwhile and feasible endeavour. Specific practices supported by the findings that positively influenced students' mathematical proficiency included consistent lesson structures, short and targeted diagnostic teaching cycles featuring high

levels of feedback, a mastery approach, and high expectations within a framework of classroom management that worked to maximise learning time. Further studies are needed to explore the impact of these practices in other Indigenous school settings, and the limitations of this study include the small sample size.

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Computer based mathematics assessment: Is it the panacea?

Angela Rogers
Royal Melbourne Institute of Technology
<angela.rogers@rmit.edu.au>

Test developers are continually exploring the possibilities Computer Based Assessment (CBA) offers the Mathematics domain. This paper describes the trial of the Place Value Assessment Tool (PVAT) and its online equivalent, the PVAT-O. Both tests were administered using a counterbalanced research design to 253 Year 3-6 students across nine classes at a primary school in Melbourne. The findings show while both forms are valid and comparable, the online mode was preferred by teachers. The affordances and constraints of using CBA in the formative assessment process are explored.

Over the past 10 years there has been a rapid uptake of Mathematics *Computer Based Assessments* (CBA) in Australian primary schools. Commercial firms have identified teachers as eager consumers in this market. Companies are acutely aware of the friction points for teachers: the challenges around creating their own formative assessments and time-consuming marking. This has led to the development of several increasingly popular CBA formative assessment "programs". Yet, for these programs be the panacea their advertising suggests, schools must be confident they provide valid formative data teachers can easily interpret and apply.

Currently in Australia, there are very few comprehensive formative whole number place value assessments for Years 3-6 students. To address this, a Rasch analysis-based methodology was used to develop a valid and reliable whole number place value paper-and-pen assessment, called the *Place Value Assessment Tool* (PVAT) (see Rogers, 2014). While the PVAT provided a detailed picture of student knowledge in the construct, the time taken to mark (5-7 minutes per student) was seen as a potential obstacle for teachers. To address this, the researcher investigated if a comparable online version of the test could be created.

Relevant Literature

Place value knowledge has been compared to the framework of a house, such that if a student's knowledge in this area is shaky, his/her understanding of mathematics as a whole is affected (Major, 2011). An understanding of place value has been shown to be closely related to students' sense of number (McIntosh et al., 1992), understanding of decimals (Moloney & Stacey, 1997), and comprehension of multi-digit operations (Fuson, 1990). Underpinning almost every aspect of the mathematics curriculum, it is an integral part of the primary school syllabus. Yet there is considerable evidence to suggest students struggle with whole number place value well into lower secondary school (Thomas, 2004; Wade et al., 2013). Research has shown that place value is often taught superficially, something that can be attributed to the lack of quality formative assessments available in this construct (Major, 2011; Rogers, 2014).

An assessment is essentially a sample of selected tasks intended to allow inferences to be made about a student's level of achievement. The strength of these inferences relies heavily on the quality of the tasks used (Izard, 2002). An assessment which includes a selection of items that are too easy, or too difficult, will not provide teachers with a complete picture of each student's knowledge. Similarly, an assessment that does not comprehensively cover the required content may cause the omitted content to be devalued by teachers (Webb,

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2007). In both cases, the inaccurate inferences drawn from these assessments, adversely influence the quality of instruction. Formative assessment is a process that provides teachers with information that can be used to support individual student's future learning (Popham, 2018). It is one of the most effective, empirically proven, processes that teachers can use to improve student performance.

An important consideration when developing assessments is practicality (Masters & Forster, 1996). If an assessment instrument does not justify the time or money required for its administration and marking, it will not be implemented by schools. Doig (2011) noted that some educators (despite appreciating the quality of data they received) avoided using interview-based assessments simply because of their administration time. As a result, many schools consider paper-and-pen tests a more practical assessment option, particularly with older students. Proponents of interview-based assessments disagree, stating clinical interviews provide higher quality assessment information and enhance teacher knowledge of common misconceptions in mathematics (Clements & Ellerton, 1995). While mathematics assessments have traditionally been delivered via paper-and-pen or interview (Griffin et al., 2012), the accessibility of technology has seen test developers investigate the many opportunities provided by CBA (ACARA, 2021).

CBA's major advantage is it delivers traditional assessment in a more efficient and effective manner (Bridgeman, 2009). CBA has the potential to save teachers time marking test papers and means results can be used to guide instruction in a timelier manner (Tomasik et al., 2018). Yet, as Thompson and Weiss (20011) explain, many school's technological capabilities fail the standard required to successfully implement CBA, leading to test administration problems (McGowan, 2019). Thus, while CBA has great potential in schools, further logistical work is required to ensure its success.

Much research associated with CBA has explored the comparison of traditional paper-and-pen based tests with their CBA equivalent (e.g., Wang et al., 2007; Thompson & Weiss, 2011). Wang et al. (2007) conducted a meta-analysis of 44 mathematics-based assessments comparing paper-and-pen and CBA versions of the same test. Overall, they reported that the mode of administration did not have a substantive effect on the students' performance (ES = -0.059). These comparisons aimed to determine whether online and paper versions of the same test could be used interchangeably. This is an important practical consideration, as comparable tests allow schools the flexibility to choose the most appropriate mode for their context. Yet, as Popham (2018) suggests, the decisions around test selection rely heavily on the assessment literacy of teachers and school leaders.

Popham (2018) defines assessment literacy as an "individual's understanding of the fundamental assessment concepts and procedures deemed likely to influence educational decisions" (p.13). An assessment literate teacher makes informed choices around the assessments they use, and accurately applies the results to guide their instruction. Research has shown that assessment literacy is not usually a focus of teacher education, meaning most teachers have poor levels (Stiggins, 2006). While providing teachers with assessment literacy professional development has been shown to be effective (Xu & Brown, 2016), without access to this, teachers are left to develop these skills 'on the job'. As CBA is a relatively new mode of assessment, it is realistic to assume that teachers need support to develop their assessment literacy in this mode. As Popham (2008) points out, being provided with assessment data is only the beginning of the process – teachers need the assessment literacy skills to understand a test's construction so they can successfully *interpret* the data.

One proven method of test construction is *Item Response Modelling* (IRM), which has well-established methods for analysis (Wright & Masters, 1982). IRM measures the

relationship between student achievement and item difficulty on the same scale (Wright & Stone, 1979). IRM has been successfully applied to a variety of test modes and used in large-scale assessments through to high-quality classroom-based assessment tools including PAT-M (Australian Council for Educational Research, 2012) and the *Scaffolding Numeracy in the Middle Years* (SNMY) assessment (Siemon et al., 2006). A popular IRM model, devised by Rasch (1960) is used in this research. Rasch analysis is based around the interplay of candidates and items in an assessment. While analysis of assessments traditionally generates a score that summarises the number of items correctly answered by students, Rasch considers the students who correctly answered each item (Izard, 2004). Rasch examines the extent to which the item distinguishes between those who are more and less knowledgeable (Izard et al., 2003). That is, the model assumes that less knowledgeable students have lower probability of answering a difficult item compared with those who are more knowledgeable (Rasch, 1960). Items that are considered not to follow this pattern do not fit the Rasch model and are generally removed from a test. This process verifies that the test content is meaningful and appropriate so that useful inferences can be made about the knowledge of candidates (Izard et al., 2003). Rasch allows different tests to be located on the same scale and allows test designers to determine if they are of comparable difficulty. The next section describes how quantitative Rasch based methods were used to compare the PVAT and PVAT-O, and the qualitative methods used to gather insights from teachers.

Methodology

PVAT-O Creation

Multiple technologies including *HyperText Markup Language* (HTML5), *Javascript*, and *PHP: Hypertext Preprocessor* (PHP) were used to create the PVAT-O assessment. The mathematical content and format of each PVAT-O item was as close as possible to the equivalent PVAT items. However, some items required the inclusion of computer-based features. For example, a ‘drag and drop’ feature was used in items requiring students to place numbers in order from smallest to largest and ‘radio buttons’ were used in multiple choice items.

The Counterbalanced Trial

The online and paper and pen PVAT trial was conducted at School C, a Catholic Primary school in metropolitan Melbourne where approximately 11% of students were from English as an Additional Language or Dialect (EAL/D) families (ACARA, 2020). All Year 3 to 6 students (N = 253) from nine classes took part in the trial (Male= 47%, Female= 53%). The trial took place over a 2-week period in the school library and was supervised by both the researcher and the classroom teacher. The trial was conducted using a counterbalanced measures design (Shuttleworth, 2009). Half of the students in each class (randomly selected) completed the PVAT-O, whilst the other half of the class completed the paper-and-pen PVAT. Exactly one week later, the students completed the alternate version of the test. This research design was used to minimise factors such as learning effects and order of treatment, adversely influencing the results of the trials (Perlini et al., 1998). Only 227 students (Male= 45%, Female= 55%) completed both forms, due to absences and technical issues.

Teacher Surveys

A short survey was given to the nine Year 3 to 6 classroom teachers (Female=100%). The purpose of this survey was to gain an indication of the teacher’s preferred testing mode.

When the survey occurred, the teachers had not yet received their student's results from the PVAT-O database, but it was explained they would receive each student's raw score in a spreadsheet. Due to the small sample size, the survey data was interpreted by the researcher and reported as individual responses (Neuman, 2006).

Rasch Analysis

The paper PVAT tests were scored and coded by the researcher. The PVAT-O was scored by the PVAT-O database and then rechecked by the researcher to ensure consistency and accuracy. In order to determine if the PVAT and PVAT-O could be considered valid tests and comparable in their mean item difficulty and mean student achievement (Kolen & Brennan, 2004), three Rasch analyses were conducted:

- *Run A* was conducted to re-confirm that the paper-and-pen PVAT was a valid and reliable test. The items which fit the model were used to create an anchor file for Run C. This allowed the PVAT and PVAT-O items to be placed on the same scale.
- *Run B* looked at the PVAT-O items in isolation. Rasch analysis was used to determine which PVAT-O items fit the model and determine if it was an internally consistent test.
- *Run C* investigated if the PVAT and PVAT-O could be placed on the same uni-dimensional scale and thus determine if they were comparable in item difficulty and student achievement.

The anchor file from Run A was used to fix the difficulty estimates of the PVAT items that fit the model. This allowed the PVAT-O items to be calibrated against the PVAT items (Izard, 2005). The mean item difficulty and mean student achievement for the PVAT and PVAT-O was then calculated from this run. Effect Size measures were used to quantify the standardised mean difference between the two tests (Izard, 2004). Cohen's (1969) descriptors for the magnitude of Effect Sizes, alongside the assigned ranges for each descriptor as suggested by Izard (2004), were then be used to describe the Effect Sizes in plain language.

Results

Rasch Validation and Comparison

The mean and standard deviation of the PVAT (n = 65) and PVAT-O (n = 59) items which fit the model in Run C were calculated to determine if the PVAT and PVAT-O could be considered comparable tests. The Effect Size measure was calculated to be 0.14, while the difference in student achievement between the tests was 0.01. This is described to be a "very small (0.00 to 0.14)" (Izard, 2004, p. 8) magnitude of Effect Size. This suggests there was not a substantive difference between the mean of item difficulties in the two modes, nor the students' achievement (which is to be expected, given the tests were of similar difficulty).

Teacher Survey

The class teachers (N = 9) at School C completed a brief survey asking them to indicate their preferred mode of administration for the PVAT. Seven teachers preferred the PVAT-O, while two preferred the PVAT. The seven teachers who preferred the PVAT-O stated:

> 'It will save correcting it' (Teacher #1,#2,#3)
>
> 'The results are immediate, I can use them the next day in my teaching' (Teacher #4)
>
> 'If the computers all work, online is much better' (Teacher #5)
>
> 'I don't have to correct it…and I can use the results tomorrow' (Teacher #6)
>
> 'The corrections would save me a lot of time and effort' (Teacher #7)

The two teachers who indicated they preferred the PVAT mode stated:

> 'Correcting them myself gives me a sense of their understanding' (Teacher #8)
>
> 'I'm always concerned students will lose their responses' (Teacher #9)

The small sample of teachers completing this survey limits the inferences that can be made from the data. However, within this group of teachers there was a clear preference for the PVAT-O mode of test administration, largely due to marking time it saved.

Discussion

Formative mathematics CBA continues to be embraced by schools, teachers and test developers. This research highlights several considerations when implementing formative CBA in classrooms: transparency, rigor, flexibility, and assessment literacy.

Transparency

While teachers in this research project were provided access to both the paper and CBA version of the test, this is not always the case. For example, in *Computer Adaptive Tests* (CAT) (Martin & Lazendic, 2018) each child is provided with a different set of items according to their responses. It is impossible for a teacher to view the combination of items individual students encounter, thus they are unable to judge their quality, appropriateness and relevance. Without this transparency, teachers are outsourcing the judgement of student knowledge to test designers. While somewhat appropriate in summative situations, eliminating teacher judgement in the formative assessment process should raise concerns for schools. Teacher #8 at School C echoed this 'transparency' constraint, indicating she was concerned about missing important diagnostic information in the PVAT-O. In response to this, the database was later adjusted to ensure teachers were provided with a summary of student responses to each item. The *Specific Mathematics Assessments that Reveal Thinking* (SMART) tests (University of Melbourne, 2012), are another platform that recognises the importance of allowing teachers to 'see' common student errors in the CBA mode. Doig (2011) reiterates this concern, noting that 'off site marking' does little to assist teachers to develop their knowledge of common student errors and misconceptions. Providing teachers with an overall raw score, rather than access to individual responses, is a major constraint of formative CBA and an issue which needs to be addressed by test designers.

Rigor of the Assessment

Wiliam (2007) states that formative assessment can effectively double the speed of student learning. Yet, as often happens in education, approaches can become diluted when commercial firms become involved. In order for schools and teachers to make informed decisions about the worth of formative CBA programs (particularly those produced commercially), it is critical teachers understand how to evaluate the rigor of a test's construction. The results presented in this paper use Rasch analysis to show both the PVAT and the PVAT-O are valid and reliable tests. For schools, this is essential information as it means the test has been empirically proven and robustly constructed. While the relatively small sample size gathered from only one school limits the scope of conclusions that can be made from this trial, very little difference was detected between the mean difficulties and student achievement of test items. Similarly, the student achievement was found to be comparable. This supports the results of the meta-analysis conducted by Wang et al. (2007), which noted that the mode of administration did not have a substantive effect on student achievement in computer-based and paper-based mathematics assessments. As Popham

(2018) suggests, schools should be encouraged to contact test developers, ask for a test's technical guide, and gather information related to the trialing, reliability and validity so they can make informed decisions about the suitability and rigor of tests.

Flexibility

Providing teachers with access to a comprehensive formative place value assessment that can be administered in two modes is considered to increase the usability and practicality of the PVAT tool. The PVAT-O was designed to support teachers by providing instant feedback on their students' achievement and save them considerable time. As the online and paper PVAT tests were found to be comparable, teachers are now able to choose the mode which works best for them and their students. This flexibility is useful, as not all schools have the technological requirements to successfully implement CBA. As Csapo et al. (2012) note, at a minimum, a school requires the capacity to allow students completing the assessment concurrent access to the Internet while still supporting the Internet requirements of the rest of the school. As Huff and Sireci (2001) correctly note, when this does not occur, the validity of the test is threatened. In the PVAT-O trial it was noted that some computers took a great deal longer than others to move through the PVAT-O. This frustrated and disadvantaged the students working on the 'slow' computers. Teachers #5 and #9 both mentioned their concerns with the fragility of the technology at their school, stating "if the computers all work…"(Teacher #5) and "I'm always concerned students will lose their work" (Teacher #9). Providing teachers with a 'back up' paper version of the test is considered a practical way to alleviate these fears.

Assessment Literacy

Popham (2018) explains that educators who are not assessment literate often make inappropriate decisions about which tests to use. Using formative CBA is a relatively new form of mathematics assessment in schools, so it is critical teachers are helped to understand the affordances and constraints of these tools. Teachers are a critical stakeholder in the formative CBA process. They are required to administer the assessment and their interpretation of the results influences its success (Jones & Truran, 2011). Seven of the nine teachers in this research described how they based their mode preference choice solely on the time it would save. Research by Melleti and Khademi, (2018) showed that for both assessment literate and illiterate teachers, time was their main concern when implementing formative assessment. Yet interestingly, assessment literate teachers considered the time they spent creating and marking assessments a necessary part of the process. Thus, it appears that when teachers do not fully appreciate the advantages of formative assessment, they consider the time spent on it untenable. This reinforces the need to develop teacher's assessment literacy skills around formative assessment, particularly in CBA (Popham, 2018). Without appropriate professional development designed to increase assessment literacy, teachers will continue to focus on selecting assessments based on their perceived ease of administration and marking, rather than the quality of the tool.

Conclusion

The demands on a classroom teacher's time have never been greater. Whilst a major affordance of CBA is the time it saves teachers, one of the major constraints is its lack of transparency. When a computer database marks student responses, a teacher's judgment and involvement in the process is removed. This research suggests in order to retain the fidelity of the formative assessment process, teachers require access to professional development

that aims to grow their assessment literacy skills. Developing these skills will encourage teachers to seek quality empirically proven assessments, and assist them to accurately interpret CBA data.

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Why that game? Factors primary school teachers consider when selecting which games to play in their mathematics classrooms

James Russo
Monash University
<james.russo@monash.edu>

Toby Russo
Spensley Street Primary School
<Toby.Russo@education.vic.gov.au>

Leicha A. Bragg
Deakin University
< leicha.bragg@deakin.edu.au>

Research examining teachers' decision making is abundant for pedagogical activities, yet a neglected area is the study of factors influencing teachers when selecting mathematical games. This article sheds light on the factors considered when teachers' select a specific game to use in their primary mathematics classroom. Data from 248 Australian primary teachers was gathered via a questionnaire and thematically analysed. Results indicated four strongly endorsed factors: Mathematics is central; Accessibility and differentiation; Classroom management; and Engagement and enjoyment. Implications are discussed of how this study can inform the decision making of educational leaders, policy makers, and game designers.

Mathematical games are an integral component of primary mathematics instruction. Although there is scant empirical evidence about the frequency of game use, one recent study of Australian early years primary teachers (Foundation-Year 2; n = 135) found that almost all study participants used games at least once per week, with half of teachers incorporating games in almost every mathematics lesson (Russo & Russo, 2020). Given the frequency of game use, we recently wrote a conceptual paper outlining five principles of educationally rich mathematical games to support teachers and pre-service teacher educators identify worthwhile games. These principles included: students are engaged; skill and luck are balanced; mathematics is central; flexibility for learning and teaching; and home-school connections (Russo et al., 2018). However, although we have made normative claims about factors teachers *should* consider when deciding which games to play, we could not identify any studies that examine those factors that teachers *do* consider when deciding which games to play. The purpose of this paper is to address this gap in the literature.

Background Literature

It is a widely held view amongst educators that mathematical games have the potential to support student learning in mathematics. Over three decades ago, Ernest (1986) put forward a rationale for using games in the mathematics classroom, suggesting that games could be used to teach a variety of mathematical ideas, and were perhaps particularly powerful for supporting student understanding of mathematical concepts; allowing for consolidation and practice; developing problem-solving skills; and, enhancing student motivation to engage in mathematics. In addition, it has been argued that opportunities to play mathematical games supports social skill development (Koay, 1996), encourages mathematical reasoning (Olson, 2007), allows for a differentiated approach to instruction (Buchheister et al., 2017; Trinter et al., 2015), and can be used to explore multiple connected mathematical ideas (Clarke & Roche, 2010). Indeed, there is empirical evidence to suggest that games are efficacious for engaging students in mathematics learning (Bragg, 2007; Campos & Moreira, 2016) and improving student learning in mathematics (Bragg, 2012b;

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Bright et al., 1985; Swan & Marshall, 2009), including for students in the early years (Cohrssen & Niklas, 2019; Elofsson et al., 2016). A recent meta-analysis exploring the effectiveness of mathematical games across all levels of education revealed that games had a medium positive impact on academic achievement compared with what were described as "traditional methods" of mathematics instruction, such as worksheets (Turgut & Temur, 2017, p. 196).

Although there has been substantial research into the impact of games on the student mathematical learning experience, especially in relation to digital games (see Abdul Jabbar & Felicia, 2015), how teachers use mathematical games in classrooms has been far less of a focus. One exception was a study by Heshmati et al. (2018), who examined the use of a game to support the teaching of fraction concepts in a naturalistic classroom based setting. The authors videotaped and analysed mathematics lessons across 14 US fifth grade classrooms during the teaching of a unit of work focussed on fractions. They found that 20% of lessons involved the use of a game to support fraction instruction for at least part of the lesson, and that games were used almost exclusively to consolidate student understanding, rather than introduce concepts. This latter finding is consistent with the literature that teachers frequently use games to support practice and the development of procedural fluency, particularly with number (Godfrey & Stone, 2013; Graven & Roberts, 2016). However, whether this is the predominant rationale for teachers using games in mathematics classrooms remains to be systematically investigated.

The current study

In order to address some of the gaps identified in the literature around primary teachers' use of games to support mathematics instruction, we invited teachers to complete a questionnaire. In total, 248 Australian primary teachers responded. We have published our initial, predominantly quantitative, findings focussed around primary teachers' motivation for and frequency of game usage, their game execution within lesson routines and structures, and their perceptions of the efficacy of games to achieve particular pedagogical objectives (Russo et al., 2021). Some key findings include:

- Consistent with Russo and Russo (2020), 98% of teachers reported using games at least once per week to support their mathematics instruction, whilst 79% reported using games multiple times per week.
- Teachers used games in a variety of contrasting ways to support mathematics instruction. For example, whilst three-quarters of teachers indicated they employed games multiple times per week as a warm-up to begin a mathematics class, almost half of teachers (45%) responded that they used games multiple times per week as a context for launching rich mathematical investigations.
- Reaffirming perhaps the most consistent finding in games research (Bragg, 2003; 2007; Campos & Moreira, 2016), all teachers agreed that games were an effective means of engaging students in mathematics, with 82% of teachers strongly agreeing with this statement.
- There was strong evidence that teachers preferred using non-digital games and tactile materials. When asked about their favourite mathematical game to use in a classroom, only 4% of teachers described a game which involved students or the teacher interacting with a digital technology in any capacity (e.g., calculator, random number generator, interactive number chart, supportive software), whilst only 1% selected a digital game specifically. This stands in stark contrast to the relative focus

on digital games within the education research literature (Abdul Jabbar & Felicia, 2015).

- Perhaps surprisingly, given the oft-remarked connection between games and building mathematical fluency (Godfrey & Stone, 2013), teachers indicated that they viewed games as being equally effective for developing all four proficiencies highlighted in the Australian Curriculum: Mathematics (ACARA, 2019): fluency, understanding, problem-solving, and reasoning.

The purpose of the current paper is to present additional qualitative analysis from the questionnaire data. Specifically, we focus on one specific free text response item in the questionnaire to shed light on the factors teachers consider when deciding which games to use in their mathematics classroom.

Method

Two hundred and forty-eight teachers completed the questionnaire focused on how they use mathematical games in their classrooms. Participants were spread across all years of primary education in Australian classrooms: Foundation-Year 2 (31%); Year 3-4 (25%); Year 5-6 (29%); taught across multiple year level groups (15%). Respondents were relatively experienced primary school teachers, with a median period of 10 years classroom teaching experience (mean = 13.2; SD = 9.3; Min = 1 year; Max = 51 years).

The questionnaire was administered through an online survey platform, Qualtrics. Snowball sampling was employed to disseminate the questionnaire, with the questionnaire link being distributed via email to 15 key informants based in three Australian states, as well as through social media platforms. Teachers currently teaching in an Australian primary education context were invited to complete the questionnaire. Two hundred and thirty-six teachers responded to the qualitative item that serves as the focus of the current paper. This item was: Which factors do you consider when selecting which games to play in your classroom?

Data was analysed thematically, approximating the process outlined by Braun and Clarke (2006). We began by reading and rereading questionnaire responses, the purpose being to immerse ourselves in the data. As we reread the responses, several proto-categories emerged. These proto-categories were clarified, refined, combined, and then elaborated to comprise our final ten themes (see Table 1). For example, the proto-categories ‘materials easily sourced’, ‘simple game mechanics’, and ‘time’, were eventually aggregated into the theme *classroom management*.

Results and Discussion

Table 1 displays the results of our thematic analysis, noting the number of teachers whose response was coded to each of the ten themes, as well as two quotations from teachers that help to illustrate this theme. Note that teacher responses could be coded to multiple themes. For example, the following response was coded to two themes, *mathematics is central* and *enjoyment and engagement:*

> Will these games increase mathematical awareness, do they tie into the maths lessons and how much do they promote engagement in the lesson and maths in general? Teacher Number 111 (T111)

From viewing the table, it is apparent that there were four themes frequently endorsed by participants: *mathematics is central; accessibility and differentiation; classroom management; and engagement and enjoyment*. Each of these four major themes will now be discussed, with relevant links made to the academic literature.

Table 1
Thematic analysis of factors teachers considered when selecting which games to play

Theme	Number (%) (n = 236)	Example quotations
Mathematics is central: connection to mathematical learning focus and/ or suitability of game for building conceptual understanding and procedural knowledge	183 (78%)	Linked to a specific mathematical focus, connected to the needs of the student group. (T1) How it enables the student to practice the concept that has been introduced if it is the main activity of the lesson. (T196)
Accessibility and differentiation: accessible for students and capacity for differentiation across age, mathematical performance, reading abilities	127 (54%)	How can the game be modified with enablers or extenders to cater for all students? (T63) Something that will provide a level of challenge for students working at all levels, possibly with progression or layers. (T183)
Classroom management: organisation, availability of required materials, setup time, noise level.	110 (47%)	How much equipment is needed? Do I need to make any of the resources? Are the instructions simple? Is it easy to get started/independent? (T8) The time it will take (set up, finding the materials, providing the instructions). (T82)
Engagement and enjoyment	91 (39%)	We want our kids to develop a love for numbers and maths and approach the subject without fear. Games are perfect for that reason. Kids love them. (T61) If the activity will engage the students for a sustained period of time. (T65)
Communication and reasoning: opportunities for encouraging mathematical dialogue, student reasoning and language development	24 (10%)	A way to share mathematical language, thinking and reasoning. (T2) The relevance of the maths language used in the game. (T224)
Supporting social and emotional development: opportunities for collaboration, interaction and learning how to play with others	21 (9%)	I like to make sure that games can be played with a partner to ensure students get to work together. (T76) Teams. Usually random so not necessarily ‘fair’, just like in real life! (T230)
Thinking strategically: developing strategic thinking, skillful behavior and providing opportunities to solve problems	21 (9%)	Opportunity for the move that a player makes to effect the move of their opponent/s…Opportunity for strategies to be articulated and developed (T15) One that allows students to stop and reflect on the mechanics of the game and explore ways to become more efficient in playing the game. (T35)
Skill and luck are balanced	16 (7%)	Games that allows children to experience success based on skill and also an element of luck (T49) Games that involve a bit of luck as well as skill/ strategy so that all students have a chance at winning regardless of their ability. This means that struggling students are more likely to want to keep playing the game. (T87)
Game adaptation and inquiry: opportunities to transform the game into an investigation and extend student mathematical thinking	12 (5%)	Does it support rich mathematical investigation? (T55) Does the game have the ability to be 'ramped' up over the week with further investigations into the strategy or reach a higher level of thinking. (T67)
Supporting assessment of student thinking and mathematical knowledge	4 (2%)	Quick formative assessment ‘check in’ (T2) What would assessment of game look like (T210)

Note. The mean number of themes a teacher response was coded to was 2.8

Mathematics is Central

The most frequently endorsed theme that emerged from the data related to choosing games with explicit connections to an identified mathematical learning focus, as well as games that could further student conceptual understanding of a particular concept, or their application of a particular procedure. This reaffirms the supposition that primary teachers tend to use games for specific mathematical purposes, rather than for engaging students in mathematics irrespective of the content.

Several teachers referred to the fit between the chosen game, and the intended learning intention. For example:

> The learning intention - what is it that I want the children to understand? The maths knowledge required - are there any barriers or misconceptions that might come up? (T181)

However, others indicated that the game may not always be connected to the learning objective of the current lesson, but instead be used to reinforce previous learning or as a cognitive activation device:

> Does the game support the learning intention of the lesson? Not always, however, sometimes they are selected to consolidate learning from previous units or just to get their 'maths' brains attuned. (T96)

In fact, the notion of using a game to build number fluency, practice a skill, or to consolidate student understanding was an important sub-theme to emerge that was explicitly noted by over one-quarter of teachers coded to this theme (n = 48).

> Does it help to consolidate a skill? Is it for reviewing a skill? (T70)

Interestingly, on occasion, teachers noted how the purpose of a game might evolve over time, initially using a game to build conceptual knowledge, and then using the game to reinforce understanding in subsequent lessons:

> I will teach a new game to introduce then consolidate a new skill. Once the game is understood and knowledge in the concept understanding is at a reasonable level, the game becomes a more regular warm up. (T88)

Accessibility and Differentiation

Over half of teachers noted that when considering which games to play in the classroom, they contemplated the extent to which the game was inclusive of all students and whether they could modify the game to align with the learning needs and performance levels of a diverse group of students. This is consistent with literature suggesting the flexibility of games to support differentiated instruction is a comparative strength of this pedagogical approach to teaching mathematics (Buchheister et al., 2017; Trinter et al., 2015). Some teachers specifically commented on the capacity to adjust game mechanics to optimise the level of challenge:

> Ability to differentiate to cater for different skill levels. For example, games where the rules can be changed or built on as students develop or where 6 sided dice can be replaced with 10, 12 etc. sided dice to make it more challenging. (T87)

Other teachers emphasised the need for the game to have various entry levels, so that a student's prior mathematical knowledge was not a barrier to them participating in the game:

> I ensure it is fun and engaging and has different entry points for different students based on what they understand. (T196)

Similarly, there was a reference to the value of a game having a 'low-floor, high-ceiling':

> Whether it has a low entry point and high ceiling to cater and challenge all students. (T107)

In the words of one teacher, considering "how the game could be extended or scaffolded" was principally about supporting "maximum participation" in the lesson (T210), a sentiment consistent with both high quality and equitable mathematics instruction (Sullivan, 2011).

Classroom Management

Approximately half of teacher respondents (54%) described practical considerations as being a critical factor when deciding which specific games to play in their classroom, encompassing aspects such as the accessibility of materials, the time needed to explain and set up the game, and whether student groups could play independently and remain on-task. The emphasis on classroom management is noteworthy, particularly given such factors have generally not been highlighted in the games research literature in the few empirical studies that have focussed on teachers' use of games (e.g., Heshmati et al., 2018).

Indeed, the importance of incorporating easily available materials was one of the reasons teachers tended to endorse dice and card games over more elaborate alternatives that involved the need to create, locate, or purchase specialised equipment:

> Resources. Can I use materials I already have, or does the game need special equipment? I usually go for games that use dice, playing cards, or readily available equipment over those that have a specialised game board. (T181)

To some extent, it appeared that the reluctance to use games involving specialised materials was due to the time investment needed:

> Resources. For example, games that use dice, cards, counters etc. are better than games where I have to make game boards etc. which can be time consuming; although I do do this. (T87)

As alluded to earlier, time was also mentioned in relation to minimising lost instructional time by ensuring the game is easy to set up and play.

> Materials required. Being able to be play the game and pack up in under 10 minutes. (T77)

Engagement and Enjoyment

In contrast to the theme of classroom management discussed previously, engagement and enjoyment are concepts frequently mentioned in connection to games in the literature (Attard, 2012; Bragg, 2003; 2007; Bright et al., 1985). Indeed, as reported elsewhere, our study teachers highlighted engagement as the principal pedagogical benefit of games (Russo et al., 2021), whilst other studies have concluded that the comparative advantage of games over other activities relates to their capacity to engage and maximise on-task behaviour (Bragg, 2012a; 2012b). Consequently, it is not surprising that many teachers emphasised that the game be engaging and enjoyable to play:

> Engagement - the more students that are interested in math and learn to see math as an enjoyable everyday part of life is a win in my opinion. (T166)

> It needs to be engaging and fun. (T223)

What is surprising, at least ostensibly, was that only a minority of teachers (39%) mentioned levels of student engagement and enjoyment as being relevant when choosing which games to play in their classroom. One possible interpretation of these data is that teachers primarily associate high levels of engagement with the general category of games, whilst the specific level of engagement generated by a particular game is only a secondary consideration. To put it another way, if (almost) all games are considered engaging, then this dimension might be less important when deciding which specific game to choose.

Conclusions and Implications

In conclusion, to deepen the understanding of games usage from the teachers' perspective, this paper presents an investigation of the factors influencing teachers when selecting mathematical games for inclusion in their primary classroom. Ten themes were drawn from these data, with four of the themes taking prominence. Mathematics being central to the selection of the games was the leading consideration for teachers. Hence, key to games selection was the enhancement of students' mathematical understandings. These teachers recognised the usefulness of games as a tool for mathematical learning.

More than half the teachers were cognisant about extending their students' knowledge and skills from different starting points through games. This aligns with initiatives across Australia to include differentiated teaching as a high impact teaching strategy recommended in schools (e.g., Department of Education and Training, 2017). The emphasis on differentiation in these teachers' responses provides possible evidence for policy makers of the effect these initiatives are having on teachers and their classroom decision making.

Classroom management, with an emphasis on organisational matters, was a consideration for many teachers. While educators may appreciate the mathematical value inherent in some complex, expensive, or time-consuming games, the practicalities of utilising such games were considered and potentially discounted. This factor has implications for game designers and individuals responsible for purchasing and organising educational resources (e.g., numeracy coordinators). It emphasises the significant practical considerations that primary school teachers need to consider on a daily basis, and the premium placed on simplicity and ease of access when planning and implementing learning tasks.

Enjoyment and engagement are often presented as key factors for inclusion of this non-traditional approach to teaching mathematics; and indeed were emphasised by all our study teachers as reported elsewhere (Russo et al., 2021). Although still a prevalent theme when choosing which specific game to play, as discussed earlier, it may be that engagement was a secondary consideration for many teachers because engagement is associated with games as a pedagogical category, more so than specific games. What these teachers mean by engagement requires further investigation and will be explored in future research.

Encouragingly, the findings of this research support four of the five principles of educationally rich mathematical games raised in our earlier conceptual paper (Russo, et al., 2018). Absent from these teachers' considerations was mathematical games providing opportunities for fostering home-school connections. The questionnaire was administered prior to the COVID-19 pandemic forcing school closures and children learning from home with family support. Hence, the relevance of selecting games which support home-school connections may be more pertinent to teachers today. Making a home-school connection via games will raise the status of games amongst the broader school community from merely an enjoyable pastime to a valuable educational tool to be played at home and school.

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Charting a learning progression for reasoning about angle situations

Rebecca Seah
Royal Melbourne Institute of Technology
<rebecca.seah@rmit.edu.au>

Marj Horne
Royal Melbourne Institute of Technology
<marj.horne @rmit.edu.au>

As a multifaceted concept, the learning of angle concepts takes years to achieve and is beset with challenges. This paper explores how the processes of constructing and validating a learning progression in geometric reasoning can be used to generate targeted teaching advice to support the learning of angle concept. Data from 1090 Year 4 to Year 10 students' ability to reason about geometric properties and deduce angle magnitudes were analysed. Rasch analysis resulted in eight thinking zones being charted. Students' responses to the angle items within this larger data set were analysed with a focus on how reasoning about angles developed. The result is a five-stage framework for learning angle concepts.

Teaching that is informed by effective assessment data has a significant, proven effect on learning (Goss et al., 2015). Designing targeted teaching advice that can nurture mathematical reasoning has become even more vital in light of the 2018 Programme for International Student Assessment (PISA) results (Thomson et al., 2019). Australian students' mathematical problem solving ability is in a long-term decline, equivalent to the loss of more than a year's worth of schooling since 2003. Australian students are particularly weak in the content areas of geometry (Thomson et al., 2017), a discipline that is linked to measurement and spatial reasoning.

Understandings of measurement are embedded in all curriculum in the STEM (Science, Technology, Engineering and Mathematics) areas. Concepts such as length, volume and angle take years to learn and are beset with challenges. A case in point is the learning of angle measurement. The concept of angle can mean different things in different situations. When viewed as a static image, angle is defined as a geometric shape, a corner or two rays radiating from a point, then as a dynamic image, angle is a rotation and a measurement of turn. Research shows persistent student difficulties with angle concepts, including focusing on physical appearances such as the length of the arms or the radius of the arc marking the angle when comparing angles, inability to see angles from different perspectives and contexts, and errors in measuring the angle magnitudes using a protractor (Gibson et al., 2015; Mitchelmore & White, 2000). In the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority [ACARA], n.d), the concept of angle is introduced under the sub-strands of geometric reasoning from Year 3 onwards. In Year 5, students are expected to use degrees and measure with a protractor and in Year 6, to find unknown angles. Year 7 refers to angle sums in triangles and quadrilaterals. The curriculum expectation is that the students will have the necessary understanding of angle and angle measurement to be able to reason about angle sizes by Year 6 and Year 7. It is presumed that teachers are able to make the necessary connections among and across content strands and teach for mathematical reasoning (Lowrie et al., 2012). International results obtained to date do not reflect such a reality.

With STEM becoming a key focus in education, research on learning progressions can help transform the teaching and learning of mathematical reasoning. In this paper, we survey and analyse Australian students' knowledge of and reasoning about angle measurement within a more comprehensive geometric learning progression.

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Theoretical Framework

Learning progressions are a set of empirically grounded and testable hypotheses about students' understanding of, and ability to use, specific discipline knowledge within a subject domain in increasingly sophisticated ways through appropriate instruction. They can relate to a specific instructional episode, develop a curriculum or in our case, charting mathematics learning that encompasses different but related aspects of mathematics. Our purpose is to equip teachers with the knowledge, confidence and disposition to go beyond narrow skill-based approaches to teach for understanding and mathematical reasoning.

Reasoning is a cognitive process of developing lines of thinking or argument to either convince others or self of a particular claim, solve a problem or integrate a number of ideas into a more coherent whole (Brodie, 2010). Mathematical reasoning is about constructing mathematical conjectures, developing and evaluating mathematical arguments, and selecting and using various types of representations (National Council of Teachers of Mathematics [NCTM], 2000). Mathematical reasoning encompasses three core elements: (1) core knowledge needed to comprehend a situation, (2) processing skills needed to apply this knowledge, and (3) a capacity to communicate one's reasoning and solutions. Justifying and generalising are two key characteristics of mathematical reasoning (Brodie, 2010). To justify a position, individuals need to connect different mathematical ideas and arguments to support claims and conjectures. To generalise requires individuals to reconstruct core knowledge and skills when making sense of new situations. Both help improve reasoning skills, cement core knowledge and may lead to the development of new ideas.

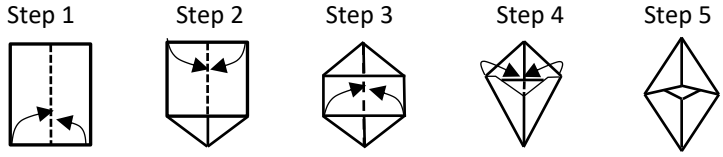
Engaging in mathematical reasoning is a social act, directed by a semiotic process (Bussi & Mariotti, 2008). Symbols (°, ∠), lines (∟, ⊥, ∡), shapes and objects serve as signs and artefacts for a particular purpose. An artefact (e.g., a folded piece of paper or written words) is a tool or an instrument that relates to a specific task to be used for a particular purpose. A sign is a product of a conjoint effort between it and the mind to communicate an intent, such as indication of a right angle. The use of signs and artefacts is never neutral but is intentional and highly subjective, linked to the learner's specific experience and requires the reorganisation of cognitive structures. From a cognitive perspective, how well a learner reasons mathematically is largely dependent on the degree of connectedness among multiple representations (artefacts), visualisation and mathematical discourse (Seah & Horne, 2019). Angle is multifaceted and can be represented in various ways. Visualisation of angle artefacts requires a dynamic neuronal interaction between perception and visual mental imagery. The viewers need to draw on past experiences and existing knowledge to make sense of the visualised artefacts. The context within which perception takes place plays a critical role in determining the type of imagery gaining attention. Individuals' beliefs about their own ability and how mathematics is practiced also play a critical role in this process. Context and beliefs are influenced by the mathematical narratives and routines learners experience. Words and terminologies produce certain visual images. For example, Gibson et al., (2015) found that whole-object word-learning bias led many pre-schoolers to judge angle size by the side length. This was also found with older children (Mitchelmore & White, 2000). During a mathematical discourse, communication can take a combination of linguistic, symbolic or diagrammatic forms. How they are being used reveal the users' thought processes and in turn shapes their thinking. Analysis of students' responses to angle measurement tasks will enable researchers to document and chart how students' reasoning about angle measurement progressed. This can then help design instructions that move students from where they are to the next level of their learning journey.

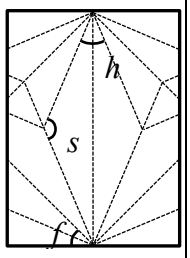
Method

Drawing on the work of Battista (2007), a draft geometric learning progression was developed that saw the development of geometric reasoning as moving through five levels of reasoning: visualising physical features, describing, analysing, and inferring geometric relationships, leading to engaging in formal deductive proof (Seah & Horne, 2019). The data presented here was taken from the Reframing Mathematical Future II study into the development of learning progression for mathematical reasoning. The participants were middle-years students from across Australia States and Territories. The first group – the trial data, was taken from two primary and four secondary schools across social strata and three States. They were asked to participate in trialling the assessment tasks to allow for a wider spread of data being collected. The trial school teachers administered the assessment tasks and returned the student work to the researchers. The results were marked by two markers and validated by a team of researchers to ascertain the usefulness of the scoring rubric and the accuracy of the data entry. The second group – the project data, came from 11 schools situated in lower socioeconomic regions with diverse populations across six States and Territories. The project school teachers marked the items and returned the raw score instead of individual forms to the researchers. They also received ongoing professional learning sessions and had access to a bank of teaching resources. There are two angle measurement tasks, Geometric Angles 1 and 2 (coded as GANG) reported here (Figure 1).

Geometric Angles 1

You will need the shape you made in class. The steps and diagrams below show how you made the shape.





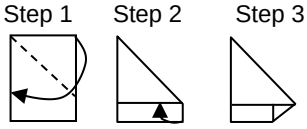
Step 1 Fold an A4 paper in half lengthwise to make a crease line in the middle of the page.
Step 2 Fold two corners to the middle at the bottom
Step 3 Fold two corners to middle at the top
Step 4 Fold the new corners on the sides at the bottom to the middle
Step 5 Do the same with the top

a [GANG1]
Phoebe made the same shape that you made using A4 paper. She said her shape is a rhombus.
Do you agree? Explain your reasoning.

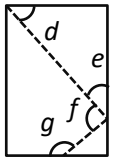
b [GANG2]
When Phoebe unfolds the paper, she found a number of crease lines. Find the marked angles on the crease line: Angle *f* = ____ Angle *h* = ____ Angle *s* =_____
Explain how you work out the angles.

Geometric Angles 2

A four-sided shape is folded from a sheet of A4 paper using the following instructions.



a [GANG3]
What is the name of this shape?
_____
Explain your reasoning.



b [GANG4]
Unfold the paper and find the size of each marked angle.
Angle *d* = _____ Angle *e* = _____
Angle *f* = _____ Angle *g* = _____

Explain your reasoning.

Figure 1. Geometric angles task 1 and 2.

Note that both tasks were used in different forms rather than administered together. Both tasks begin with a question on geometric properties followed by deductions of angle magnitudes. In GANG1, the teacher was instructed to guide the students to first fold the shape and use it to answer the angle measurement question. In this way, the difficulty in following the origami instructions was avoided. As an artefact, the folded shape also served as a context and a tool to help students comprehend the diagram depicting the crease lines. In GANG3, students were shown the steps taken to fold a shape. No further information was given. Items GANG2 and GANG4 ask students to work out the magnitude of the angles formed by the crease lines. While the tasks GANG1 and GANG3 do not ask students specifically to use angle, angle properties are one component of shape classification. The focus in this paper is on reasoning about angle magnitude in GANG 2 and GANG 4.

Rasch partial credit model (Masters, 1982) using Winsteps 3.92.0 (Linacre, 2017) was used to analyse students' responses on the larger set of geometric reasoning tasks including these for the purpose of refining the marking rubrics and informing the drafting of an evidence based learning progression. Rasch analysis of the validity of the underlying construct through the idea of fit to the model produced eight thinking zones in geometric reasoning (Seah & Horne, 2019). To validate the zones, the research team interrogated student responses located at similar points on the scale to decide whether or not there were qualitative differences in the nature of adjacent responses with respect to the sophistication of reasoning involved and/or the extend of cognitive demand required (see Siemon & Callingham, 2019).

SCORE	DESCRIPTION for GANG1	DESCRIPTION for GANG3
0	No response or irrelevant response	
1	Disagree it is a rhombus based on appearance rather than properties	Diamond or other incorrect shape
2	Disagree it is a rhombus but claim it is a parallelogram with some properties	Quadrilateral because it has 4 sides OR because it looks like a kite
3	Agree it is rhombus but insufficient or incorrect properties to define it or claims it is a parallelogram and includes all properties	Kite OR unable to name, but gives side and/or angle properties of a kite
4	Agree it is rhombus. Explanation needs to include necessary and sufficient properties, that is, it has 4 equal sides, or it is a parallelogram with one of the following properties: • Adjacent sides equal • Diagonals bisect each other at right angles or diagonals bisect the angles • Two lines of symmetry	Kite because two pairs of adjacent equal sides are equal OR because at least a pair of opposite angles equal and at least one pair of adjacent sides the same length OR because it has a pair of opposite angles equal and a line of symmetry. May include other properties.
SCORE	**DESCRIPTION for GANG2**	**DESCRIPTION for GANG4**
0	No response or irrelevant response	
1	Incorrect angles	Incorrect with little/no reasoning, may include one correct angle
2	At least 2 angles correct but no reason given, or one angle correct with correct reasoning	At least two angles correct with an attempt at explaining reasoning
3	Two angles found correctly with sensible reasons or all angles correct with insufficient reasoning	Angles correct (d = e = 45°, f = 90° or right angle, g = 135°) but reasoning sparse and incomplete
4	All angles correct with clear reasons given relating to the folding and properties. F = 45°; h = 45°; s = 135° (e.g., Folding corner to centre creates half right angle; All angles around centre of side equal so any 2 make 45°or Four angles of quadrilateral add to 360°)	Angles correct. Reasoning includes justifies d as half of the right angle in corner or as angles in an isosceles triangle, and g on the basis that the four angles of the kite shape have to add to 360°

Figure 2. Geometric angles task scoring rubrics.

In the following, we focus on students' responses to the angle items to determine their usefulness and fit to the overall learning progression framework.

Findings

Based on 1041 students' responses from the larger study, the zones of geometric reasoning were established as precognition; recognition; emerging informal reasoning; informal and insufficient reasoning; emerging analytical reasoning; property based analytical reasoning; emerging deductive reasoning; and logical inference-based reasoning (Seah & Horne, 2019). Student responses were coded so that GANG3.1 meant a response at Level 1 on the rubric to the question GANG3. Table 1 shows how the responses to the GANG questions were spread across the zones (with Zone 8 being the highest level).

Table 1
Excerpt from the variable map for geometric reasoning (n=1041).

Zone 8			GANG3.4	GANG4.4
Zone 7	GANG1.4	GANG2.4		
Zone 6		GANG2.3 GANG2.2		GANG4.3
Zone 5				GANG4.2
Zone 4	GANG1.3 GANG1.2	GANG2.1	GANG3.3	
Zone 3	GANG1.1		GANG3.2	
Zone 2			GANG3.1	GANG4.1
Zone 1				

To validate these zones, the research team interrogated student responses located at similar points on the scale to decide whether or not there were qualitative differences in the nature of adjacent responses with respect to the sophistication of reasoning involved and/or the extent of cognitive demand required. For example, GANG1.2 (disagree it is a rhombus claiming it is a parallelogram) and GANG1.3 (agree that it is a rhombus with insufficient explanation about its properties) were located in zone 4, indicating similar level of thinking. Reasoning about a kite (GANG3.4 and GANG4.4) were located in the highest level (Zone 8), perhaps revealing students' lack of exposure to this concept. The angles on the rhombus were also easier to deduce than those on the kite.

Table 2
Breakdown of student responses on geometric properties (GANG1 and GANG3).

Score	Trial Data (*n=230*)				Project Data (*n=433*)			
GANG1	Yr 7	Yr 8	Yr 9	Yr 10	Yr 7	Yr 8	Yr 9	Yr 10
	n=83	*n=90*	*n=31*	*n=26*	*n= 171*	*n= 204*	*n= 37*	*n= 21*
0	20.5	45.6	19.4	3.8	36.3	32.8	24.3	14.3
1	30.1	13.3	9.7	0	19.3	15.2	0	38.1
2	12.1	11.1	12.9	7.7	4.1	11.3	2.7	47.6
3	33.7	17.8	48.4	73.1	36.8	27.9	62.2	0
4	3.6	12.2	9.7	15.4	3.5	12.8	10.8	0
Score	Trial Data (*n=157*)				Project Data (*n=270*)			
GANG3	Yr 4	Yr 5	Yr 9	Yr 10	Yr 7	Yr 8	Yr 9	Yr 10
	n=31	*n=59*	*n=30*	*n=37*	*n= 23*	*n= 113*	*n= 32*	*n= 102*
0	22.6	23.7	27.6	35.1	0	17.7	53.1	17.7
1	77.4	66.1	27.6	32.4	17.4	21.2	31.3	28.4
2	0	10.2	13.8	29.7	69.6	21.2	15.6	32.4
3	0	0	34.5	2.7	13	31.9	0	20.6
4	0	0	0	0	0	8	0	1

Because the Rasch model is probabilistic, an in-depth analysis of students' responses to the items were conducted. A total of 663 samples for Geometric Angles 1 task and 427 samples for Geometric Angles 2 were collected. Table 2 shows the breakdown of students' responses for reasoning about geometric properties. Both cohorts performed better in the rhombus item than the kite item. The majority of students found reasoning about geometric properties difficult and on average, around 25% of students did not respond. In GANG1, students tended to define a rhombus based on its orientation or what it looks like:

Year 7: I believe wasn't wide enough to become a rhombus and the shape is a diamond (score 1).

Year 10: It can be depending on how you look at it. It could be a diamond or rhombus (score 2).

Year 9: … when you hold the shape so that the pointed parts point from left to right you would see that it is in the shape of a rhombus (score 3).

Only a handful of students accurately defined a rhombus, as having '4 equal sides'; none included the square as part of the rhombus family. Further, when the term angle was used, it was to emphasize that a rhombus has no right angle, or incorrectly stating that the shape has 'four exactly the same sides with 4 acute angles'. In GANG3, 76.4% of students provided a 2D name to the folded shape, such as triangle (16%), irregular rectangle/square (31.2%), polygon (6.4%) and quadrilateral (8.9%). None of the trial school students were able to correctly state the properties of a kite.

Nevertheless, inability to reason about geometric properties did not appear to influence the deduction of angle magnitudes. Comparison of students' responses by year level shows that project schools' performance was slightly better and that the angles in the rhombus were easier to deduce than those in the kite (see Table 3). There was still a large number of no response or irrelevant responses received from the trial data (27.8% and 38.5% in GANG2 and GANG4 respectively).

Year 9: I worked this out by counting the crease of each angle (wrote 3, 3, 2 in GANG2).

Year 4: I measured each line and quartered it (wrote 2 cm, 3 cm, 4 cm, 5 cm in GANG4).

Other students (47.8% and 28.9% respectively) either wrote the name of the angles as acute or obtuse or were only able to give the magnitude of one angle.

Table 3
Breakdown of student responses on angle magnitudes (GANG2 and GANG4)

Score	Trial Data (*n=230*)				Project Data (*n=433*)			
GANG2	Yr 7	Yr 8	Yr 9	Yr 10	Yr 7	Yr 8	Yr 9	Yr 10
	n=83	*n=90*	*n=31*	*n=26*	*n= 171*	*n= 204*	*n= 37*	*n= 21*
0	14.5	44.4	29	11.5	47.4	39.7	35.1	9.2
1	67.5	35.6	35.5	42.3	33.9	27	35.1	9.5
2	6	7.8	22.6	7.7	7.6	9.3	5.4	14.3
3	3.6	10	9.7	23	6.4	10.3	10.8	4.8
4	8.4	2.2	3.2	15.4	4.7	13.7	13.5	61.9
Score	Trial Data (*n=157*)				Project Data (*n=270*)			
GANG4	Yr 4	Yr 5	Yr 9	Yr 10	Yr 7	Yr 8	Yr 9	Yr 10
	n=31	*n=59*	*n=30*	*n=37*	*n= 23*	*n= 113*	*n= 32*	*n= 102*
0	83.9	20.3	27.6	37.8	17.4	18.6	43.8	5
1	16.1	47.5	27.6	10.8	43.5	38.1	34.4	9.8
2	0	25.4	13.8	27	8.7	23	9.4	21.6
3	0	5.1	34.5	10.8	21.7	17.7	6.3	33.3
4	0	1.7	0	13.5	8.7	2.7	6.3	30.4

Many of the irrelevant responses for GANG4 were from the primary years 4 and 5, suggesting that these students may not have learned this concept. The trial data showed that some of the responses were far from the correct answers. Students either solved the problem based on physical appearance - 'it looks like a right angle… (90°, 43°, 180° for GANG2)', made obscure comments such as 'use a pencil (90°, 110°, 155°)', or wrote '60°, 70°, 90°, 140° the sum of all the angles = 360°' (GANG4). A further 29 trial school students admitted to using a protractor for GANG2. This may be due to the teacher's oversight or assumption that it was inaccessible by the students or because they have no strategies otherwise. Despite its availability, only two students were able to provide the correct answers.

A 45° angle was the easiest to deduce by using right angle as a benchmark. Using existing angle knowledge as benchmark did not always work however and the Year 9 and Year 10 students tended not to provide a reason for GANG4:

Year 7:	You work out the angles by knowing where 90° is and if the angle is smaller then you take a given between 0° and 90°. If the angle is bigger than 90° and smaller than 180° then you guess what the angle might be. I then checked with a protractor to see how far off I was (34°, 96.5°, 135°).
Year 10:	They all need to equal to 180 (wrote 20°, 30°, 130°)
Year 9:	*d* and *e* has the same size angle as you can see, *f* as everyone knows that it is 90° because it's a right angle and *g* is an obtuse, which is 180° (wrote 45°, 45°, 90°, 180°).

Discussion

Angle is the foundation for much of geometry and trigonometry and applicable in many daily activities, yet many students did not demonstrate understanding of the concept nor ability to reason about angle size. Looking within the overall geometric framework at the student responses to the question requiring reasoning about angle measurement in more detail gave an indication of the development of reasoning about angle.

Students operating in Zones 1 and 2 of the geometric learning progression usually did not show evidence of identifying the meaning of angle in any useful way. When they did use the term angle it was in reference to a right angle, often incorrectly. They did not use angle properties at all in identifying shapes. Students in Zone 3 were identifying right angles and in Zone 4 some of the students were referring to acute and obtuse angles though they were still not correctly giving many angle magnitudes with the exception of a right angle. By Zone 5 the students were attempting to reason about the angle magnitudes and were identifying the magnitudes of some of the angles correctly, usually in relation to a right angle. Diagrams, calculations and connecting language were beginning to appear. In Zone 6, the students were correctly identifying angle magnitudes, but their reasoning tended to relate just to the right angles and was incomplete. The few students who responded in Zones 7 and 8 were able to correctly identify the angles and explain their reasoning using a combination of diagrams and calculations integrated with words.

Relating this to the overall learning progression for geometric reasoning indicates that for reasoning about angle measurement there appeared to be five stages

1. Informal reasoning based on appearances (Zones 1-3): This encompasses the development of the concept from thinking of angles as lines or lengths through to identifying angles as corners and visually recognising 90° angles.
2. Informal and insufficient formal reasoning (Zone 4): Reasoning about magnitudes as being greater or less than a right angle and assigning magnitudes accordingly.
3. Emerging analytical reasoning (Zone 5): Deducing and arguing 45° angles in relation to a right angle, often with an accompanying diagram or calculation.

4. Relational-inferential property-based reasoning (Zone 6): Correctly identifying angles and giving reasoning for at least some of them usually with some attempt at using diagrams and connecting language, often with some calculation.
5. Emerging deductive and logical inference-based reasoning (Zone 7-8): Correct identification of angles reasoned with supporting diagrams, calculations and integrating connecting language.

The descriptions here are in the context of the questions that were asked. The final stage would be moving to full deductive reasoning and proof, but we have no evidence of this stage as the questions did not seek a response at this level. Nevertheless, the results show that many Australian students in our sample across all states are unable to do what the curriculum expects them to do. Learning progression research allows researchers to identify what learners can do, and what needs to be done to move their learning forward. The stages as described here contributed to the development of advice for teaching reasoning about angle measurement. Further research is needed to investigate this progression and expand it more fully to encompass the whole of angle measurement.

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Conceptualising 3D shapes in New Zealand primary classes

Shweta Sharma
The University of Waikato
<ss555@students.waikato.ac.nz>

This paper explores three multilingual students' (9 to 11-years-old) conceptual understanding of three-dimensional (3D) shapes as displayed through peer and classroom interactions in two New Zealand primary classes. Bakhtin's dialogic theory and Garfinkel's ethnomethodology inform the theoretical framework. The paper presents two excerpts from audio-video recorded data. Findings suggest that the students use their multilingual capacities to convey their meanings of geometry shapes as they engage in peer and classroom interactions. The paper recommends that it is crucial to explore prosodic features of the language to facilitate the meaning-making process during teaching and learning of geometry.

Multilingualism constitutes the overt or covert existence of multiple languages in mathematics classrooms (Barwell et al., 2019). This presence of various languages in mathematics classes is widely acknowledged as a *resource* for promoting the understanding of mathematical concepts (Adler, 2010). As a result, the strategy of code-switching between the language of instruction and other languages present in students' repertoire is claimed to promote mathematical understanding (Planas & Setati-Phakeng, 2014). It has been argued that code-switching can enable students to blend their multilingual capacities and successfully participate in mathematical activities (Setati & Moschkovich, 2013); however, it has been argued that the nature of language (if it is verb-based or noun-based) influence the ways in which students understand and display their mathematical ideas (Borden, 2013). For example, Borden found that there is no Mi'kmaw (an aboriginal language of Mi'kmaw communities in Nova Scotia) word for the concept of "flatness", that we take-for-granted in mainstream mathematics. Acknowledging the scarcity of research exploring multilingualism in geometry classes, the paper aims to investigate how multilingual students talk about geometric shapes in primary classes. Moreover, in the work of those who have examined language in mathematics education (e.g., Kaur, 2015; Ng & Sinclair, 2015), the exploration of dynamicity of language that incorporates the prosodic patterns of stress and intonation is often ignored. Ward (2019) argued that these prosodic features of language convey meanings and provide social significance to the words in any setting. Moreover, speakers of different languages employ these prosodic patterns differently to signify their focus of interaction (Ward, 2019). For example, Ward and Al Bayyari (2010) found that Arabic speakers construct their utterances in low pitch to signal their intention of continuous listening to the speaker, which is often perceived as rude behaviour by English language speakers.

Thus, acknowledging the dynamic nature of language in contemporary multilingual geometry classes, the present paper aims to address the following research gaps: (i) to explore the processes through which multilingual students construe and display their understanding of geometry shapes as they engage in classroom interactions, and (ii) to develop a critical understanding of how the multilingual context of contemporary geometry classes influences the process of development of geometry concepts.

Two excerpts from two New Zealand primary schools are presented here to address the research question: *How do multilingual students (9 to 11-years-old) discursively construct and reconstruct two-dimensional (2D) shapes and three-dimensional (3D) shapes in New Zealand primary classes?*

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Theoretical Framework

The present study draws its theoretical underpinnings from Bakhtin's (1981) *dialogic theory* and *ethnomethodology* (Garfinkel, 1967). According to Bakhtin, the specification of the meaning is dependent upon the preceding and succeeding dialogues within the dialogic space. Moreover, two opposing language forces operate simultaneously at different levels of interaction. The centripetal force or the "unifying language" (Bakhtin, 1981, p. 269) aims to guarantee mutual understanding of the meanings of utterances by crystallising their meanings, within the domains of prevalent dominant discourses. Concomitantly, the diversifying force or 'heteroglossia' as Bakhtin (1981, p. 270) defined, attempts to decentralise the already established meanings of the utterances by embedding individualised meanings into the language. It is the ongoing interplay of unifying and diversifying language forces in a specific circumstantial context as well as the socio-cultural milieu that informs the particular sphere of communication. Exploration of what is said, when it is said, and how it is said can enable the researcher to tap into these heteroglossic and unitary language forces. Interpretation of what is said, when it is said, and how it is said can be achieved by exploring the *indexical nature* (patterns of stress and intonation) of the language use (Barwell et al., 2019). The ethnomethodological approach of the present study allowed me to explore these indexical properties of the language use as it unfolds within interaction in day-to-day life events.

Undertaking ethnomethodology with Bakhtin's dialogic theory as a theoretical foundation helps us to acknowledge that knowing is construed as an ongoing action that takes place within ongoing interactions. Therefore, in this paper, I aim to explore the processes through which meanings of geometric shapes are appropriated and developed from moment to moment during classroom interaction, on the one hand; while developing a critical understanding of dominant mathematical discourse that influences the process of meaning-making of geometry shapes in multilingual mathematics class, on the other.

Methodology

In this paper, I report on three multilingual students' (9-11-years-old) discursive constructions of 2D and 3D shapes in two primary schools (School A and School B). Participants from diverse ethnic and linguistic backgrounds volunteered to participate, and informed consent was obtained from participants and their parents before participation. Data were primarily gathered through classroom observations in School A, and audio-visual recordings of the whole class and group interactions in six geometry lessons School B, as well as field notes in both settings. Short semi-structured interviews were conducted with the two classroom teachers to seek clarification about the lessons. Six (two from School A and four from School B) short audio-recorded focus group interviews were also conducted with the students to explore their understanding of shapes and their properties.

For data analysis, participants' *utterances* were considered as the unit of analysis. To explore students' discursive constructions about 3D shapes, video- and audio-recorded data were viewed repeatedly to identify the relevant key moments. Only moments where students either identified or described the shape or its properties were considered as *key moments*. The key moments were then subjected to two levels of analysis; (i) micro-level and (ii) macro-level. At the micro-level of analysis, several Conversation Analysis (CA) techniques were employed to explore the circumstantial organisation of talk-in-interaction that aids in the conceptual development of the geometric concepts of shapes and their properties. The interactions were transcribed using a simplified version of Jefferson (2004) transcription

conventions (see Appendix A for transcript key). The transcribed data enabled me to identify the subtle prosodic patterns that participants used to convey their meanings with-in talk-in-interactions. The macro-level analysis used an adapted version of Paul Sullivan's (2012) analytical approach, coding each key moment in terms of participants, genres and discourses used, and emotional registers. This coding enabled me to identify the dominant discourses that influence the meaning-making process of geometric shapes.

Analysis and Discussion

This section presents the analysis of two key moments (from School A and School B) in which students discursively constructed "cube" as "3D Square", and a 'triangular prism' as "3D Triangle".

Micro-level analysis

Key moment 1: "yeah- just a three-d square." The first key moment is presented from the focus group interview conducted in School A. The focus group interview was held on the same day after the third lesson on shapes had been taught. In previous lessons, students were taught about 2D and 3D shapes, and their properties. The focus group interview (with a group of five students) was audio-recorded, and transcribed data is presented in Excerpt 1. R denotes the researcher in the transcript. Lily is a monolingual English speaker. Amir and Liu are bilingual students with Arabic and Chinese as their respective home languages. In Excerpt 1, students were asked to talk about shapes.

Excerpt 1

7	Lily:	so:: (.6) we counted how many ↓edges so ^if it was
8		(1.0)a^ like a SQUARE(.)it had like
9		twe↑lve(.6)edge:s
10	R:	okay
11	Amir:	^yeah^
12	Lili:	a:n::: twelve co:rner:[s
13	Amir:	^no^
14	R:	so was it a square?
15	Amir:	cube.
16	Lily:	yeah (1.0)↑cube(.)which is really same as a square
17	Amir:	^its just three d^
18	R:	is it is it same as square
19	Amir:	[yeah] its #just the three d square#
20	Lily:	[yeah]
21	R:	whats exa:ctly three d
22	Amir:	↓a three d [>is when it pops ou:t<
23	Lily:	[a three d is like when it pops out?
24	Amir:	↓yeah three dimensiona:l
25	Lily:	li↑ke(.) a square if you draw it like this
26	R:	yeah
27	Lili:	he it (.5) ↑wont be: a three ↑d: itll just be a
28		nor:mal squa:re?
29	Liu:	^its like [this^
30	Lily:	and then a three D:: is when you do >that an then
31		another square inside ↑then <you join them up>
32		toge↑ther? ((Lily drew the cube there on a piece of paper))
33	Lily:	an then a normal square is just like(1.0) four
34		line:s
35	Amir:	Yup

The first question for the focus group interview was not directed to any particular student. Lily self-selected (Line 7) and started with pointing out the property of 3D shapes (i.e., "edges"). However, in her next utterance (Line 8), she used the name of a 2D shape ("square") to signify the 3D shape (cube). Following Lily's turn, in Line 11, Amir showed agreement. However, he constructed his utterance in a low pitch, a prosodic feature used in the Arabic language to convey continued listening and to encourage the speaker to continue their talk (Ward & Al Bayyari, 2010). In Line 12, Lily used stretching the syllables of her utterance, (i) to hold the floor, and (ii) to look for the right words to express her thoughts (Hellermann, 2005). Amir again used the low pitch to convey his intention for continued listening, yet he displayed his disagreement with Lily's suggestion (Line 13) as he said "no". He did this to take part in the discussion without overpowering Lily, the speaker. In Line 14, the researcher asked Lily if the shape the group was referring to was a square. This time Amir self-selected and stated that the shape was a cube, not square (Line 15). He used a falling tone to display (i) his dominance over the knowledge, and (ii) his intent to finish the interaction about this shape. His assertion was met with agreement from Lily (Line 16) with "yeah"; however, she again paused for one second after saying "yeah" (Line 16). Her use of high pitch with the word "cube" (Line 16) indicates her interest in sustaining the topic (Walker, 2017), unlike Amir. She constructed her utterance to show that her use of "square" is correct as both terms- "square" and "cube'", imply the same shape. Again, in Line 17, Amir self-selected and used lower pitch voice to indicate his agreement with Lily's statement again without interrupting the flow of conversation. In Line 18, the researcher once again asked if both names imply the same shape. To this question, both Amir (Line 19) and Lily (Line 20) started answering. However, Lily stopped as Amir continued. Amir argued that cube is "just the 3D square" (Line 19). He used a creaky voice to claim his authority over the knowledge with certainty (Ward, 2019). Lily (Line 20) again approved Amir's statement with "yeah"; however, she did not provide any explanation of why she agrees that a cube is "just the 3D square". It is noteworthy that, in Line 22, Amir used faster speech along with a lower pitch voice again to signify his authority over his knowledge (Ward & Al Bayyari, 2010). In Line 23, Lily again self-selected and she constructed her utterance using *high rising terminal* (HRT), denoted by '?', a conversational solidarity marker used in New Zealand English, which is used to check whether the other members of the group agree with her (Warren, 2016). In Lines 25, 27, 28, 30-34, Lily constructed her utterances to justify her previous claim that a "cube, which is really same as a square" (Line 16).

The presented analysis shows that Amir (a multilingual student) often made use of prosodic patterns of his Arabic language in his use of English to convey his understanding of shapes to his listeners. Moreover, through his Arabic language patterns of stress and intonation, he displayed his authority and confidence about his knowing of geometry shapes. Ward and Al Bayyari (2010) noted that Arabic ways of supporting the speakers' utterance with the use of low pitch and faster talk are often perceived negatively as a sign of either anger or disinterest by English speakers.

Key moment 2: "what's a triangle three-d? A triangular prism!" The second key moment is presented from the audio-visual recording from the first lesson at the School B. During this lesson, the teacher provided the students with a task called "Shapes in everyday life", and asked them to identify the shapes in the picture given to them. The teacher divided the class into groups for this task. After completing the task, she asked each group to come and present the shapes that they had identified. As they reported the shapes, the teacher wrote the names of the shapes on the whiteboard. The teacher asked a group of three students (Alyssa, Tane, and Olivia) to talk about the shapes. They identified one shape as "triangle

3D". The second key moment (Excerpt 2) is extracted from the transcribed classroom discussion that followed. In Line 547, the teacher reads the names of the shapes from the task sheet. In Line 548, she used a high pitch with "what" to draw students' attention to the coming question (Walker, 2017). Moreover, she stretched the word "triangle" while emphasising "three d". In this utterance, she acknowledged students' conception of three-dimensional shape as "3D triangle". However, she displayed her intention to direct students' attention towards using the geometry term for the identified shape. In Line 549, Ethan (with English as his first language), raised his hand to answer and began to speak without permission from the teacher. The teacher ignored his utterance (Line 550), and selected Yue (bilingual student with English and Chinese) to take the next turn. Yue answered that the shape is a "cube" with a flat pitch. It has been argued that Chinese bilinguals often use flat pitch while using English (Pickering, 2001).

Excerpt 2

547	Teacher:	so they ve got(0.2)square(0.5)two d:(1.0)triangle.
548		three d:(0.5) ↑what is: a tri::angle three d
549	Ethan:	it. is. [a:
550	Teacher:	[<can anyone remember> what (1.0) a tri (1.0) Yue?
551	Yue:	cube
552	Teacher:	CU::BE(0.5)um kori cu:↑be is (1.0)a cube is a bit
553		Different (.)um::: Matiu ((teacher smiled and pointed to Matiu))
554	Matiu:	tri:angular (0.5)a[::
555	Tane:	[prism
556	Matiu:	prism
557	Teacher:	triangular prism gre:at.
558	Garry:	I WAS ABOUT TO SAY Cone (1.0)

In Line 552, the teacher emphasised the word "cube" by using both increased volume and stretching. She used these prosodic features for two purposes: (i) to get Yue's attention at the start of her utterance, and (ii) "um" as a hedging device (Schegloff, 2007) to produce her next utterance that would implicitly reject the suggestion (Line 552). The teacher selected Matiu (bilingual student with Te Reo Māori and English) as the next speaker (Line 553). Matiu, in Line 554, used stretching and a pause to hold the floor so that he could recall and state the full name of the shape. As Matiu could not recall the full name of the shape, Tane self-selected and constructed his utterance (Line 556) in alignment with the Matiu's utterance. The teacher accepted Tane's response and started writing on the whiteboard as Matiu constructed his utterance (Line 556) in agreement with Tane's response. It should also be noted that the teacher responded positively to Matiu's and Tane's response (Line 557). The teacher used a falling tone with "great" to signify the completion of the task of naming the 3D triangle in geometry language (Jeong, 2016). However, Garry (a bilingual Filipino student) in Line 558, used high volume majorly for two purposes: (i) to draw the teacher's attention to his suggestion of 'cone' as the name for shape in question, (Gries & Miglio, 2014); and (ii) as an attempt to continue the discussion on the possible geometry term for the shape by engaging in a parallel talk, a characteristic of bilingual Filipino students (Speicher, 1993). Speicher (1993) showed that Filipino students often engage in parallel or simultaneous talk to offer their explanation without further delay.

It is evident that though Yue (a bilingual Chinese student) used English as a medium to state her response, she was still learning to use intonational patterns used in English. Moreover, Garry, (a bilingual Filipino student) used multilingual capabilities of the English language and Filipino language to display his intentions. The micro-analysis of this key moment also draws our attention to the different ways multilingual students employ their

language repertoires from a different language to convey their meanings as they participate in classroom interactions.

Macro-level analysis

For this level of analysis, each key moment was coded in terms of: (i) participants, (ii) discourses used, and (iii) emotional registers used (see Table 1). The analysis of the two key moments suggests that a constant struggle between unitary language and heteroglossia can be observed at the overlapping dimensions of language and discourses.

Table 1
Macro-level analysis of Key Moments

Participants	Key moment	Discourses	Emotional registers used
Lily, Amir, Researcher, Liu	1. *" a three-d square."*	Everyday language (e.g., corners); Geometry specific language (e.g. square)	Authoritative (e.g., Amir- line 17), Uncertainty (e.g., Lily- line 16),
Teacher, Yue, Ethan, Matiu, Tane, Garry	2. *"a triangular prism."*	Everyday language (e.g. 'triangle 3D'); Geometry specific language (e.g. triangle)	Authoritative (e.g., Tane- line 555),

On the dimension of language, both classes catered to multilingual students with varying degree of proficiency in their different languages, including English. New Zealand Curriculum (Ministry of Education, 2007) encourage and uphold the value of diversity of languages (p. 10), thus, promotes the use of *Te Reo Māori* in classes. However, the use of English as the medium of instruction in both multilingual classes highlights the unitary language forces. It is interesting to note that the heteroglossic language forces are evident in the ways in which students made use of their prosodic features from their multilingual repertoire. For example, the use of HRT by Lily (Line 28 in Excerpt 1) to check the conversational solidarity with other members highlights the influence of *Te Reo Māori*, (an Indigenous language of New Zealand with an official status gained in 1987) on her English (Stubbe & Holmes, 2000). Similarly, Yue's utterance (Line 551, Excerpt 2) display the ongoing interplay of (i) centrifugal force embedded in her use of intonations (use of flat pitch), and (ii) the centripetal force of using English as a medium of communication.

On the dimension of discourse, two different discourses are at play in both the key moments. In both the key moments, participants used everyday language and geometry-specific language to display their understanding of three-dimensional shapes. In Excerpt 1, Lily started her utterances using geometry-specific language (e.g., "square" and "edges"). The use of geometry-specific language directs our attention to the embeddedness of unitary language force in her utterances. It is possible that she used these specific terms to keep her utterances in alignment with the dominant geometry discourse. However, the use of "corners" (Line 12, Excerpt 1) shows the embedded heteroglossia as the word corner" is laden with geometry meanings as well as everyday meanings. Moreover, the use of terms like "normal square" and "3D square" in her later utterances also draws our attention to the ongoing dialogic tensions between the centripetal and centrifugal language forces. The phrase of "normal" with "square" uses both everyday language and geometry-specific language. "Normal" implies an everyday understanding of "square", however, by specifying "square" Lily shows her geometry understanding of shape as a four-lined shape. Similarly, in Excerpt 2, the teacher's utterance (Line 548) also highlights the presence of heteroglossia and unitary language force. The phrase "triangle three d" emphasises the understanding that

it is a triangle shape, which is three-dimensional. Thus, the geometry unitary language forces are used to define the shape as "triangle", yet the meaning of "three d" implies a solid shape, highlighting heteroglossia by providing it with everyday meaning. The intent of the teacher's utterance (Excerpt 2, Line 548) was to promote the use of geometry-specific language, that is, to direct students' utterances to align with the dominant geometry discourse.

The analysis highlighted two noteworthy findings. First, it was found that multilingual students can make use of subtle yet significant prosodic features from their repertoire of multiple languages to display their meanings during peer and classroom interactions. Second, the students construct three-dimensional shapes in reference to the two-dimensional shapes that they know. It is interesting to note that the students and teacher did not question the idea of "3D square" or "3D triangle"; instead, the meanings of these terms were discursively constructed in those particular moments. The analysis showed that the use of these terms in students' utterance is confident, signifying their authority over their knowledge. Thus, it can be argued that students used these discursive constructions not only to make sense of shape but also to negotiate the meaning of it as they engaged in the conversation.

Conclusion

The paper explored discursive constructions of the multilingual (9 to 11-year-old) students as they engaged in group and whole-class interactions. The paper reported on two excerpts from two New Zealand primary classrooms. The use of multilingual repertoire by multilingual students draws our attention to the growing need to develop understanding of these nuances to better support the practices of teaching and learning. Moreover, Bakhtin's dialogic theory enabled me to explore the discursive constructions and reconstructions that students used to display their understanding of geometric shapes. It is evident that a variety of meanings may emerge as the interaction proceeds. The present analysis contributes to the knowledge base in geometry education classroom-based research, specifically in relation to multilingual classrooms. Moreover, this present exploration of the multilingual aspects of primary classes will be fruitful in developing a diverse knowledge base for teachers and researchers for promoting effective teaching and learning practices.

Appendix A- Transcript key

↑ Higher Pitch	↓ Lower Pitch	> < Faster talk	^ Whispering	. Falling Intonation	(.) Silence for 1/10th of second	# Creaky Voice
: Stretch	[ ] Overlaps	< > Slower talk	Underline- Emphasis	? Rising Intonation	CAPs Volume Increase	(n.0) Silence for n seconds

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"*I think it's 3D because it's not 2D*": Construing dimension as a mathematical construct in a New Zealand primary classroom

Shweta Sharma
University of Waikato
<ss555@students.waikato.ac.nz>

The paper explores one episode from a larger study, where a multilingual student (10-year-old) described her understanding of what makes a shape 2D or 3D. Bakhtin's dialogic theory and Garfinkel's ethnomethodology inform the theoretical framework. Transcribed data of the episode is presented, which is analysed at micro-level and macro-level. The analysis revealed three major findings. First, the analogy of "flat vs fat" for describing shapes may not be useful. Second, the meanings of the terms such as fat are shaped by the interaction of unitary language and heteroglossia. Third, prosody embedded in utterances contribute to the meaning constructions during mathematical discussions. A few implications are presented.

The mathematical construct of dimension plays a crucial role in developing foundational skills in mathematics, more so for construing understanding of two-dimensional (2D) shapes, three-dimensional (3D) shapes and their properties. This paper presents an episode from a larger study from a New Zealand Year 5/6 geometry classroom, where a student described her understanding of what makes a shape 2D or 3D. In school geometry, 2D shapes are the plane shapes that have only two dimensions that are length and breadth. Whereas 3D shapes are the solid shapes that can be held, have thickness/depth along with length and breadth. These definitions of 2D and 3D shapes focus on the dimension as a measurement attribute of an object. In mathematics education research, very few studies have explored students and teachers' understandings of dimension. Lehrer et al. (1998) found that students construct dimension as a property of thickness of an object, a finding supported by Morgan (2005). Recently, Panorkou and Pratt (2016) investigated 10-year-old students' construction of ideas about the dimension, and provided additional understanding of dimension. They argued that children expressed that objects/spaces with lower dimensions can move within objects/spaces of higher dimension. Tossavainen et al. (2017) studied pre-service teachers' understandings of the area and its dimensional aspect. They argued that although the concept of area is central in elementary mathematics, the aspect of two-dimensionality is hardly considered in teaching and learning of shapes and their areas. In addition, they highlighted that the use of the same word for the boundary of the shape as well as the space within the shape might add to the difficulty in construing dimension as an important attribute for understanding shapes. For example, Bezgovšek Vodušek and Lipovec (2014) have shown that in Slovenian language, the boundary of circle is not considered as a 2D shape, and is called *krožnica*; whereas, a disk is a 2D shape of a circle, and is called *krog*. Interestingly, these two terms *krožnica* and *krog* highlight the understanding of shapes from Euclid's boundary notation perspective (Manin, 2006; Skordoulis et al., 2009). According to this perspective, the points are the boundaries of lines, lines are boundaries of the surface, and the surface accounts for the boundary of the solid object (Skordoulis et al., 2009). As a result, the dimension of, let us say, *krožnica* (circumference of a circle) and *krog (circular region)* would be different. However, in English, a "circle" is used to signal both the boundary and the area (as a disc) enclosed by the circle, which may complicate the process of understanding dimension as a mathematical construct.

Interestingly, research investigating students' understanding of dimension in multilingual classrooms are even rarer. Multilingualism research has often focussed on either

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mathematical terminology (e.g., Adler, 2002) or grammatical patterns of mathematical registers (see Kotsopoulos et al., 2015). This focus on mathematical terms and syntax negate the role of prosodic features of language use that might contribute to meaning-making. Ward (2019) argued that patterns of stress and intonation in language provide impact to words, their meanings, and their social significance. For example, Hay et al. (2008) have shown that New Zealand speakers often use a High Rising Terminal (HRT) intonation in their speech to show solidarity with the listener and to check if the listener is following the speaker, instead of asking a question. Thus, to explore students' understanding of dimension while acknowledging the superdiverse context of New Zealand, this paper aims to answer the following research question: *How do Year 5/6 multilingual students discursively construct and reconstruct their understanding of dimension during classroom interactions?*

Theoretical Framework

The theoretical framework for the larger study was informed by Bakhtin's (1981) dialogic theory and Garfinkel's (1967) ethnomethodological approach. Bakhtin (1981) argued that language provides us with a dialogic space, which opens shared space for all participants to generate meanings as they engage in dialogue, that is in this dynamic space, all possible meanings are considered in a continuum of meaning construction. The specification of meaning is dependent upon the preceding and succeeding dialogues. Bakhtin argued that this negotiation of meanings occurs in the realm of the constant struggle between unitary language and heteroglossia that operate concurrently at different levels of interaction. The *unitary language* (unifying language force) account for the system of norms that dictate the accurate use of language, with the aim to guarantee mutual understanding of the meanings of utterances by crystallising their meanings, thus, limiting the occurrence of divergent meanings of the utterances. At the same time, *heteroglossia* (diversifying language force) attempt to decentralise the already established meanings of the utterances by embedding the use of language with individualised meanings. It is the ongoing play of these unifying and diversifying language forces in a specific circumstantial context as well as the socio-cultural milieu that informs the specific meaning of an utterance within a sphere of communication (Barwell, 2018). Exploration of what is said, when it is said, and how it is said can enable access of these heteroglossic and unitary language forces. To explore what, when, and how an utterance is said, the paper made use of the ethnomethodological approach. The ethnomethodological description, therefore, aims to provide a detailed description of how members make sense of any activity as it unfolds in its everyday manner. Undertaking this theoretical approach helps us to acknowledge that knowing is construed as an ongoing action that takes place within the ongoing interactions. This paper aims to explore the processes through which a participant displayed and developed her conception of dimension from moment to moment as she participated in classroom interaction, on the one hand; while developing a critical understanding of dominant discourses that influenced this process of meaning-making in multilingual mathematics class, on the other.

Methodology

This paper presents one episode from a larger study. In this episode, a 10-year-old student discursively constructed her understanding of dimensions in a New Zealand primary classroom. Informed consent was sought from the participants. Six geometry lessons on shapes and their properties in one Year 5/6 New Zealand classroom were observed, and field notes were taken. Participants included 15 students (with nine multilingual students) and

their teacher. Data pertaining to students' languages were collected using a small questionnaire that was filled in by the parents. The observed six lessons were also audio- and video-recorded using two directional cameras, one eye-gear, and five audio-recorders. One camera was kept in the front of the classroom, and one at the back. The eye-gear/glasses with an inbuilt camera was used to record any moment of interaction that caught attention. Five audio-recorders were kept on the tabletops to records students' interaction as they worked on group tasks. Each lesson lasted for 45 to 50 minutes. In addition to audio-visual data from the lessons, three short (10-12 minutes) semi-structured interviews were conducted with the teacher, which were audio-recorded, transcribed, and sent back to the teacher for member checking. Each teacher interview focussed on seeking clarifications about the lessons or activities if there was any question. Four focus group interviews (with four students in each group) were also conducted once all six lessons were taught. Each focus group interview lasted for 18 to 20 minutes. Students' work samples were also collected.

The audio-video recorded data was the primary data set for the study, and participants' utterances as units of analysis. Field notes and repeated watching of six video-recorded lessons enabled the identification of the relevant moments, where participants displayed their understanding of dimension. This paper presents one such moment of classroom interaction when a student, Elie (pseudonym), and her teacher displayed their understanding of dimension as they engaged in classroom interaction. These relevant moments were then analysed at two levels: micro-level and macro-level (see Figure 1). The micro-level analysis formed the basis for macro-level analysis.

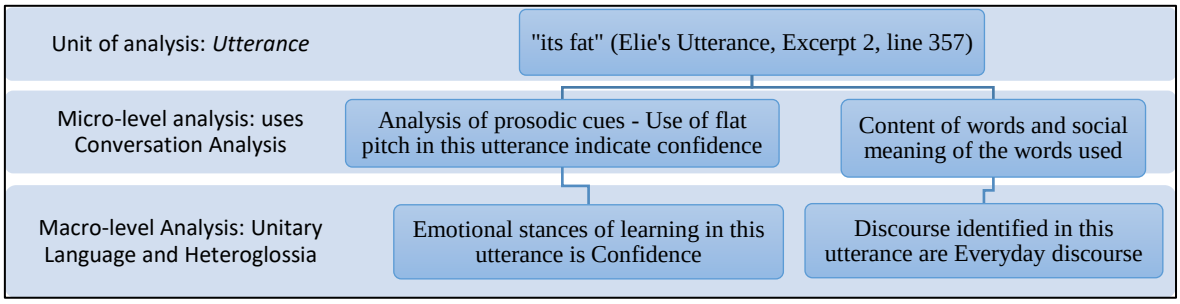

Figure 1. Example of data analysis procedure

At micro-level analysis, the selected moment was analysed using selected features of Conversation Analysis. The moment was transcribed using a simplified version of Jefferson's (2004) transcript convention. The analysis of participants' utterances explored how participants constructed their utterance using linguistic (including words and grammatical forms) and paralinguistic (that includes prosodic features of the pitch, silence along with gestures) features to convey their intended meaning and action (e.g., declaring their understanding, asking a question, seeking confirmation). Thus, the analysis at the micro-level focused on *what* and *how* discursive constructions were made. Following the micro-level analysis, the same moment was analysed at the macro-level. At this level, based on the prosodic analysis at the micro-level analysis, emotional stances embedded in participants' utterances were identified (Sullivan, 2012). In addition, the words used in the utterances with their intended socio-historical meanings within the utterance enabled the identification of the discourses (Sullivan, 2012).

Findings

During the classroom observations, it was noted that students often stated their understanding of dimension using a flat vs. fat analogy, as expressed in one of the students' utterance:

> "D is dimension. Two d is flat and three d is fat. Three d has a lot of stuff. Like a three d has some stuff in it. Two d is like flat and it has nothing. It's like his, his body was like he just, it's like squished over from the car" (Ozan, Focus Group Interview 1)

In this moment, the meanings that a student may imply when she distinguishes between 2D and 3D shapes as she participated in whole-class interaction are explored. During this lesson, the teacher organised students in groups and provided them with sticks and glue to make the shapes that they already knew. After the group work, the teacher invited one student at a time to describe the shape that they had made using "the language of geometry" (Teacher, Field notes, Lesson 2). In the moment presented in this paper, she invited Elie to describe the shape she made using sticks and glue. Questionnaire data revealed that Elie is a bilingual student with more proficiency in English than *Te Reo Māori* (an indigenous language of New Zealand, which gained the official status in 1987). She made a hexagonal shape (see Figure 2). The teacher then asked Elie if her shape was 2D or 3D (See transcript below, line351).

Figure 2. Elie making shapes with sticks and glue.

351	Teacher:	Elie just hang on a minute (.) is it three d: or two d: (1.0)
352	Elie:	um:: I think its three d because °its not (.) a two d°
353		((she was holding the shape and rolling it around her finger))
354	Teacher:	put it down on a on the grou:nd (1.0) is it (.) flat (.) or fat (0.5)
355	Elie:	its fat (1.5)
356	Teacher:	its fa:t (.) is it ↑coming ou:t towards you (1.0)
357	Elie:	((looks at the shape holding it near the eye level))
358	Teacher:	=okay lay it on the grou[nd (1.5)
359	Kimi:	[°no its flat°
360	Teacher:	its its okay. so: its not actually coming out of the ground or going through
361		the grou:nd (.)so we call so we call (.) ↑we call that a two
362		d? (0.5) okay [so:(.2)
363	Elie:	[↓uhm::

The teacher realised that Elie had not mentioned if the shape was 2D or 3D (line 351). To this question, Elie responded that the shape that she had made was 3D (line 352-353). Elie used a flat pitch for the first half of her utterance and a whispery voice (whispering voice is denoted by degree sign, ˚) for the second half. Research has shown that English speakers may use a flat pitch to display their authority or confidence (Ward, 2019). However, a whispery voice at the end of utterance may indicate a lack of confidence (Ward, 2019). Thus, the use of flat and whispery voice in the same utterance may indicate that Elie was partially confident of her claim. Field notes show that the teacher explained the difference between 2D and 3D shapes, as "*two d is flat. three d is fat. two d, straight onto the ground,*

three d, you can hold it, its fat, its solid" (Field notes, Lesson 1). The video-recorded data showed that Elie was holding the shape and spinning it around her fingers. It is possible that Elie understood the shape that she made was 3D as she could hold it. In the following utterance, the teacher asked Elie to put the shape on the ground (line 354). As the teacher did not repeat Elie's previous utterance or used markers like "good girl", it is probable that the teacher evaluated Elie's response as incorrect (line 354). Moreover, she stretched "ground" (stretching is denoted by a colon, :) to emphasise it, paused for one second (gap in utterances is denoted by (1.0)) probably to provide Elie with a cue. The field notes inform that during this activity, the teacher often stated that if the shape was coming out of the ground, it was 3D, otherwise, 2D. It seemed the teacher intended to use the same idea to help Elie to identify the shape as 2D. The teacher rephrased her question and asked Elie if the shape was flat or fat (line 354). The teacher did not emphasise 'fat' or 'flat' in her utterance. This lack of emphasis may imply that the teacher was expecting Elie to recall the 'flat vs. fat' distinction of shapes. It was noted in the field notes, and video-recorded data (Lessons 1 to 4) that the "flat vs fat" analogy was often used in this class to describe 2D and 3D shapes. To this question, Elie (line 355) responded that the shape was fat. Elie's flat pitch suggests that Elie was sure of her answer (Ward, 2019). The teacher waited for 1.5 seconds before constructing her turn, and then repeated Elie's response (line 355-356); however, she stretched "fat" for emphasis. Hellermann (2003) has shown that silence in between turns can be interpreted as the current speaker's (in this case, the teacher) orientation to the previous speaker's (in this case, Elie) utterance as a dispreferred response. Moreover, the teacher used different intonation patterns (line 356) with the same words (see fa:t) used by Elie (line 355). The use of different intonational patterns with the same words often imply contrast rather than agreement (Hellermann, 2003). Thus, it seems that the teacher again evaluated Elie's response as incorrect. Therefore, she again provided Elie with feedback to reconsider her response (line 356). The video-recorded data inform that Elie held the shape at her eye level instead of verbally responding (line 357) to the teacher's feedback in the previous turn. This may be interpreted as Elie's way of restating that the shape is 3D as she could hold the shape in her hand. Noticing this, the teacher (line 358) asked Elie to put it on the ground. As the teacher was talking to Elie, Kimi (a female, 10-year-old, Tongan student) self-selected and offered a repair on Elie's turn. Kimi structured her response in whispery voice (line 359), so that she did not interrupt the flow of conversation (Hay et al., 2008), a different way of using whispery voice than English speakers. The teacher attempted to build an understanding of the shape as 2D with Elie (line 360-362). She used the High Rising Terminal (HRT) (denoted by a question mark, ?) (line 362) as a way to overcome a barrier to comprehension and build solidarity (Hay et al., 2008). Therefore, through her utterances, the teacher attempted to develop a mutual understanding with Elie, as she explained that the shape was not "coming out of or going through the ground" (line 360-362). Interestingly, the teacher used "so we call" (line 361) twice in her utterance; the use of this phrase could be interpreted as her acknowledgement of the possibility of non-confirmation from Elie. Ward (2019) has noted that high onset (denoted by ↑) is often used in conversations to mark a change or draw attention to the topic of conversations. Thus, the teacher's use of high onset (line 361) with "we call" may be interpreted as intended action to change the topic of discussion.

It appears that the teacher realised that Elie was probably not convinced with her explanation; thus, the teacher attempted to change the topic of discussion. Elie picked up the cue in her utterance as in the following turn (line 363), Elie used 'uhm' as a hedging device probably to convey that she is not convinced (Drew, 2013). Ward (2019) has shown that low/falling pitch (denoted by ↓) may also be interpreted to show declining interest in

continuing a discussion. Thus, Elie's use of low pitch in this context may be interpreted as her way to indicate that she was not interested in carrying on with the conversation.

The micro-level analysis suggests, first, multilingual students may use prosodic features of repertoire of their multiple languages. Second, the micro-level analysis suggest that the analogies of "flat vs fat" shapes, and "shapes coming out of the ground" as 3D shapes may not be helpful for some students to understand dimension as a mathematical construct. Similar to Elie, another student also displayed his thinking about fat and flat while making shapes using play-dough. He stated that "it's like all three d you can't like make…not make a fat" (from another relevant moment of group interaction, Lesson 2).

During the presented moment, the teacher focussed on using geometry-specific language to describe the shapes that students made. However, what counted as the geometry-specific language for participants was constructed during the classroom interaction. The macro-level analysis focussed on the interaction of unitary language and heteroglossia to explore what discourses contributed to the meaning of geometry-specific language. Two dominant discourses can be identified in the moment analysed here: Eurocentric-Academic Discourse and Everyday Discourse. The use of terms like "2D", "3D" displays the use of Eurocentric-Academic Discourse, whereas the use of "coming out of ground" display the use of Everyday Discourse. Interestingly, the use of "flat vs fat" analogy may suggest the use of Eurocentric-academic Discourse as well as Everyday Discourse. The heteroglossia can be in located in the different meanings that can be drawn from these two terms. For example, "flat" can imply either smoothness of surface without any depth from the Everyday Discourse perspective; or a very thin object like paper cut-outs that are often used as resources in geometry classes for teaching 2D shapes from the Eurocentric-Academic Discourse. The use of term "fat" could mean thick, thin, or something that can be held in Everyday Discourse, and in case of geometry teaching within the use of Eurocentric-Academic Discourse may mean 3D shapes. In this moment, it is interesting to note that the unitary language force supported different discourses during different micro-moments within this interaction. The unitary language force supports the Eurocentric-Academic Discourse in teacher's utterance (line 354) as she asked Elie if the shape was flat or fat. Through this utterance, the teacher seemed to use the analogy of "flat vs fat" for identifying shapes as 2D and 3D. The Eurocentric-Academic Discourse supports the use of "flat" for 2D shapes and "fat" for 3D shapes. However, in the following utterance, Elie stated that the shape is fat (line 355). Elie's utterance highlights the heteroglossia embedded in her utterance. Based on the micro-level analysis, it seems that Elie construed the shape that she made as 3D as she could hold the shape and could see its slight thickness. It seems that at this moment, the unitary language force supported the use of Everyday Discourse instead of Eurocentric-Academic Discourse for keeping the flow of conversation. The interaction of unitary language and heteroglossia within the use of "flat vs fat" analogy highlights that these words are laden with geometric meanings as well as everyday meanings, which highlights the heteroglossia.

The micro-level and macro-level analysis of the moment presented here highlights three main findings. First, the analogies of "flat vs fat" shapes and "shapes coming out of the ground" may not be helpful for some students to understand dimension as a mathematical construct. Second, the macro-level analysis suggested that the meanings of terms "flat" and "fat" are constructed within the conversational moment, and the meaning of the term may be shaped by any of the discourses supported by the unitary language forces at a particular moment of interaction. Thirdly, the analysis suggest that prosodic features play a crucial role in meaning construction. Importantly, multilingual students may use prosodic features from their multiple languages, which may be differently used in English.

Discussion and Conclusion

In mathematics education research, the construct of "dimensions" has rarely been explored, even when the studies have focussed on geometric shapes and their properties (e.g., Lowrie et al., 2017; Seah & Horne, 2019). It appears that the mathematical understanding of dimension is often taken to be understood without explicit teaching. This study adds to the research literature exploring students' understanding of dimension at the Primary level. The analysis revealed that the student may use a "flat vs fat" analogy for explaining what D stands for in 2D and 3D. This finding is consistent with the research done by Morgan (2005) and Lehrer et al. (1998). Both these studies reported that students and teachers often describe "dimension" as one of the mathematical words that concern with the thickness of the shape. However, the analysis presented here suggests that the use of "flat vs fat" may not be useful for describing the dimensional property of shapes. The difficulty may be attributed to the two different understanding of dimension embedded in definitions of 2D and 3D shapes. First, defining 2D shapes as planar shapes and 3D shapes as solid shapes underscore the understanding of dimension from Euclid's boundary notation perspective (Manin, 2006). Second, defining 2D shapes as having length and breadth; and 3D shapes with length, breadth, and height underscore the understanding of dimension from a measurement perspective. This construction emphasises dimension as a measurement attribute and does not underscore the need for the "planes" to understand dimension. The analysis presented here underscore a need for developing a comprehensive understanding of what dimensions imply, as a mathematical construct, in curriculum documents; so that its understanding can be translated into teaching and learning of shapes including dimensions. Moreover, a comprehensive understanding of dimension may also help teachers and students to acknowledge the context within which the idea of dimension is used, and for what purposes. Hence, the study suggests future opportunities for further research in this area.

The second finding revealed that the meanings of the terms such as fat are shaped by the interaction of unitary language and heteroglossia. From a Bakhtinian perspective using unitary language and heteroglossia, it can be argued that the meaning of the utterances is dependent upon the discourse supported by the unitary language force within the milieu of discourses available in any particular interactional moment. The unitary language is a *theoretical* language force that tends to homogenise the meaning of the utterance to facilitate the flow of interaction (Barwell, 2018). It was evident in the data that the unitary language force may support either of the discourses depending on the interactional context. Therefore, providing different meanings to the same word as and when embedded in different discourses. Barwell (2013) claimed the participants may treat an everyday term mathematically during a particular interaction. Thus, the students engaged in using their everyday language in the form of mathematical language, embedding everyday words with mathematical ideas.

The third finding from this study highlights the role of prosody that contributes to the meaning constructions as participants engage in mathematical discussions. For example, the study supports the findings that of the use of low, high, flat pitch are some of the interactional devices that participants use to draw listeners' attention to the focus of their utterances (Reed & Michaud, 2015). Also, the study suggests that multilingual speakers engage in the practice of languaging that involves the use of prosodic features conventions from the linguistic repertoire of different languages along with the words used; instead of just engaging in a practice of code-switching to meet their needs to communicate their understandings in a particular interactional context. The research focussed on language as a resource perspective (Adler, 2002; Moschkovich, 2007) often ignores the role played by the prosodic repertoire

in contributing to the meanings conveyed in the utterance. Therefore, this study calls for further research on interactive practices that may support teaching and learning of geometry. The study also suggests that it is important for teachers to become aware of the subtleties of prosodic features of language that have an impact on the meaning-making process and learning of mathematical ideas.

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The development and validation of two new assessment options for multiplicative thinking

Dianne Siemon
Royal Melbourne Institute of Technology
<dianne.siemon@rmit.edu.au>

Rosemary Callingham
University of Tasmania
<rosemary.callingham@utas.edu.au>

Lorraine Day
University of Notre Dame Australia
<lorraine.day@nd.edu.au>

The capacity to recognise, represent, and reason about relationships between different quantities, that is, to think multiplicatively, has long been recognised as critical to success in school mathematics in the middle years and beyond. Building on recent research that found a strong link between multiplicative thinking and algebraic, geometrical, and statistical reasoning, this paper will describe the development and validation of two new assessment options for multiplicative thinking and discuss the significance of this for the teaching and learning of mathematics in the middle years of schooling.

Introduction and Theoretical Background

Multiplicative thinking has long been recognised as a necessary foundation for fractions, rate, ratio, percentage, and proportional reasoning in the middle years (Harel & Confrey, 1994; Siegler et al., 2012; Vergnaud, 1988). However, at least 30% and up to 55% of Australian Year 8 students have not developed this critical facility (Siemon, et al., 2018a). While research suggests that formative assessment can be a powerful means of improving student learning (Black & Wiliam, 1998), it would appear that this is more difficult to implement than previously thought (Smith & Gorard, 2005; Swan & Burkhardt, 2014; Wiliam & Leahy, 2014). Hodgson et al. (2014) suggest that one of the reasons for this may be that "formative assessment has been described generically rather than in subject-specific terms" (p. 168). But even where evidenced-based, subject-specific formative assessment materials have been developed, they are not necessarily taken up where schools feel pressured to prepare for high stakes assessment (Wiliam et al., 2004) or teachers lack the depth of knowledge needed to provide effective feedback (Hodgson et al., 2014).

Research-based formative assessment materials to support the development of multiplicative thinking were provided by the *Scaffolding Numeracy on the Middle Years* (SNMY) project in 2006. The materials include two validated assessment options and a *Learning Assessment Framework* (LAF) for multiplicative thinking that incorporates an evidenced-based learning progression and targeted teaching advice. They are appropriate for use in Years 4 to 9 and offer a valid means of identifying starting points for teaching and tracking learning over time (Siemon et al., 2006).

While the SNMY materials have been used quite widely in coaching and professional development programs, their use in secondary schools is not widespread. One of the reasons given for this is that secondary teachers do not see that multiplicative thinking is something that is relevant to what they believe they have to teach (Siemon, 2016; Siemon, Banks, et al., 2018). A phenomenon that Arnett et al. (2018) have referred to in terms of the 'job to be done', even though a large proportion of the mathematics curriculum at this level is dependent upon multiplicative thinking (Siemon, 2013).

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, pp. 369-376. Singapore: MERGA.

Mathematical reasoning is another aspect of the curriculum which is not seen as a focus of mathematics teaching in middle years even though it is recognised as an important proficiency in the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment & Reporting Authority, 2016). A funding opportunity in 2014[1] afforded the possibility of investigating the development of evidenced-based learning progressions and teaching advice for algebraic, geometrical, and statistical reasoning that could be seen to be more related to the curriculum and thereby more relevant to the work of secondary school mathematics teachers. Known as the *Reframing Mathematical Futures II* (RMFII) project, this also provided an opportunity to explore the extent to which *multiplicative thinking* (MT) was related to *mathematical reasoning* (MR) by including a number of tasks from the SNMY project in the trials of the MR assessment tasks and by collecting data on both MT and MR from project schools who had not participated in the earlier *RMF-Priority* project in 2013 (Siemon, 2016).

The outcomes of the RMFII project have been reported elsewhere (Siemon et al., 2019; Siemon, Callingham et al., 2018), but as the analysis of RMFII trial data suggested a strong relationship between MT and MR, a secondary analysis of these data together with combined data from the RMFII project and archived data from the original SNMY project was conducted to test the extent to which this link could be empirically established. This process resulted in the development and validation of a single, integrated scale for multiplicative reasoning that incorporated the scale for multiplicative thinking and the scales for algebraic, geometrical, and statistical reasoning (Callingham & Siemon, 2021).

At around the same time, the *Growing Mathematically – Multiplicative Thinking* (GM-MT) project was initiated by the Australian Association of Mathematic Teachers for the purpose of trialing a Teacher's Manual that could be used as a stand-alone guide to support the use of the SNMY formative assessment materials in secondary schools. The project team comprised the Chief Executive Officers of the Australian Association of Mathematics Teachers (past and present), three members of the RMFII team (the authors of this paper), and a representative of Australian Curriculum and Reporting Authority. Given evidence of the strong relationship between MT and MR, it was agreed that this opportunity would be used to trial two new assessment options for MT that included MR items from the single scale for MT and MR. As a result, an application was made to amend the ethics approval in place for the ongoing data analysis work of the RMFII project to cover this aspect of the GM-MT project. The purpose of this paper is to describe the processes involved in developing and validating the new options and, in doing so, to address the research question: To what extent is it possible to develop valid assessments of multiplicative thinking that incorporate aspects of algebraic, geometrical, and statistical reasoning?

Research Approach

The work to be reported here was made possible by the *Reframing Mathematical Futures Priority* project and the RMFII project both of which explored the efficacy of using the SNMY materials in secondary schools alongside the development of the evidenced-based formative assessment materials for mathematical reasoning. As indicated above, the details of this work have been reported elsewhere, however it is important to acknowledge all three projects were framed in terms of a social constructivist view of learning that acknowledges

[1] The Australian Mathematics and Science Partnership Program was funded by the Australian Government Department of Education and Training from 2013 to 2017

the need to identify and build on what is known (e.g. Cobb & Yackel, 1996; Shepherd & Penuel, 2018).

The RMFII project used a design-based research approach (Barab & Squire, 2004; Cobb et al., 2003) involving iterative rounds of assessment and the use of Rasch modelling (Bond & Fox, 2015) to scale assessment items from easiest to most difficult for the purposes of developing, testing, and refining learning progressions for mathematical reasoning (Siemon, Callingham, et al., 2018). A similar approach was used in the GM-MT project to evaluate the two new assessment options for MT. Interested schools were recruited through Australian Association of Mathematics Teachers in 2019 and asked to administer and assess one of the options using the scoring rubrics provided and return the de-identified results to the project team via an excel spreadsheet. Initially 25 schools agreed to participate in this process and use the other option as a pre-test at a later date, but COVID restrictions limited the extent to which schools could contribute to the GM-MT data set and provide pre- and post-test data in 2020.

Item Selection

The two options, referred to as Option 3 and Option 4, had to be compiled such that they could be statistically linked to the existing SNMY data set for validation purposes. With these constraints in mind, the tasks (each of which comprised at least one item) were chosen from the pool of 113 validated assessment items used in the SNMY and RMFII research projects. A number of tasks from SNMY Options 1 and 2 were included to provide links among the projects. Consistent with the structure of the existing SNMY Options, an extended task and a number of shorter tasks were included in each of the new Options. As there were strong conceptual links between the SNMY and algebraic reasoning, the new extended tasks both came from the RMFII. *Trains* (Option 3) used a series of increasingly complex questions to develop generalisations about the relationships between the number of wheels and the train design. *Board Room Tables* (Option 4) considered the relationship between the number of tables in a rectangular arrangement and the number of people that could be seated. Tasks from the SNMY pool were chosen because of clear links to geometric or statistical reasoning. *Stained Glass Windows* was set in a geometric context of a triangular tessellation. *Canteen Capers* drew on the Cartesian product to identify the number of possible combinations available from a school canteen, which has links to statistical reasoning and probability. Conversely, tasks from the RMFII project were chosen because of explicit use of multiplicative thinking, such as drawing names from a hat and expressing the answer as a fraction (SHAT8) and designing a package to hold a given volume of soft drink (GBEV1). All tasks, with the exception of *Skin Rash* (SRASH) and SHAT8, had multiple items. The two new Options had no overlapping items to maximise their utility as pre- and post-tests over the short period of time.

Two draft options were created (referred to as draft Option 3 and draft Option 4) and piloted in a small-scale trial for feasibility.

Pilot study

Although the numbers from the initial trial were small (n = 38; for draft Option 3 and n = 32 for draft Option 4), the Rasch analysis provided sufficient indicative information about the behaviour of both the complete draft Options and the individual items to decide whether or not they were working as intended. Each option was Rasch analysed separately to provide information about the extent to which the items worked together coherently to provide a

scale. Both assessments provided good fit to the Rasch model and showed high reliability. These findings indicated that the items used were suitable for alternative assessment Options.

Table 1
Summary Statistics for Individual Assessment Options

Option (No. of Items)	Infit (Items)	Infit zstd (Items)	Outfit (Items)	Outfit zstd (Items)	Reliability
Option 3 (17)	0.99	0.01	0.94	0.03	0.93
Persons (n = 38)	0.97	-0.04	0.94	0.12	0.87
Option 4 (19)	1.01	0.00	0.99	-0.03	0.90
Persons (n = 32)	1.04	0.11	0.99	-0.01	0.80

Note. Ideal values for Infit and Outfit are 1.00, and zstd = 0.00. Ideal reliability coefficient = 1.00.

Overall, draft Option 4 was much harder than draft Option 3. When appropriate cut scores were applied to identify zones, this option had no items in Zone 1 and only one item in Zone 2. A revised Option 4 was developed with one of the more difficult SNMY tasks (*Tiles, Tiles, Tiles*) replaced by an easier task (*Butterfly House*).

One issue that emerged was that, of the items developed for geometrical reasoning, few made explicit links to MT. As a result, two new questions *Enlarging Nets* (GENLG) and *Park Map* (GMAP) were developed to address perceived gaps in the geometric aspects of MT (i.e., scale and enlargement) at an easier level than those included in RMFII. These changes were incorporated into the revised Options that were then trialed as part of the GM-MT project with students from Year 5 to Year 10 in late 2020. Tables 2 and 3 show the revised task and item selection

Table 2
Tasks and Items for Option 3 Trial

Task	Source	Item Codes
Adventure Camp	SNMY	ADCA, ADCB
Stained Glass Windows	SNMY	SWGA, SWGB, SWGC
Relations	RMFII-Alg	AREL1, AREL2, AREL3
The Beverage Company	RMFII-Geo	GBEV1RA, GBEV1RB
Skin Rash	RMFII-Stats	SRASH
Trains	RMFII-Alg	ATRNS1, ATRNS2, ATRNS3, ATRNS4, ATRNS5, ATRNS5A, ATRNS6
Enlarging Nets	New	GENLG0, GENLG1, GENLG2, GENLG3, GENLG4

Table 3
Tasks and Items for Option 4 Trial

Task	Source	Item Codes
Butterfly House	SNMY	BTHA, BTHB, BTHC, BTHD
Canteen Capers	SNMY	CCA, CCB
Lemonade	RMFII-Alg	ALEM1, ALEM2
Hat Chance	RMFII-Stats	SHAT8

Spy Squad	RMFII-Geo	GSPSQ7, GSPSQ8, GSPSQ9
Board Room Tables	RMFII-Alg	ABRT2, ABRT3, ABRT4, ABRT5, ABRT6, ABRT7, ABRT8
Park Map	New	GMAPA, GMAPA1, GMAPB, GMAPB1, GMAPC, GMAPC1, GMAPD

These options were trialed by the schools participating in the GM-MT project.

Trial Analysis

As the purpose of the project was to extend the usefulness of the LAF, questions from that project were used as the anchor for the two assessment options. Because there were no common items across the two forms, a link file of student responses to the items that came originally from the LAF was created from archived SNMY data. Then all responses to Options 3 and 4 and the created link file were merged into a complete data set, so that the options were solidly linked through common items. Finally, to ensure that the existing LAF scale could be validly compared to the new scale from Options 3 and 4, an anchor file was created from the link items so that the new scale was, in effect, using the same ruler. Overall, there were 4494 responses included to provide maximum data about the scale.

Rasch analysis was undertaken using Winsteps v. 4.7.1.0 (Linacre, 2020). Summary statistics for the overall scale are shown in Table 4.

Table 4
Summary Statistics for Anchored Scale from Options 3 and 4

	Infit	Infit zstd	Outfit	Outfit zstd	Reliability
Item (n = 50)	1.00	-0.26	1.02	-0.07	1.00
Person (n =4494)	0.98	-0.03	0.97	0.09	0.82

Note. Ideal values for Infit and Outfit are 1.00, and zstd = 0.00. Ideal reliability coefficient = 1.00.

Following this trial, all the items were behaving as expected and the revised scale was interpreted using a process of ‘segmenting the variable’ (Wilson, 1999) as reported elsewhere (e.g. Callingham & Siemon, 2021).

Although the detail is too small to be seen clearly, a small part of the Wright map produced by the software (Linacre, 2019) for all trialed items is shown in Figure 1 to provide a sense of the approach used and the relationship between the MT items (blue), the RMFII-Alg items (yellow), RMFII-Geo and new geometrical reasoning items (green), and the RMFII-Stats items. The scale on the left-hand side is in logits, the unit of Rasch analysis. Items at the bottom of the map are easy whereas those at the top are difficult. Similarly, persons located towards the bottom of the map have performed less well than persons located at the top. Where persons appear at the same logit values as an item, they have a 50% chance of achieving the score allocated to that item. the Zones are marked by horizontal boundary lines. These borders are not “hard” borders. Rather, the zones provide an indication where students are in relation to the development of MT.

It is noticeable that the Geometry items are more difficult for students with no items appearing in the lower two Zones. This may be due to a lack of familiarity with geometric contexts, rather than inherent difficulty. Alternatively, the kinds of reasoning in geometry occurring in Zones 1 and 2 may rely less on numerical reasoning and more on visualisation.

In these Options only two statistics reasoning items were used, although aspects of the items from the SNMY project did draw on statistical thinking.

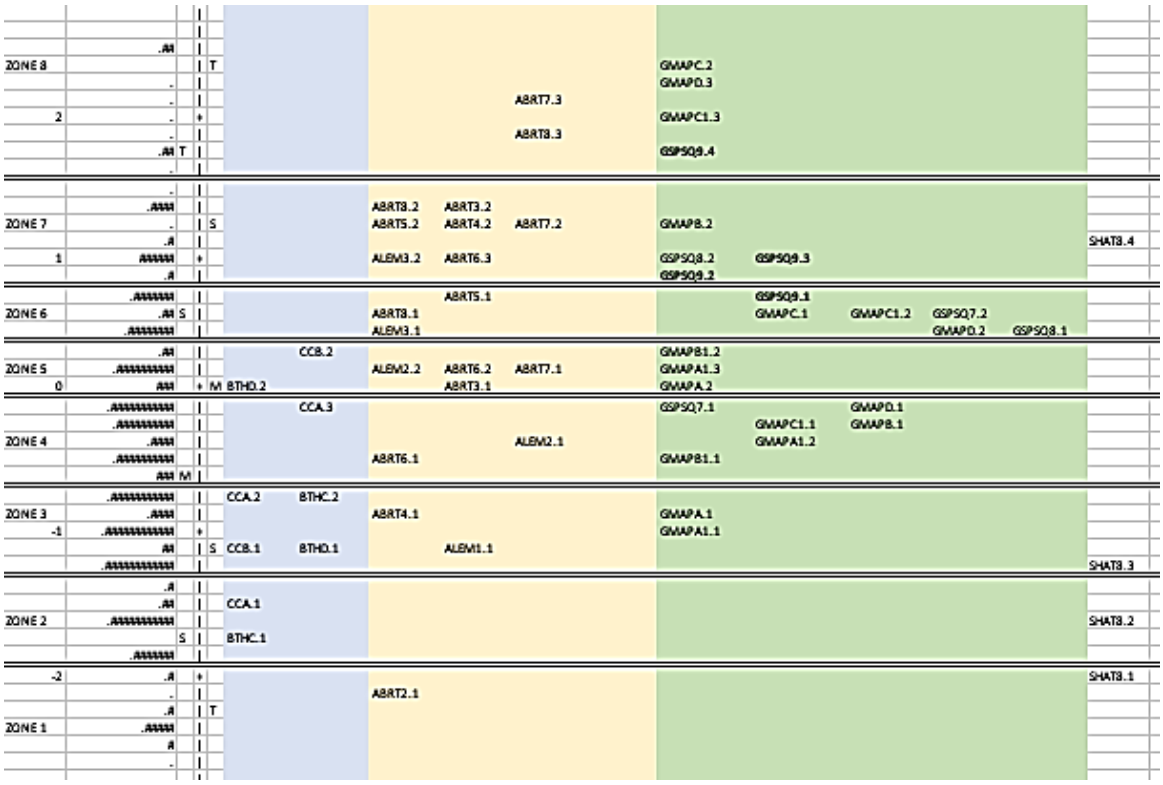

Figure 1. A portion of the Wright map from the GM-MT second trial

As part of the GM-MT project, participating schools were asked to use the assessment options as pre and post assessments to evaluate the efficacy of a targeted teaching approach to multiplicative thinking using the existing teaching advice from the Learning Assessment Framework for Multiplicative Thinking (LAF). While COVID restrictions significantly affected the number of schools who were able to provide matched data sets, the results suggest that the assessment options as trialed were working reliably and could be used to evaluate learning over time.

Discussion and Conclusion

The analysis reported in this paper has shown how assessment tasks used in previous research could be combined to create two new assessment options for multiplicative thinking that relate multiplicative thinking to algebraic, geometrical and statistical reasoning. Overall, the new scale performed in a manner remarkably similar to the existing LAF, meaning that the empirical thresholds could be retained, and the new assessment options can be used with confidence to place students within a Zone with sufficient accuracy to support targeted teaching. This is significant because secondary teachers are much more likely to see the importance of multiplicative thinking when they can visibly see its relationship to what they believe they have to teach, that is, algebra, geometry, measurement, statistics, and probability, and how this relates to mathematical reasoning more generally.

While further research and analysis is needed to test the extent to which the new options are more difficult than the existing SNMY options, this raises some questions. For instance, if it is established that they are more difficult, should the new options be 'flagged' as more appropriate for secondary students even though some primary school students participated in the project? As the GM-MT study was targeting the lower years of secondary schooling, is there any benefit in revising the existing SNMY options to include some of the more difficult RMFII and geometry items to better reflect the full extent to which MT is required for mathematical reasoning more generally? Should some easier reasoning type questions be developed for Year 4 to Year 6? These questions suggest there is room for more research in this area but the next step in the current process is to use the data obtained from the GM-MT trial to review and extend the original Learning Assessment Framework (LAF) for Multiplicative Thinking and to test the efficacy of using the revised framework to support a targeted teaching approach to multiplicative thinking in the middle years in a larger student population.

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Professional development and junior secondary mathematics teachers: Can out-of-field teachers benefit too?

Rebekah Strang
Griffith University
<r.strang@griffith.edu.au>

To foster effective teaching and quality education, professional development (PD) is imperative. This paper explores the perspectives of junior secondary mathematics teachers in two Queensland schools as part of a broader case study that investigated the perspectives of junior secondary mathematics teachers on the transition of Year 7 to secondary schooling in Queensland. The study explored existing PD practices and sought to identify perspectives of how PD can be better utilised to improve teacher quality. The theoretical lens of teacher identity was adopted, and the key findings indicated a lack of existing PD practices; the need for further PD; and a process for implementing mathematically focused PD in the future.

Professional development (PD), in any field of study, focuses upon changing and improving teacher quality and student achievement of those teachers already in the teaching profession; where individuals can update, revise, and reflect upon their current practices (Roland et al., 2014). Usdan et al. (2001) suggest that no singular facet of school reform is more impactful than the idea that student learning depends almost exclusively on the quality of teachers. Foster et al. (2013) identified PD as one of the few approaches by which teacher quality can be improved. This view is further supported by Kimmel et al. (1999) who state that "PD has become a key component for reform in teaching, learning and curriculum change" (p. 241); thus, in most educational jurisdictions PD must be undertaken on an ongoing basis. As the demand for quality education and the need to foster effective teachers is as crucial as ever, the challenge to incorporate timely and appropriate mathematics PD into schools is becoming increasingly apparent (Curtis, 2013). Not only will PD achieve this improvement of knowledge and broaden the scope of strategies utilised to teach mathematics effectively, it is also useful for supporting out-of-field teachers.

In recent times a shortage of teachers qualified to teach mathematics in secondary schools has occurred with a subsequent misfit between appointment, qualifications, and experience (Australian Mathematica Sciences Institute [AMSI], 2014). This shortage has led to secondary teachers teaching subjects that they are not qualified to teach, a practice known as teaching *out-of-field* (Hobbs, 2012b). Data from AMSI (2014) identified that teaching positions in mathematics are harder to fill than any other teaching positions and suggest that nearly 40% of mathematics teachers in junior secondary schools in Australia are currently underqualified to teach mathematics. Sharplin (2014) found that the experience of teachers and the degree of fit or misfit between teachers and their teaching load is a key consideration for successful student outcomes. Steyn and Du Plessis (2007) state that "out-of-field teachers struggle to teach effectively, which influences their perceptions on their professionalism, quality teaching and the extent of the success of their development in teaching as a profession" (p. 149). The literature reveals two predominant schools of thought in relation to out-of-field teachers. First is a proactive standpoint, whereby the focus remains on the improvement of the qualifications and content knowledge of out-of-field teachers to combat their lack of knowledge. Second, from a reactive standpoint, where schools are forced to remedy staff shortages by employing out-of-field teachers as the only viable option with little plan for PD. The underlying issue with the reactive approach is that utilising any registered teacher to teach mathematics classes does not work to eradicate the use of out-of-

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field teachers nor improve their knowledge (Lederer, 2004). As out-of-field teaching is complicated and prevalent in mathematics education in Australia, it is an important consideration when investigating PD.

Theoretical Framework

The theoretical framework underpinning this research is *teacher identity*, also known as "professional identity" that incorporates a range of teacher characteristics and considers their knowledge, ideals, principles and beliefs as educators and their pedagogical approach to teaching (Bennison, 2014). Teacher identity is an effective lens to examine the perspectives of teachers as it provides a robust foundation to successfully investigate qualitative data. This theoretical lens is particularly beneficial for this research as the information that is being sought from mathematics teachers and key stakeholders derives from their perspective and sentiments of themselves as educational professionals. Hobbs (2012a) suggests that teacher identity can work in conjunction with self-efficacy to reflect an individual's belief in their capacity to influence their environment in relation to motivation, behaviour, and particular performance accomplishments. Furthermore, a teacher's awareness of their own teacher identity affects their PD requirements, their ability and inclination to manage educational change, and how they innovate in their teaching practice (Beijaard et al., 2000). Therefore, to explore existing PD opportunities and to ascertain whether further PD is warranted and sought by the participants, the research question addressed in this paper herewith is: *Are current PD practices appropriate to support junior secondary mathematics teachers and out-of-field teachers of mathematics?*

Method

Data was collected from ten teachers and key stakeholders in two secondary schools in Queensland via a case study approach. Case studies offer opportunities to observe the emergent patterns and characteristics of a phenomenon with a view to establish generalisations regarding the wider population (Bassey, 1999). Although initially each school was to be considered as a separate case, the two schools have instead been considered together to form one case study. This change of approach occurred after the participants had been interviewed, in the analysis stage of research, since most of the responses were similar.

Participants

The qualitative data was gathered from ten participants comprising classroom teachers, the mathematics Heads of Department, and other key school stakeholders such as school leaders. The sample size consists of six participants from School 1; and four participants from School 2 to provide a suitable and practical data set (Kumar, 2014). Table 1 outlines the background of each de-identified participant. The participants were recruited by each school and the background of each participant was not known prior to each interview, thus the researcher had no influence over the selection of participants. Each participant was asked questions pertaining to the same issues, with the aim of encapsulating an emergent pattern to maintain quality interview data (Diefenbach, 2009).

Table 1
Demographics and Qualifications of Participants

Participant	Age Range (years)	Time at School Site	Current Teaching Position	Qualifications
School 1				
Peter	40-50	6 years	Mathematics HOD	B.Ed. Grad Dip Ed (Math)
James	20-30	2 years	Y8 Maths Co-Ord	B.Eng (Civil). Grad Dip Ed (Math/Phys)
Rose	30-40	11 years	Y9 Maths Co-Ord	B.Sc. Grad Dip Ed
Violet	50-60	22 years	Y8-12 Maths Teacher	Dip Teach, then upgraded to B. Ed
Leigh	50-60	2 years	Y7 Maths Co-Ord	B. Ed (Primary P-10). Flying Start.
Dylan	30-40	2 years	Y7 Maths Teacher	B. Ed (Primary). Flying Start.
School 2				
Mary	40-50	9 years	Y7 Echo Facilitator	Dip Teach, then upgraded to B. Ed
Caroline	20-30	4 years	Y7-9 Maths Teacher	BPsySc. Grad Dip Ed. Masters Candidate
Dougal	40-50	3 weeks	Y11/12 Maths Teacher Dir of Academic Perf	B.Ed (Math/Science)
Carly	50-60	1 year	Y7-9 Maths Teacher	B.Sc / B.IT. Grad Dip Ed.

Note. Flying Start denotes a teacher who is primary trained brought in from state primary schools to teach Year 7 students in the secondary setting; Echo denotes a special program which aims to enhance gifted students.

The participants in this study were interviewed and were required to critically reflect and analyse their own identity as a teacher. The interviews conducted were semi-structured and lasted approximately thirty minutes. The participants were invited to provide their perspective about existing PD available to them and to ponder PD opportunities sought in the future. While there were set questions of focus supported by the literature, at times, further questioning was required in response to participant remarks, thus fostering an adaptable interview structure tailored to each participant (Wiersma & Jurs, 2005). In addition, interviewees were given the opportunity to discuss topics not suggested by the researcher.

Methods of Data Analysis

Interviews were transcribed by the researcher and data was analysed methodically via the six phases of analysis approach developed by Braun & Clarke (2006) where the researcher: 1) becomes familiar with the data; 2) generates initial codes; 3) searches for themes; 4) reviews the themes; 5) defines and names the themes; and 6) produces the report. Key themes were identified throughout the transcription process via the thematic coding approach. Once the data was coded, the procedure of repeat reading was completed to validate or challenge previous observed patterns.

Findings and Results

PD is a vital approach to improving the knowledge and pedagogical methods of mathematics teachers and this is a continuous process (Linder, 2011). As such, PD forms the main premise for pedagogical improvement. One of the participants, Dougal, was involved with PD at both school sites and had conducted PD for other schools in the past, and as such, his approach to PD is a core consideration. Dougal professes the importance of PD and stated that teachers should be trying to improve themselves by attending PD or investigate other ways to enhance their pedagogy. However, it seems that mathematics PD was undertaken sporadically in both schools and this is consistent with the view of Curtis (2013) who outlines the difficulty in implementing timely and appropriate mathematics PD into schools. See Table 2 depicting details on the context of the last PD undertaken by participants.

Table 2
Topic of Last Professional Development Attended

Participant	Topic Area
School 1	
Peter	General PD focused upon the Australian Curriculum (ACARA).
James	PD on Leadership held with Teacher Education Centre of Excellence (TECE).
Rose	No PD completed recently. Requested PD in external assessment marking for Year 11.
Violet	PD attended with Dougal (in house 3 weeks ago). Last year requested PD on mathematics aimed at developing students – however not approved.
Leigh	PD on ACARA and alignment with the school textbook (generic).
Dylan	PD on Behaviour Management.
School 2	
Mary	PD on mathematics games (Concrete games in all topic areas).
Caroline	PD undertaken recently with Dougal.
Dougal	No formal PD undertaken recently.
Carly	Last attended mathematics PD at previous school and it was run by Dougal. This was school separate to both schools partaking in the study.

Dougal confirmed that he develops and presents PD for his staff incorporating an optional 15 to 20 minutes PD every week, or often larger sessions of PD in prescribed staff meetings. Since Dougal had only commenced at School 2 three weeks prior to being interviewed, it is likely that the PD he mentioned is only in its infancy. In his PD sessions, Dougal outlines that he always incorporates the use of technology. Technology can prove beneficial when utilised effectively in the classroom to teach mathematics and Peter supports this notion and suggests that if it is used properly it can be advantageous, although proper training in such technology is crucial. However, two participants (Dylan and Rose) suggested that they were unaware of how effective technology could be and felt that further training would be beneficial. While many teachers had admitted to undertaking some PD recently whether it is in other disciplines, it is concerning that eight respondents of the ten had not completed any mathematics PD in the past year meaning they had not worked to improve their mathematical content and pedagogical knowledge.

Desired Professional Development

The participating teachers were asked about the type of PD they would like to undertake, a variety of responses emerged. Table 3 outlines the participants desired mathematical PD. Peter felt that teachers would benefit from further PD on practicing mathematical problems. As the HOD of School 1, Peter stated that he goes "to some of my faculty meetings and say, 'do this problem', because I know that half of the teachers can't do it because they're not senior teachers". Peter believed that teachers must practice their mathematical skills and utilise a variety of ways to reach an answer, to ensure they are completely proficient.

Dougal agreed with Peter and further mentioned that he wanted to have more opportunities of teaching mathematics to the younger year levels to improve his pedagogic methods and ways to engage the younger students. Dougal felt that it is important to be "able to walk into any level of student and know the curriculum well enough to just jump in and teacher it really", and to achieve this, further exposure and access to junior secondary mathematics classes, and PD focused upon more simplified pedagogic methods would be useful. Furthermore, Dougal and Peter were also in alignment and believed in the benefits of the coaching model, where teachers can discuss processes and methods of teaching mathematics, and consequently are able to learn from each other.

Table 3
Desired Professional Development

Participant	Desired Professional Development
School 1	
Peter	Practising mathematical problems.
James	N/A.
Rose	Senior mathematics, essence of ACARA, STEM: the expectations from school administration.
Violet	Microsoft Excel and use with graphing.
Leigh	Apps in mathematics.
Dylan	Further exploration of tips and tricks in mathematics.
School 2	
Mary	Time management, better utilisation of time to ensure that all of the ACARA is covered.
Caroline	Senior mathematics, specifically the topic of algebra.
Dougal	Improvement of timetabling to have access and teach mathematics to younger year levels.
Carly	More difficult mathematics including Maths B.

Three participants (Rose, Caroline, and Carly) showed a strong desire to improve their mathematical content abilities in relation to senior mathematics. Rose wanted to expand her mathematical thinking to incorporate the senior mathematics curriculum "in terms of seniors, I feel like I'm in a really dark room and am just feeling my way around". Rose wanted to learn "everything about senior maths, because when it comes to senior maths, the topics are very topic driven" and currently all there is to rely on is "one or two books and one or two teachers that we can toss ideas around with". Caroline specifically identified that she would like PD on "harder algebra topics from a senior teachers' perspective, so that I can see where they need to get to". Carly's viewpoint coincided with this stance and additionally mentioned that "I don't have a clear idea of what they do in Maths B higher up the school, so because of that I don't know where the kids need to go".

Hindrances to Professional Development

Whilst it is evident that further PD is desired by the participants of this study, there are some hindrances that have emerged that inhibited staff members from actively undertaking further PD. These difficulties seem to stem from top tier school administration as well as the HOD for mathematics (Peter), and the Director of Academic Performance and Innovation (Dougal), who have control over PD opportunities. The first problem is the apparent lack of mathematics PD as most teachers reported limited opportunity. Dylan noted, "not so much, it's usually like there's a PD, but not really ones of (mathematical) interest". Dylan reported that PD must go through administration and/or the Head of Department for mathematics before reaching the teachers, "Peter sometimes says 'there's a PD of this coming up would you be interested in doing it'? So usually, they must sift through it from up here and then they pass it through". If this is the case, perhaps some opportunities are missed by departmental leaders simply because they are not aware of all PD opportunities. Also, if the school administration sifts through all PD opportunities, perhaps some are being withheld intentionally due to the cost of attendance to the school.

A further reason that could explain the reduced opportunity for PD is that School 2 prefers to undertake PD in house. Caroline reported, "since the new principal, there seems to be a shift in the way that PD is considered. They don't really want us to go out to PD, they want us to have internal PD instead of us going out to PD's it's more internal". Caroline further stated that only senior staff members attend external PD's "so it's only if you're like the Head of maths or Head of Echo. Only those type of people that do it, more the senior staff. So, everyone just teaches each other, which seems to be the way that it is". While

internal PD is an excellent method to increase the knowledge and pedagogical approaches to mathematics; perhaps this format in conjunction with external PD would be useful to broaden the horizon of the teachers.

Another issue hindering the opportunity to attend PD is the quantity of paperwork involved and the slow application response rate, an issue for School 1. Violet noted "the paperwork here is insane. So, if you find a PD that you want, you have to fill out the paperwork, take it to admin, then it's about a two-week turnaround before it's a yes or a no to go". Sonya also suggested that favouritism occurs where some teachers are more favoured and approved to undertake the PD more so than others. Violet also supported this claim "I did apply to go last year, but I got knocked back from admin. That was about teaching maths to lower-level kids. And it seems that some people are favoured more towards PD's than others in the school".

Professional Development for Out-of-field Teachers

The existence of out-of-field teaching has been identified in both participating schools with 50% of the respondents acknowledging this. Rose, Leigh, Mary, Caroline, and Dougal highlighted this issue and its occurrence. Rose mentioned that is has "been a huge problem for us, because we don't have enough maths teachers, the actual maths trained teachers are taken to teach senior maths because of the content knowledge, expertise and the amount of people qualified to teach certain subjects". Furthermore, Rose, Caroline and Leigh specifically mentioned that the mathematically untrained teachers are teaching mathematics to students in the lower year levels. Rose indicated "all the junior classes were given to teachers that are out of this department or out of this field". Caroline believed that these untrained mathematics teachers typically were from "Physical Education (PE), Science, and English backgrounds". While it is a common practice to combine the Mathematics and Science disciplines so that they are often taught by the same teachers, PE and English has limited to no relevance.

Rose felt this is particularly disadvantageous to students as she believed that Year 7 and Year 8 are the most important years of schooling, "they are teaching Year 7 and 8 (I've said this to my principal as well) they are the most important years". Furthermore, Leigh outlined the inconsistency of utilising out-of-field teachers, in that "they get that maths class for a year, and then next year they won't get another maths class – almost guaranteed. It's just their timetable and how it all fits in and gels". As out-of-field teaching is a common occurrence at these schools, PD is needed for all teaching staff no matter the discipline in case they are required to teach mathematics the following year. Leigh noted that PD is vital for out-of-field teachers but is problematic as workload changes occur each year.

Both schools are aware of the need to reduce the use of out-of-field teachers and have taken steps to combat the shortage of specialist teachers such as implementing changes to timetabling to match the mathematics trained teachers with the mathematics classes. While this is a positive step in the right direction in terms of diminishing the use of out-of-field and untrained teachers, this does not completely correct the issue as it still relies on having enough specialist mathematics teachers to meet the timetabling requirements.

A tactic that School 2 is opting for is a differentiated approach so that in conjunction to ensuring its teachers are mathematically trained, they are actively training and up-skilling the untrained mathematics teachers. Dougal explained this in house training to involve team teaching where Dougal does the teaching, and the staff member observes by watching and learning. With one new staff member specifically, in conjunction to the coaching model, "I am sitting down one-on-one and doing a lot of work with her at the moment to up-skill her".

Overall, it seemed that while some PD relating to mathematics has been undertaken, much further incorporation in both schools is warranted considering that PD is one of the few approaches to improve teacher quality (Foster et al., 2013). This is particularly important to this study as the Junior Secondary teachers interviewed are perhaps not completely confident in their abilities along with the use of out-of-field teachers in mathematics.

Conclusion

In summary, the findings indicated that eight of the teachers have not undertaken mathematics PD in the past year. Although School 2 is in the process of improving this, the focus is the incorporation of digital technology. Whilst a step in the right direction, the findings suggest that little emphasis has been placed upon other pedagogical strategies to improve mathematical content knowledge and teaching. The findings also suggested that School 2 favoured in house PD which again limits the scope of the PD available. This is contrary to the findings by Roland et al. (2014) who suggests that PD which incorporates a variety of teaching professionals such as teacher candidates, current teachers from the discipline and internationally educated teachers, allow professionals to collaborate as members of the broader educational community providing a more robust and enriching learning experience. School 1 was not of this belief and had recently employed Dougal for external PD specifically, however, it was unclear as to how long this had been occurring and again if technology is of focus.

The findings indicated that the deficiency of PD is attributed to several factors including: the lack of opportunities provided by school administrators; lack of time to attend PD; and tedious paperwork involved in the application process. Perhaps if a more streamlined, automated administrative process were adopted, mathematics teachers would be able to attend more PD. It is evident that there is a need for future teacher learning, and the findings indicated specific topic areas that the teachers would like to explore such as problem solving and extending their understanding to explore the senior mathematics curriculum. Several teachers also felt that the tactic of pairing teachers with their more senior counterparts to learn from each other and practice their existing mathematical skills would be beneficial.

The use of out-of-field teachers occurred in both schools; however, proactive steps are in action to reduce this and thus improve the quality of teaching. School 1 is working to improve timetabling, whilst School 2 has incorporated a different approach by up-skilling its existing teachers. Perhaps by multiskilling junior secondary mathematics teachers to teach mathematics to senior students, there would be more availability of staff to fill teaching roles, and this will be beneficial for timetabling which is an issue identified at both schools. This would also reduce the likelihood of employing out-of-field teachers to teach mathematics. However, the shortage of qualified mathematics teachers in Queensland remains a significant problem. Overall, it appears that teachers want to develop and extend their existing mathematical knowledge and pedagogical practices; however, there is limited scope for this development.

Implications and Limitations

It was found that although PD is a requirement to sustain teacher registration, very little PD is undertaken in mathematics education and rectifying this will aid to not only improve the content knowledge of teaching but will also expose teachers to a variety of methods and strategies to improve their pedagogic approaches to teaching mathematics. PD will also help to expand the skill level of existing mathematics teachers, for instance, if Junior Secondary

teachers are trained in senior mathematics teaching there will be more teachers available to fulfil the mathematics teaching jobs and the use of out-of-field teachers would be less likely.

This paper was part of a wider study that explored the self-reported perspectives of Junior Secondary mathematics teachers considering the Year 7 transition to secondary schooling in two Queensland schools. As such, only one facet of this research focused upon PD specifically. Therefore, this research was limited, in that further exploration of PD is required to provide a wider consensus to existing PD practices in schools. It is evident that PD is crucial to foster high quality mathematics teachers, and as such, further exploration and research into effective mathematical PD practices would be beneficial to all stakeholders.

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Perspective taking: Spatial reasoning and projective geometry in the early years

Jennifer S. Thom
University of Victoria
<jethom@uvic.ca>

Lynn M. McGarvey
University of Alberta
<lynn.mcgarvey@ualberta.ca>

Nicole D. Lineham
University of Victoria
<nlineham@uvic.ca>

Spatial reasoning is seen as increasingly important in STEM fields. Within mathematics, geometry is a potential site to study and support young children's spatial reasoning. In this paper we revisit Piaget and his colleagues' theoretical perspective on children's development of geometry concepts and take note of projective geometry in that theory. We outline critiques of Piaget and Inhelder's (1967) theory of topological primacy and then situate the criticisms within current spatial reasoning literature. We suggest a return to research on projective geometry holds promise for exploring and expanding opportunities that promote spatial reasoning in the early years.

For more than two decades, the push for STEM (Science, Technology, Engineering, Mathematics) skills worldwide has called attention to the importance of spatial skills, and specifically the role spatial reasoning plays in each of these domains as well as STEM-related fields. Spatial reasoning is integral to spatial skills and more generally, spatial ability, can be defined as "the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 147). Studies indicate that spatial ability is a critical attribute for entry into and success in STEM professions (Wai et al., 2009). Moreover, everyday activities such as assembling furniture, packing a suitcase, or using a web-based mapping system to get from one location to another not only require spatial know-how but also spatial know-why.

In fundamental ways, spatial reasoning shapes what we do, how we experience the world, and the ways we make sense of and think within it. Yet while spatial reasoning underlies all STEM domains, it is mathematics, in particular, that enables examination and communication of spatial concepts (Smith, 1964). Arguably then, spatial reasoning and spatial skills can be explored and developed in depth within the domain of mathematics.

In this paper, we report on key conceptual areas informing our inquiry regarding the spatial reasoning involved in projective geometry. Focusing on spatial reasoning as generally defined and recognized within STEM, we identify its relevance within mathematics education, and specifically, its relationships to geometry in the early years of mathematics education. Following this discussion, we make the case for spatial reasoning being taught through projective geometry—a largely forgotten area of research. We provide a brief summary regarding Piaget and his colleagues' theorizing of young children's conceptions of projective space, including key critiques. We then bring the three criticisms of Piagetian theory forward to 2021 and situate them within current spatial reasoning literature. What results is a change of theoretical perspectives and the emergence of new sightlines for early years research in mathematics education.

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Spatial Reasoning, Mathematics Education, and the Case for Geometry

There is a strong link between spatial skills and STEM professions (Mix & Cheng, 2012), and increasingly studies reveal connections between spatial skills and mathematics performance (Gilligan et al., 2017). Such skills observed with young children appear to be critical predictors of mathematics achievement—even beyond measures of verbal and quantitative scores, throughout schooling and into STEM-field careers (Cheng & Mix, 2014). Individual differences in spatial ability are apparent as early as the preschool years. For example, children who build with blocks, put together puzzles, and play with shapes tend to have stronger spatial reasoning skills than children who do not (Verdine et al., 2014).

Once viewed as a static and innate attribute of intelligence, spatial ability is now proving to be malleable (Uttal et al., 2013). These findings suggest early skill development could enhance mathematics performance for students. In fact, spatial training on tasks such as mentally rotating objects has led to improved performance in mathematics for 6- to 8-year-olds (Cheng & Mix, 2014). While findings differ, one conclusion is that some spatial skills are more likely than others to impact performance, such as constructing a mental number line (Lakoff & Núñez, 2000), solving missing term equations (Lourenco et al., 2018), and scaling related to proportional reasoning (Gilligan et al. 2018).

Spatial skills entail many constructs such as spatial thinking, spatial sense, visualisation, spatial cognition, and spatial orientation. There is a debate as to which skills are relevant. In this paper, we use Newcombe et al.'s (2013) categorisation of spatial skills, and specifically, tool making. *Tool making* refers to skills involved in manipulating, transforming, and creating objects, as well as using these objects as tools. Doing so requires dynamic spatial (reasoning) processes such as rotating, bending, and scaling. More than ever, research is connecting these process skills and tools with mathematics performance, especially in the early years (Mix et al., 2017).

However, despite evidence for spatial reasoning and spatial skills being essential to mathematics, especially in the early years, a clear absence of spatial skill development persists in K-12 mathematics classrooms (Woolcott et al., 2020). Given that it is mathematics "through which we communicate ideas that are essentially spatial" (Clements & Sarama, 2011, p. 134), it is concerning that mathematics curricula do not emphasise spatial concepts, processes, skills, and thinking (Davis et al., 2015).

Geometry, which underlies most if not all mathematical thinking (e.g., Bronowski 1947), is the curricular area with the greatest potential for providing educational experiences in spatial reasoning (Lowrie & Logan, 2018). However, geometry receives the least attention in schools K-12 (e.g., Larkin et al., 2016). Geometry content is often limited to sorting and naming 2D shapes (Clements & Sarama, 2011), yet young children are motivated by, capable of, and need opportunities to apply, analyse, and investigate geometric transformations of 2D and 3D shapes through mental rotation, use of symmetry, multiple representations, and de/constructing parts (Frick et al., 2014).

The importance of spatial reasoning within mathematics and the supporting role that geometry plays, prompted us to turn to educational research in projective geometry initiated over a half century ago. While studies in this area were essentially abandoned in the 1990s, we assert projective geometry employs extensive spatial processes, many of which are underexplored. As such, re-examining and extending research focused on young children's projective thinking is vital to early years mathematics.

Projective Geometry

Projective geometry involves the relationship between objects and images and their mappings or projections onto other surfaces. For example, what geometric properties are maintained between an object and the shadow it casts. Spatial transformations, such as rotation, translation, scaling, and shearing are central to projective geometry. Historically, projective geometry grew out of attempts by artists and architects to use perspective to draw or paint (i.e., project) the 3D world onto a flat surface. Today, we take for granted the ability of artists to draw with perspective. Yet, compared with Euclid's geometry which is over 2000 years old, perspective drawings only appeared 600 years ago, during the Renaissance. A key difference between Euclidean geometry, on which most of school geometry is built, and projective geometry, is that while lines remain lines and points remain points, in the latter, lengths, angles, and areas are not all preserved under projection. Figure 1 illustrates a 2D presentation in which the 90-degree angles, edge lengths, and surface areas of a cube are not preserved. We know the image represents a cube because we have learned to read, interpret, and thus see it as possessing the geometric properties that can only be observed when actually holding the physical object itself. This common example illuminates how projective geometry lies at the very intersection between perceptual and representational space, and as such, holds tremendous potential for young children's spatial reasoning in mathematics. Current studies on how children come to make sense of projective concepts are virtually nonexistent in the literature. As such, our inquiry starts with the research of Piaget and his colleagues on young children's conception of space.

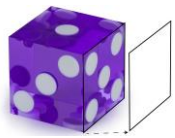

Figure 1. Projective image of a cube

Children's Spatial Reasoning within Projective Geometry

The spatial reasoning research by Piaget and his colleagues (e.g., Inhelder, Meyer) preceding and during the 1950s generated many further studies in the decades that followed up until and including the 1990s. Here we highlight key aspects of the theory concerning projective geometry, related studies, and critiques of the research.

The Child's Conception of Space

Piaget (1953) described young children's discovery of spatial relationships as spontaneous geometry. Central to Piaget and Inhelder's (1967) theory on how children come to make mathematical sense of space is that unlike a child's perceptual space which directly reflects their sensorimotor schema of a spatial environment, geometric representations of space result from their ongoing building up of motor and internalised actions into logical operational systems. Piaget (1953) contended that children's geometric conceptions follow a deductive or axiomatic progression, opposite in sequence to the historical development of the mathematics. That is, at 3 years old, the child first makes internal or topological distinctions about a particular figure (i.e., open and closed structures, interiority and exteriority, proximity and separation). By age 7, they begin to construct the projective concept of the straightness of a line. Then, at 9 or 10 years of age, the child understands problems involving perspective, such as angle of vision and point of view, and Euclidean

concepts relating figures with other figures through angles, sides, parallelism, and distance. It is only when the child reaches this point that they have "complete conception of how to represent space" (Piaget, 1953, p. 79). Known as the topological primary thesis, Piaget (1953) asserted that young children's spatial reasoning was associated with their geometric representation of space as "another example of the kinship between psychological construction and the logical construction of science itself" (p. 75).

Piaget and Inhelder (1967) suggested that children begin to engage in projective geometry when they no longer view or represent objects in isolation but in relation to different viewpoints, including perspective in their drawings. For example, children at 7 to 8 years old can correctly infer by drawing what the doll's perspective or line of vision of the object is, when the doll is standing on the table and an object is placed in a certain direction in front of it. A similar experiment involved predicting the shape of an object's shadow when the object is placed in a certain position between a light source and a screen. Piaget (1953) concluded, based on the findings from this task, that the coordination of different viewpoints does not occur until the child is 9 or 10 years old. Further, Piaget and Inhelder (1967) theorised that differently from children's spatial constructs which are perceptual and experiential, or grounded in single viewpoint, their conception of basic projective relations requires their conscious constructing of reference systems through operationally connecting and coordinating all possible perspectives.

Critiques of Topological Primacy Theory

Subsequent studies by researchers who replicated Piaget and Inhelder's experiments have, for the most part, confirmed their findings (Laurendeau & Pinard, 1970; Page, 1959). However, while not dismissed outright, Piaget and Inhelder's (1967) theory on topological primacy is not supported. We discuss three key criticisms of the theory which inform this initial stage of our research on children's spatial reasoning through projective concepts.

First, children's conceptions of space may not follow the logical order of topological ideas then projective and Euclidean concepts. Research findings which revealed varied results in drawing experiments (Dodwell, 1963; Lovell, 1959) open the possibility that all three types of geometric concepts might occur simultaneously and over time, children develop the ideas by further integrating and synthesising them. For example, drawings by 4-year-old children which were not predominantly topological suggest it might not be topological features that allow children to draw homeomorphic copies. Instead, it could be their coordinating of projective and Euclidean properties which enables topological properties to be maintained (Martin, 1976). Other research by Rosser et al. (1988) suggests an alternative developmental sequence wherein children progress in their spatial reasoning through three levels: from reproducing a set of geometric figures exclusively by encoding (i.e., given a set of shapes and then matching them to the original); to building the same configuration from memory; to then matching by building an identical configuration of geometric shapes from memory after a rotation or taking another's perspective. In their study, preschool children achieved the first two levels but not the third. Rosser et al's (1988) study also emphasise the need for research to focus on how children's projective processes relate to their thinking and activity in geometry.

This point leads to the second critique of topological primacy theory regarding young children's engagement as they develop projective ideas (Clements & Battista, 1992; Rosser et al., 1988). The contention concerns the overemphasis in Piagetian theory on identifying logical errors which then precludes examination of projective concepts that may not yet be fully fledged, articulated, and perhaps are altogether different. For example, Frick et al.

(2014) demonstrate how children, as young as 3 years old, engage in perspective-taking tasks when allowed to move the objects around or when provided with a model of the room. Here, the children encode the location of small objects through the use of landmarks. Additionally, these studies indicate that children's development of projective space may not only require operationally connecting and coordinating all possible perspectives, but also forming and integrating an external framework as part of their reference system for spatial organisation.

Third, while children's geometric conceptions of space may not be direct reproductions of their sensorimotor perceptions of the environment, at the same time, it is unlikely that their representations are purely logical operational systems. Clements and Battista (1992) identified this point as a key aspect that researchers had not yet examined in any depth. Drawing on the work of Fischbein (1987), Clements and Battista (1992) argued that space representations (e.g., a square shadow of a cube die projected on a screen) are more complex than exclusively abstract properties of space (e.g., four edge lengths and 90-degree vertices). Rather, children's conceptions of space entail "sensorimotor and intellectual skills organized into a system of beliefs and expectations that constitute an implicit theory of space. Most important, intuitions thus constructed are enactively meaningful... because they express the direct behavioral meaningfulness of an idea" (Clements & Battista, 1992, p. 426). Today, even more, it is necessary for researchers to examine how Piagetian theory and theories that emphasise the intuitive and complex nature of cognition can inform children's spatial reasoning through projective concepts.

Changing Perspectives

We now take the three critiques and consider them further by situating each of them within more current and general spatial reasoning literature. In doing so, we distinguish complementary theoretical perspectives which not only offer possibilities for how we might observe anew the ways young children reason spatially in projective geometry, but also prompt sightlines for reconceptualisation.

1. Children's conceptions of projective space may not follow the logical order proposed by Piaget and Inhelder (1967).

This point calls into question several aspects of Piaget and Inhelder's (1967) notion of topological primacy such as ages, stages, linear sequencing, mutual exclusion of the three types of geometries, and contexts. Moving deeper and changing tact, what might it mean if young children's conceptions of projective space were not characterised once and for all, as a predetermined sequence, or prescriptive stages?

In *Spatial Reasoning in the Early Years: Principles, Assertions, and Speculations*, Davis et al. (2015) argue for theoretical perspectives and ways of interpreting spatial reasoning in mathematics which not only move beyond isolating observable and measurable aspects, but at the same time, more closely reflect learners' cognitive activities as they engage in-situ where mathematics teaching and learning happen. Here the authors characterise spatial reasoning as:

> a clearly discernible whole that cannot be fully comprehended by reducing it to its components. Such forms arise in the entangled interactions of many aspects, agents, or subsystems – and, within those interactions, new and unpredictable possibilities can arise. Those possibilities, in turn, can affect and occasion the entire system's current and future properties and behaviors. (Davis et al., 2015, p. 140)

Both the description and circular model proposed by Davis et al. (2015) (see Figure 9.1, p. 141) reflect spatial reasoning as neither linear nor hierarchical but instead dynamically

emergent and ever-changing. This perspective which complements Piaget and Inhelder's psychological axiomatic order, facilitates more nuanced research, both theoretical and empirical, and enables alternate ways to understand young children's spatial reasoning in projective geometry. We illustrate and discuss these aspects next.

2. There is a need for research that examines children's projective concepts which may not yet be fully fledged, integrated, or articulated, and perhaps are altogether different.

Taking an emergent view of young children's spatial reasoning in projective geometry implies that while their cognitive activities may be unpredictable in foresight, for example, the forms they take or the ways they manifest moment to moment in everyday learning settings such as mathematics classrooms, they can potentially be understandable in hindsight. Here the value of both artifacts and acts of children's spatial reasoning is emphasised (Thom & McGarvey, 2015) as well as the prospect of gaining insight into the moments and contexts during which children bring projective ideas into being.

Within this theoretical perspective, we revisit Piaget's (1953) experiment with children predicting the shape of the shadow projected by an object (e.g., a six-sided die). Studying the emergence of their spatial reasoning demands paying even closer attention to children's cognitive activities as they engage with the object, its projected image, and different points of view. Using Davis et al.'s (2015) descriptive terms, several discernable and possibly co-emergent *transformings* include *movings* (e.g., *rotations*), *alterings* (e.g., *dialating/contracting, distorting/morphing*), situatings (e.g., *dimension-shifting, orienting, and locating*), and *(de)constructings* (e.g., *de/re/composing, sectioning*). Elements of *understanding* that could arise involve *interpreting* (e.g., *diagramming, comparing, relating*) and *sensating* (e.g., *perspective-taking, visualising, imagining, tactilising*).

Further, we see still other and potentially different opportunities to examine young children's conceptions of projective space within today's contexts. Projective geometry underlies many different designing and map-making activities associated with computer modelling, 3D printing, digital photography and editing, perspective drawing, engineering and architectural plans, as well as other imaging applications. These activities of *designing* and *map*-making, along with *projecting* are also identified by Davis et al. (2015) in their circular model as emergent competencies.

3. While children's geometric conceptions of space may not be direct reproductions of their sensorimotor perceptions of their environment, at the same, it is unlikely their representations are purely logical operational systems.

It is worth repeating that what makes projective geometry striking, complex, and unique is how the concepts are inextricably perceptual and representational. Relevant theories that allow for inquiry into young children's perceptual and representational conceptions of projective space include those which enable cognition to be viewed as dynamic, contextually contingent and body-centred, whereby logical forms of knowing are not separate from perceptually-guided activity (Varela et al., 1991).

Perspectives such as those rooted in enactive and/or embodiment theories, take cognitive structures and activities to be co-implicated by our biological bodies and our social-cultural ways of knowing. That is, what we come to know, how we think, and that to which we choose to attend is influenced by how our material bodies move through space and in relation to other bodies, as well as historical and cultural significances (Nemirovsky et al., 2020;

Varela et al., 1991). Cognitive scientists increasingly show the vital role our body plays in the conceptual development of both simple and seemingly abstract mathematical concepts (Marghetis & Núnez, 2013).

Past and current studies in mathematics education elucidate bodily aspects of young children's spatial thinking such as the spontaneous and deliberate ways they use their bodies to express concepts and develop meanings, though not necessarily related to projective geometry, for 2D and 3D figures and transformations (e.g., Bussi & Baccaglini-Frank, 2015; Thom, 2018). These include gestures, movement, sound, speech, rhythm, and drawing(s). Thus, it seems reasonable to assume that the body and perception are not simply the means by which children progress to more formal projective thinking, but rather, the means with which their conceptual thinking depends, emerges, and evolves. Here lies tremendous potential for research to expand understanding of spatial reasoning in projective geometry, in terms of critical spatial skills, processes, and tools that young children 'know' as well as how they demonstrate and develop these perceptually, logically, informally, and formally, mentally and physically.

Acknowledgments

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The reification of the array: The case of multi-digit multiplication

Kristen Tripet
The University of Sydney
<kristen.tripet@science.org.au >

The array is a powerful tool that builds students' understanding in multiplication. Students' interactions with the array changes through the course of an instructional sequence, which can be viewed as a process of *reification*. In this paper, I report the findings of a research study conducted with 45 Year 5 students in Sydney. The study explored students' changing use of the array through the course of an instructional sequence on multi-digit multiplication. Design Research methods were used to track students' use of different forms of the array and the functions that these forms served. Three key stages were identified in the process of reifying the array in multi-digit multiplication.

Mathematical representations play an important role in the development of understanding (Hiebert & Carpenter, 1992; Pirie & Kieren, 1994). Representations make visible that which is abstract, thus making more abstract concepts accessible to students (Gravemeijer, 2004). Students' abilities to work with mathematical representations flexibly and their capacities to interpret and connect representations are key to the process of building mathematical understanding (Goldin & Shteingold, 2001; Gravemeijer, 1999). As students interact with formal, external representations they can more easily observe connections and relationships between mathematical concepts. Those observed connections and relationships form students' own internal representations of concepts (Goldin & Shteingold, 2001).

The array has been recognised as a powerful representation that allows access to the important theoretical constructs of multiplication (Barmby et al., 2009; Battista et al., 1998). This two-dimensional representation of multiplication highlights equal groupings and shows how the composite units build on each other to produce a whole (Steffe, 1994). Curriculum documentation presents the array in various forms as an important tool in the teaching of single- and multi-digit multiplication (ACARA, 2017).

There has been substantial research on the array with single-digit multiplication but there exists limited research on its usage in multi-digit multiplication (Barmby et al., 2009; Young-Loveridge & Mills, 2009). Despite this limitation, studies have shown that the array affords students access to important multi-digit multiplication understandings, including the distributive property (Barmby et al., 2009, Izsak, 2004; Young-Loveridge & Mills, 2009) and the associativity property (Ding et al., 2013). What is less evident in the literature is how students' interaction with the array in multi-digit multiplication evolves over the course of an instructional sequence as their understanding of the multiplicative structure develops.

In this study, I examined the power of the array as a *representation of* a contextual situation through to a *representation for* mathematical reasoning as enacted by the students through their mathematical activity over the course of an instructional sequence. Gravemeijer (1999) described this changing use of representations as a process of *reification*, where mathematical activity takes on object-like character as a result of student activity. According to Gravemeijer (1999), there are two stages to the process of reification. First, students' activity is bound in the context of the problem, a stage Gravemeijer refers to as the *referential level*. The second stage is the *general level*, where students' interpretations and solutions operate separately to the contextual imagery.

To address the research gap, and to inform curriculum design and teaching practice, the following question focused the research: *How does students' use of the array develop from*

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a representation of a contextual situation to a representation that is used for more generalised mathematical reasoning in multi-digit multiplication?

Theoretical Framework

The theory of Realistic Mathematics Education (RME) was used to guide the design of this research. RME is founded on the belief that mathematics is not a closed body of knowledge to be transmitted. Rather, it is an exercise in which learners are active participants (Van den Heuvel-Panhuizen, 2003), whereby one 'reinvents' conventional mathematics for themselves (Gravemeijer, 2004). In the context of the classroom, students engage in tasks that require them to develop their own tools and strategies as they solve experientially real problems. This is the process of mathematisation. Students form and organise new knowledge and develop their own mathematical insights (Van den Heuvel-Panhuizen & Drijvers, 2014). The aim of RME is to support students' progressive mathematisation, or level-raising (Gravemeijer et al., 2003). To achieve this, learning experiences are based on three important design heuristics: experientially real contexts, guided reinvention (a process where students reinvent conventional mathematics through active teacher guidance), and emergent modelling. Most relevant to this paper is the heuristic of emergent modelling, which describes how students' interactions with models develop and change through the course of an instructional sequence.

In RME, a *model* is a broad term that encompasses varied representations of mathematical concepts and structures (Van den Heuvel-Panhuizen, 2003). Models are not designed as ready-made representations trying to make mathematical concepts concrete. Rather, models are developed out of contexts (Gravemeijer, 2004) and support students in the process of progressive mathematisation. The model, as Van den Heuvel-Panhuizen (2003) explains, serves as a bridge. On one side of the bridge are the informal understandings bound within the context of the problem, and on the other side are the formalised mathematical concepts. It is students' interactions with the model that allow them to cross this bridge.

The nature of the model changes through students' activity. It moves from a *model of* a situation to a *model for* mathematical reasoning (Gravemeijer, 2004; Van den Heuvel-Panhuizen, 2003) with different forms of a model serving different functions (Saxe, 2002, 2004). Initially the model is closely connected to the context of the problem: it is a model of a particular situation and students use it to make sense of the problem at hand. As students work with the model over multiple experiences, they build an appreciation for the mathematical concept or structure that the model embodies. Their understanding of the model becomes more generalised, and it becomes a model for mathematical reasoning. The model is *reified*. As Gravemeijer (2004) explains,

> the model first comes to the fore as a model of the students' situated informal strategies. Then, over time, the model gradually takes on a life of its own. The model becomes an entity in its own right and starts to serve as a model for more formal, yet personally meaningful, mathematical reasoning (p. 117).

Methods

The methodological approach for this research needed to allow the researcher to observe first-hand students' reasoning and interaction with the array. To meet this aim, Design Research methods were employed (as described by Cobb & Gravemeijer, 2008). Three

research phases were enacted: preparatory thought experiments, teaching experiments and a retrospective analysis.

The preparatory phase formed the foundation of the project. A detailed analysis of relevant literature was the basis for anticipatory thought experiments (Gravemeijer, 2004). This phase clarified the learning goals, documented the starting points for instruction and then, from this, delineated a predicted learning pathway.

The teaching experiment phase of the project was conducted in two different Year 5 classes. Both classes were from non-government schools in Sydney; 23 students in the first class and 22 in the second class, comprised a sample size of 45 students. The researcher adopted the role of the teacher in each teaching experiment. This approach allowed the learning environment and teaching across both experiments to be controlled and enabled the researcher to experience first-hand the events of the classroom, thus enriching the ongoing cycles of data analysis and experimentation. The students' regular teacher was also present in the classroom and helped facilitate student activity. The same instructional sequence was taught in both classes. The sequence was implemented over a two-week period and comprised of four teaching episodes. Each teaching episode spanned two or three one-hour lessons and was characterised by a focus on a distinct mathematical concept, presented through the context of a problem. Each teaching episode is described later in the results section of this paper.

The retrospective analysis situated the classroom learning process into the "broader theoretical context as a paradigmatic case of a more encompassing phenomenon" (Cobb & Gravemeijer, 2008, p. 83). It was in this phase that a grounded theory (Glaser & Strauss, 1968) on the reification process of the array was formed.

Data collection and analysis

The data collected needed to elicit evidence of students' reasoning with the array, shifts in their reasoning, and how these shifts were supported and organised. Based on this, three key forms of data were collected: student work samples from the teaching episodes, transcribed video recordings of classroom activity, and field notes compiled by the researcher and class teacher during classroom lessons.

The analysis of data occurred over two phases of the research. The Constant Comparative method (Glaser & Strauss, 1968), adapted to the needs of Design Research as illustrated by Cobb and Whitenack (1996), was used during the teaching experiments. Students' use of the array was tracked across the teaching episodes and descriptions of students' usage were grouped in two ways: according to the individual students and then according to the solution method used. This enabled observation of whether the model held power for individual students, which would be evident through the moving from a *model of* the contextualised situation to a *model for* more generalised mathematical reasoning.

The second round of data analysis was conducted as part of the retrospective analysis, which mapped the process of the array moving from a *model of* a contextualised situation to a *model for* more generalised mathematical reasoning. Saxe's (2004) form-function framework was used to explore the different forms of the array used by students and what function each form of the array served. The framework helped explain how students' use of the array shifted over the course of the instructional sequence, to serve differing functions. The *form* of the array was defined as specific visual features, and its *function* was defined as the way the students chose to interact with the array in their work. Three forms of the array were observed across the instructional sequence: arrays with all individual parts visible, a pre-partitioned array, and an open array. The dataset was grouped according to the three

forms of the array so that commonalities could be identified, and so that shifts in students' form-function use over the course of the instructional sequence could be noted. The dataset was then re-grouped, this time based on the array's function. Grouping in this way served to confirm the commonalities that were identified, students' evolving use of the array, and to highlight any anomalies. The final step in the analysis was to explore the form and function of the array based on students' diverse conceptions and strategies. To do this, data were grouped based on the form and function of the array that the students chose to use as they developed solutions to the problems.

Results

The results section describes the visual form of the array used and the student-chosen function that each array served. The students' use of context is also recorded as the process of reification is mapped. Examples of students' work is used to illustrate each teaching episode. The work of these students was typical of what was observed across both classes.

Teaching Episode 1 – Zoe and Lucille

The first teaching episode introduced the students to the context of a bakery that sold cupcakes. Students were presented with the following narrative: A baker makes and sells eight different flavours of cupcakes. The cakes are baked in a tray that has four rows with six cakes in each row. He bakes one tray of each flavour. How many cupcakes does he bake each day?

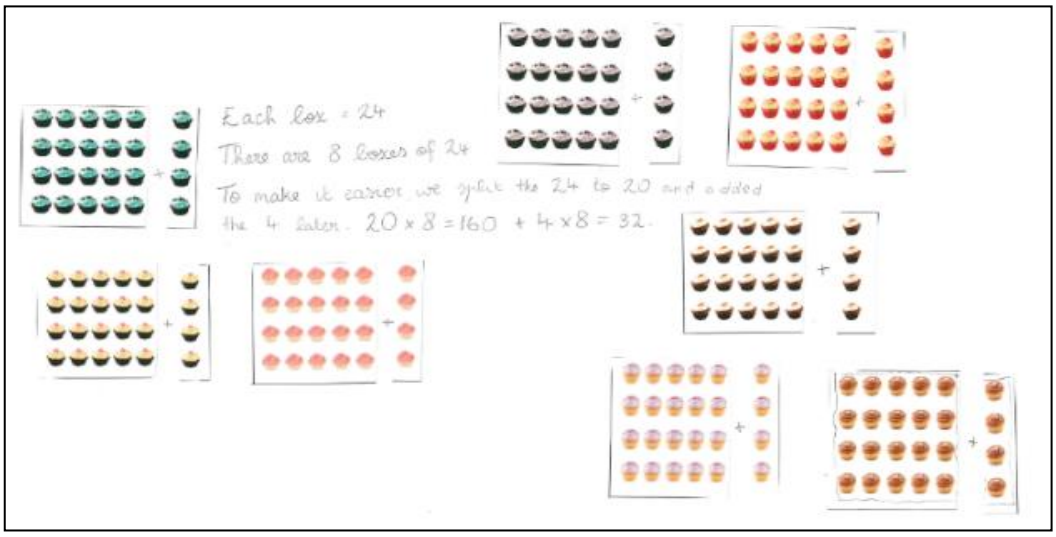

Figure 1. Zoe and Lucille's work sample from Teaching Episode 1

Zoe and Lucille's solution and justification were bound within the context of the problem. They represented their strategy using arrays which were presented as actual cakes. The function of the array in this form was to support calculation. Each tray was considered individually and was partitioned into groups of 20 and 4 (Figure 1). Lucille's verbal explanation of their strategy highlighted that the context was relevant to their thinking as they solved the problem: *But it is not like you are really cutting a row off, like, you can't. They are in a tray, so, yeah, you can't actually do it. But it is just how we worked it out.*

Teaching Episode 2 – Ryan and Dylan

The class was shown a picture of 16 filled cupcake boxes sitting on a bench in a 4 × 4 array, and students were told that each box held 12 cakes. The array was somewhat abstracted as the individual cakes were not visible. However, a further diagram shown to students revealed that the cakes in each box were configured in three rows of 4.

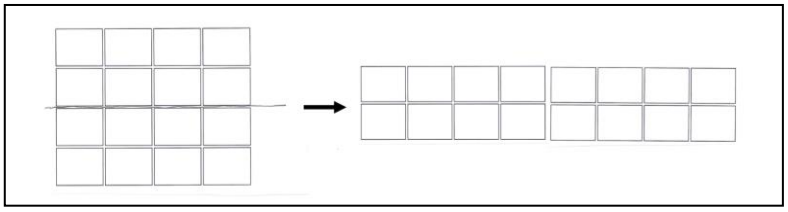

Figure 2. Ryan and Dylan's work sample in Teaching Episode 2

Ryan and Dylan partitioned the array based on place value and noted that the result for this collection of cakes, 16 × 12, was the same as the total number of cakes in the first teaching episode, an array of 24 × 8. This led into investigating a second mathematical goal—why did 16 × 12 = 24 × 8? Recognising that 16 could be halved to make 8 and that 12 could be doubled to give 24, the pair divided the array in half and rearranged it to transform 16 × 12 into 24 × 8 (Figure 2). Through their mathematical activity, they established a new function for the array: the array could be manipulated. The array moved from a static tool to a dynamic one. Through their working, the boys reasoning remained connected to the context, as illustrated in the following comment from Ryan: *You could join two of the boxes together to make 24 then… wait, that's 8 groups…yeah…that's 8 groups because 8 twos are 16.*

Teaching Episode 3 - Amelie

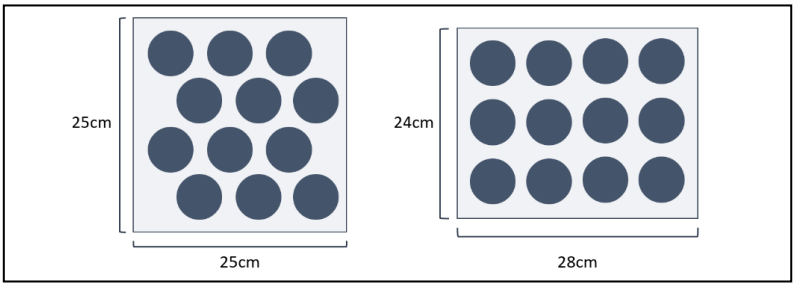

Figure 3. Comparing the area of two trays in Teaching Episode 3

In the third teaching episode, the students were shown the trays inside different cupcake boxes, and they discussed why one array was skewed and the other was not (see Figure 3). Students hypothesised that the skewed array was smaller in area and therefore would be cheaper to make. This hypothesis was the focus of the teaching episode.

Amelie, a student from Class 2, reasoned that 28 × 24 would be bigger, arguing that 2 could be taken from 28 and added to 24 resulting in the "equivalent" equation 26 × 26 (which is larger than 25 × 25). While 28 × 24 was indeed bigger, Amelie came to see that her reasoning was incorrect, and a new mathematical goal emerged: why was 28 × 24 not equivalent to 26 × 26? To achieve this goal, Amelie worked independently from the context of the problem. She regressed from an open array to a more familiar form of the array, a grid

array with all parts visible. The function of the array in this form was a sense making tool for the multiplicative structure. Amelie created a 28 × 24 array from grid paper, then cut off two columns from the 28 and taped them to the bottom of the array (Figure 4). She noticed that a 2 × 2 corner was missing, which left her puzzled. To understand what was happening, Amelie explored some other calculations, including 12 × 8. She recognised that, when attempting to form a square, a square corner with the dimensions of the number removed would be missing. This process helped Amelie realise that additive compensation could not be used in a multiplicative context.

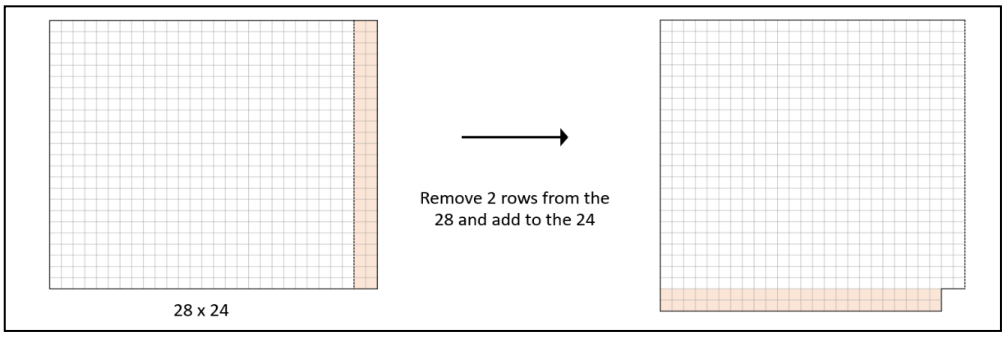

Figure 4. Amelie's strategy for comparing 28 × 4 and 26 × 26

Teaching Episode 4 – Hannah and Ava

The final teaching episode continued the narrative of the bakery and presented students with a multi-step problem: the total cost of 24 trays of cakes packed into boxes of 12 and sold at $28 per box. This context could not be easily represented as an array, as the problem presented a rate-based context. The intention was to see if students' strategies were limited by the context, or if they moved beyond the context to use the array as a calculation tool.

The majority of students from both classes used the array, partitioning it into place value parts to form simpler calculations. This is illustrated by Hannah and Ava's work. The girls reasoned that partitioning into place value parts created calculations that were easy to perform. Hannah and Ava were working abstractly with the array and made no reference to the context of the problem in their recording or justifications.

Abstract thinking, disconnected from the context, was evident in most students' work. While in earlier teaching episodes students referred to calculating with 'boxes', in this episode a shift was made to calculating with numbers: *We timesed [sic] 64 by 20 which is really just like doing 64 times 2 and then adding a zero. And then we just timesed [sic] 64 by 2, and then doubled again to get 64 times 4.* Abstract thinking was realised through students' mental calculations, as illustrated by one student's solution to 32 × 25: *25 is a friendly number because you just multiply it by 4 to get 100, so you divide the 32 by 4 to get 8, so it is just the same as 8 × 100.*

Discussion

The process of the array's reification can be understood by examining the forms of the array that students selected to use and the function that each form served. Students chose to use different forms of the array within a single problem and oscillated between multiple forms across the instructional sequence. At the start of the instructional sequence and when a problem was first posed, students used a form of the array that was closely connected to the context of the problem. In the same way, their interactions with the array were

contextually bound. The function of the array in this form was to support calculations. This is indicative of the *referential* level in the process of reification (Gravemeijer, 2004). By the end of the sequence, students' reasoning with the array was more abstract and generalised and removed from the context of the problem; they had progressed to the *general* level in the reification process (Gravemeijer, 2004).

An interim level was observed in this process which I have termed *structuring*. In the process of making sense of the multiplicative structure, students worked independently of the context with a previously understood form of the array. They were no longer working at the referential level, nor were they generalising. The array had not yet become a tool for more formal mathematical reasoning as students were not engaged in reflection, explanation and justification. Student activity was focused on sense-making through an exploration of the multiplicative structure, removed from the context of the problem.

Central to this process of structuring was the flexibility for students to move between different forms of the array. On several occasions when using a form of the array connected to the context of the problem, students were faced with their own insufficient or incomplete internal representations (Goldin & Shteingold, 2001). In these instances, students would 'fold back' (Pirie & Kieren, 1994) to the simpler form of the external representation: the array with all parts visible, as illustrated by Amelie's working. Students would use this form of the array to explore the multiplicative structure and to make sense of what was happening mathematically. The evidence suggested that students were creating new connections and strengthening existing connections between their internal representations and, in so doing, building a deeper, or 'thicker', mathematical understanding (Pirie & Kieren, 1994). This process of thickening understanding was removed from the context of the problem.

At this structuring level, students also needed to work independently of the context of the problem in order to make sense of the mathematical properties of the array. As powerful as a context can be in enabling students' access to mathematical ideas, it can also be a hindrance. Students' strategies can be bound within the context of a problem (Ambrose et al., 2003) and the array may not be recognised as a multiplicative representation. This does not mean that contextual situations should not be used to introduce mathematical content. However, students must have the opportunity to work independently of the context in order to connect the array representation with the mathematical concept being explored. It is the array representation, not the context, that highlights important theoretical properties of multiplication.

Conclusion

The process of reification of the array in multi-digit multiplication highlights how students progress from using the representation as a model of a particular situation to using the array for more generalised mathematical reasoning. Mapping this process contributes to the growing body of knowledge on how the array supports the development of understanding and provides guidance to curriculum designers and practitioners. Students need the opportunity to use the array to explore the multiplicative structure. In their explorations, students should not be restricted to one form of the array. Rather, flexibility is needed. Students should be afforded the opportunity to select and use different forms of the array, recognising that different forms will serve different functions.

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Leading mathematics: Doings of primary and secondary school mathematics leaders

Colleen Vale
Monash University
<colleen.vale@monash.edu>

Anne Roche
Monash University
<anne.roche@monash.edu>

Jill Cheeseman
Monash University
<jill.cheeseman@monash.edu>

Ann Downton
Monash University
<ann.downton@monash.edu>

Ann Gervasoni
Monash University
<ann.gervasoni@monash.edu>

Penelope Kalogeropolous
Monash University
<penelope.kalogeropolous@monash.edu>

Sharyn Livy
Monash University
<sharyn.livy@monash.edu>

James Russo
Monash University
<james.russo@monash.edu>

Effective middle leading of mathematics is a complex task as it requires a focus on improving learning outcomes for students. This study gathered information about the activities of middle leaders of mathematics using a survey of primary and secondary mathematics leaders. Both primary and secondary mathematics leaders more often focussed on interacting with students in the classroom and participating in team planning meetings. Secondary leaders mentored teachers more often than primary leaders. Time to conduct some of the less frequent but more effective leadership activities needs to be provided.

Previous studies of instructional leadership have theorized the role and responsibilities of middle leaders (Kemmis et al., 2014, Lingard et al., 2003, Sexton, 2018). Studies of mathematics leading have reported on particular projects involving mathematics leaders (for example, Grootenboer et al., 2015b). Few studies have reported on what leaders of mathematics in schools actually do. In this paper we report on the activities of mathematics leaders in primary and secondary schools in Victoria, Australia, in order to understand the support that needs to be provided by school and system level leaders.

Background

Middle leading is a complex task (Kemmis et al., 2014). It involves teaching, administration, managing, and curriculum and pedagogical development (Sexton & Downton, 2014). Grootenboer et al. (2015a) argued that middle leading is significant because middle leaders are located between the school leader and teachers and therefore participate in both the leadership and teaching practices of the school. Also as they are teachers, typically middle leaders are aligned philosophically with their teacher colleagues and therefore are able to collaborate with teachers in their day-to-day practice. Finally, middle leading is significant because it is a practice that involves "the sayings, doings, and relatings of leading rather than the characteristics and qualities of middle leadership" (Grootenboer et al., 2015a, p. 18). Driscoll (2017) argued that the focus of middle leaders' practice should be on teacher development to improve the learning outcomes of students.

Lingard et al. (2003) claimed that effective pedagogical leading engages teachers in collaborative, critical, and reflective discussion about their practices and students' learning. Productive leadership relies on school leaders providing the support and opportunities for

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middle leaders to create a collaborative culture and practice. Martinovic and El Kord (2018) conducted a review of the literature on leading mathematics in schools. Two of the studies reviewed focused on the 'doings' of leaders. Masters (2010) reported that middle leaders analyse samples of student work, co-plan with teachers, co-teach lessons, review efficiency of teaching, and celebrate professional learning. Calderone et al. (2018) reported qualities of middle leaders that included their expertise in teaching, and their practices of leading such as actively listening, encouraging success of colleagues, facilitating communities of learning, confronting barriers in school culture and structure, and striving for authenticity in teaching, learning and assessment.

The role of leaders of mathematics (and other subjects) is not specified for government schools. In Victorian public primary and secondary schools,' various titles are used for middle leaders of mathematics. For example, Learning Specialist, Numeracy Leader, Numeracy Coordinator, Maths Domain Leader, Numeracy Learning Specialist, Professional Learning Community Leader, Maths Curriculum Team Leader, and Maths Leader. The Australian Standards for teachers include descriptors of competencies and knowledge for lead teachers concerning professional learning and engaging with colleagues, parents and community and do specify roles or activities for teachers at the level of lead teacher (AITSL, 2017). These include planning and developing professional learning for colleagues (6.1), initiating collaborative relationships (6.2), implementing professional dialogue to improve outcomes of students (6.3), and lead strategies to support professional learning opportunities for colleagues (6.4) (AITSL, 2017). In Victoria, the framework that describes levels of "Instructional shared leadership" expects that school leaders will lead teaching and learning. They "model and demonstrate high levels of pedagogical practice" and "align instructional planning and curriculum planning with the goals of the school" (Department of Education and Training [DET], 2019).

It is therefore not clear what mathematics leaders are expected to do. In this study we invited mathematics leaders in Victorian government schools to provide information about their leadership activities. The research questions were:

- *What leadership activities do school mathematics leaders do and how often?*
- *What are the similarities and differences in the leadership activities of mathematics leaders in primary and secondary schools?*
- *How much time is allocated to primary and secondary mathematics leaders to do this work?*

The Study

This study is part of the Numeracy Leaders' Needs Analysis (Vale et al., 2020) designed to understand the contexts of teachers who have the responsibility for leading improvement in mathematics teaching and learning. The Numeracy Leaders' Needs Analysis set out to identify the activities, knowledge, wishes, goals, and challenges of mathematics leaders in primary and secondary schools in order to identify their professional learning needs as well as to seek their preferences for their professional learning. In this paper, we report on the activities and time allocation for leaders.

The *Numeracy Leaders' Needs Analysis* questionnaire gathered responses online through Qualtrics. The questionnaire included 24 items with a mixture of Likert items, ranking items, multiple-choice items and open-ended items.

There were three Likert items about the frequency of various leadership activities. The sub-items consisted of a range of possible activities drawing on findings from qualitative studies (for example, Driscoll 2017, Cheeseman & Clarke 2005, Sexton & Downton 2014)

and their authors' professional experiences. The sub-items were organised into three sets to reflect the main contexts in which middle leaders work (Grootenboer et al., 2015a):

- Leadership in the classroom (Question 1, includes seven sub-items)
- Leadership beyond the classroom (Question 2, includes twelve sub-items)
- Managing and administration (Question 3, includes four sub-items)

These three items used a seven-point Likert scale from 'Not at all' (1) to 'Very often' (7). The items were checked for face validity by one author and two volunteers. Descriptive statistics, that is, frequencies were calculated for all closed items, including by school sector and regional location of the school. Means and standard deviations were calculated for the Likert items, and a two-tailed t-test conducted to compare the frequency of leadership activities between primary and secondary leaders of mathematics.

One hundred and ninety-six (196) people responded to the questionnaire. The majority (71%) worked as numeracy leaders, specialists, or teachers in primary schools. About a quarter (23%) worked as leaders or teachers in secondary schools. The other participants (6%) included leaders or teachers working in, or with, Special Education schools or with networks of schools. The proportion of responses from primary and secondary leaders approximately corresponds to the proportion of primary and secondary schools in Victoria (69% and 31% respectively). About two-thirds of respondents (65%) were from metropolitan schools and one-third from non-metropolitan schools (35%). Respondents included leaders from very small primary schools with fewer than 50 students (4% of primary leaders) to large primary and secondary schools with more than 1000 students (2% of primary respondents and 26% of secondary respondents).

Findings

Data about the number of years teaching and leading mathematics is provided first followed by findings regarding the doings of primary and secondary leaders and then the time available to do these leadership activities.

Teaching and leadership experience

It was also important to understand the extent of their teaching and leading experience as factors that may influence their activities as leaders of mathematics (see Table 1). Twenty-nine (29) of respondents were not currently the school mathematics leader. Almost all the leaders, 99% of primary leaders and secondary leaders responding to the questionnaire had more than 3 years' teaching experience. However, 30% of primary leaders and 23% of secondary leaders had been leading mathematics for less than one year. A higher proportion of secondary mathematics leaders had been leading mathematics for more than three years (33% compared to 22%).

The two least frequently conducted activities by both primary and secondary leaders were "Co-teach mathematics alongside teachers and review lesson" and "Model mathematics lessons for other teachers" (see Table 2). Secondary leaders tended to "Observe and talk with students about their learning during mathematics lessons, and provide feedback for the teacher" more often than "Collect, analyse and discuss student work samples with the classroom teacher." For primary leaders they tended to analyse student work slightly more often than conducting peer observations.

Table 1
Number of years teaching and leading mathematics

	Teaching (n=196)		Leading mathematics (n=167)	
	Primary n (%)	Secondary n (%)	Primary n (%)	Secondary n (%)
Less than 1 year	1 (0.7)	0 (0)	39 (30.4)	9 (23.0)
1-3 years	1 (0.7)	1 (2.2)	55 (42.9)	15 (38.4)
4-9 years	44 (29.3)	17 (37.0)	28 (21.9)	10 (25.6)
10-15 years	34 (22.7)	13 (28.3)	3 (2.3)	4 (10.3)
Longer than 15 years	70 (46.7)	15 (32.6)	3 (2.3)	1 (2.6)

Table 2
Leading mathematics in the classroom (Q1)

	Primary mean (SD)	Secondary mean (SD)	t-test p
a. Model mathematics lessons for other teachers.	3.7 (1.9)	3.7 (1.9)	0.89
b. Co-plan individual mathematics lessons with classroom teacher(s).	4.6 (2.0)	4.1 (1.9)	0.15
c. Collect, analyse and discuss student work samples with the classroom teacher.	4.2 (1.8)	3.9 (1.7)	0.32
d. Co-teach mathematics alongside teachers and review lesson.	3.5 (2.0)	3.6 (1.9)	0.88
e. Observe and talk with students about their learning during mathematics lessons, and provide feedback for the teacher.	4.0 (2.1)	4.0 (2.0)	0.98
f. Use instructional walks to talk to students about their learning during a mathematics lesson.	5.1 (2.0)	5.4 (1.7)	0.24
g. Teach small groups of students for intervention or extension.	4.0 (2.3)	4.2 (2.4)	0.56

Leading mathematics beyond the classroom

The most frequent activity for both primary and secondary leaders when leading outside the classroom was “Participate in team mathematics planning meetings” (see Table 3). The independent two-tailed t-test found that secondary leaders (m=6.1) conducted this activity more often than primary leaders (m(s)=6.1, m(p)=4.9, t=-3.722, p<0.01). Secondary leaders also “Mentor teachers of mathematics” more often than primary leaders (m(s)=5.3, m(p)=4.2, t=-2.670 p<0.01). A third significant difference showed that secondary leaders more often “Design and lead mathematics assessment programs in the school” than primary leaders (m(s)=5.1, m(p)=4.2, t=2.810 p<0.01). These three activities were the three most frequent activities for secondary leaders. The second most common activity for primary leaders was “Facilitate or conduct professional learning for teachers of mathematics,” an

activity in the top four for secondary leaders.

A fourth significant difference was found for one of the least often activities. Primary leaders more often "Participate in a network of mathematics/numeracy leaders" than secondary leaders (m(p= 3.4, m(s= 2.8, t=1.938, p<0.01). Secondary school mathematics leaders were asked to indicate the number of non-specialist mathematics teachers that is, out-of-field teachers of mathematics, in their school. A third (33%) identified between one and three teachers who were teaching mathematics out-of-field and a further third had four or more teachers of mathematics who were not qualified to teach mathematics. Mentoring non-specialist teachers was among the least frequent activities for secondary leaders (m=3.2, SD=2.3), however the high standard deviation indicates that this varies more than other activities and likely reflects the number of non-specialist teachers at their school.

Table 3
Leading beyond the classroom (Q2)

		Primary mean (SD)	Secondary mean (SD)	t-test p
a.	Mentor teachers of mathematics.	4.4 (2.0)	5.3 (1.6)	0.00**
b.	Facilitate or conduct professional learning for teachers of mathematics.	4.7 (1.9)	4.7 (1.7)	0.98
c.	Participate in team mathematics planning meetings.	4.9 (2.1)	6.1 (1.2)	0.00**
d.	Facilitate meetings for assessment moderation.	4.0 (2.0)	4.3 (1.8)	0.44
e.	Facilitate formative assessment meetings to analyse student work.	3.7 (2.0)	3.7 (1.9)	0.91
f.	Facilitate meetings to analyse assessment data to refine and adjust curriculum based on identified needs of students.	4.2 (1.9)	4.6 (1.6)	0.15
g.	Design and lead mathematics assessment programs in the school.	4.2 (1.9)	5.1 (1.7)	0.01*
h.	Engage parents and community in the school's mathematics program.	2.9 (1.7)	2.8 (1.5)	0.86
i.	Facilitate meetings to evaluate strengths, weaknesses, and opportunities for improving teaching of mathematics/numeracy.	4.0 (1.9)	4.2 (1.6)	0.51
j.	Lead the design of goals for improving mathematics/numeracy teaching.	4.4 (2.0)	4.6 (1.9)	0.42
k.	Mentor teachers about opportunities for numeracy learning in other subjects.	3.4 (1.8)	3.2 (1.9)	0.49
l.	Participate in a network of mathematics/numeracy leaders.	3.4 (2.0)	2.8 (1.8)	0.04*
m.	Mentor non-specialist teachers of mathematics	NA	(2.3)	

*p<0.05 ** p<0.01

Managing and administration.

For the four items that asked leaders about their management and administration tasks both primary and secondary leaders frequently "Manage access to and purchasing of mathematics resources" (m(p)=5.3, m(s)=5.0) and "Manage mathematics assessment programs" (m(p)=4.5, m(s)=4.9; see Table 3). There were no significant differences for any of the four activities.

Most frequent doings

When comparing the frequency of activities across each of these leadership domains primary leaders most often "Manage access to and purchasing of mathematics resources" (m=5.3, SD=1.9), "Talk to students about their learning during a mathematics lesson," (m=5.1, SD=2.0) and "Participate in team mathematics planning meetings" (m=4.9, SD=2.1). Secondary leaders most often "Participate in team mathematics planning meetings" (m=6.1, SD=1.2), "Talk to students about their learning during a mathematics lesson" (m=5.4, SD=1.7), and "Mentor teachers of mathematics" (m=5.3, SD=1.6) (see Table 4).

Table 4
Managing and administration (Q3)

	Primary mean (SD)	Secondary mean (SD)	t-test p
a. Organise professional learning facilitated by external experts.	3.2 (2.0)	3.1 (1.8)	0.83
b. Manage access to and purchasing of mathematics resources.	5.3 (1.9)	5.0 (2.2)	0.36
c. Timetable and organise allocated planning time (APT).	3.2 (2.4)	3.3 (2.1)	0.82
d. Manage mathematics assessment programs.	4.5 (2.0)	4.9 (1.8)	0.17

Leadership support

School leaders can support mathematics leaders by providing time to complete mathematics leadership activities and responsibilities. Many of the primary and secondary leaders were provided less than two hours per week to complete their leadership activities (42% and 50% respectively) (see Table 5).

Table 5
Number of hours per week allocated for the School Mathematics Leadership role (Q17)

	Primary n (%)	Secondary n (%)
Zero hrs	28 (18.7)	7 (15.2)
< 2 hrs	35 (23.3)	16 (34.8)
2.1 - 4 hrs	16 (10.7)	9 (19.6)
4.1 – 6 hrs	13 (8.7)	9 (19.6)

6.1 – 8 hrs	8 (5.3)	2 (4.3)
8.1- 10 hrs	10 (6.7)	1 (2.2)
10.1 -20 hrs	33 (22.0)	2 (4.3)
> 20 hrs	8 (5.3)	0 (0.0)

Whilst the distribution of time release for leading mathematics corresponds with the number of teachers of mathematics for both primary and secondary leaders, it is not surprising that many leaders have been limited in the opportunity to frequently conduct many of the activities included in the instrument.

Discussion and Conclusion

The activities of middle leaders of mathematics reflect the complexity of this role which includes teaching, working with their teaching colleagues as well as conducting administrative tasks as reported previously (Grootenboer et al., 2015b; Sexton & Downton, 2014). For both primary and secondary leaders talking with students about their learning was one of the most frequent activities and so was participating in team planning meetings. It is not clear from this study what was actually involved in these planning meetings and whether they took a leadership role in these planning meetings to encourage teachers to develop evidence-based practice (Grootenboer et al., 2015b). It seems unlikely, as facilitating meetings to discuss formative assessment of students was among the least frequent activities for both primary and secondary middle leaders. Secondary leaders frequently mentor other teachers rather than use other strategies for professional learning for non-specialist and beginning teachers such as peer observation, co-planning and co-teaching, or conducting professional learning activities. The limited use of these activities by both secondary and primary leaders indicates a need for their professional learning.

Middle leaders are expected to lead the improvement of mathematics teaching in their school, but they have very limited time allowance to enact some of the more effective practices to achieve their vision for teaching and goals for student learning (Roche et al., , 2020). School leaders need to be encouraged to provide more time for middle leaders to develop collaborative practices (AITSL, 2017) and shared meanings of effective practice (Kemmis et al., 2014). Participating in network meetings with other middle leaders of mathematics was not a frequent activity. An implication from this study is for local system leaders of mathematics to be encouraged to provide opportunities for middle leaders to meet to learn from each other and support each other (Proffitt-White, 2017) to effect strategies for improving the teaching of mathematics in their schools.

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What sense do children make of "data" by Year 3?

Jane Watson
University of Tasmania
<Jane.Watson@utas.edu.au>

Noleine Fitzallen
La Trobe University
<N.Fitzallen@latrobe.edu.au>

Statistical terms are used in everyday language and, at times, used in non-statistical ways. It is often assumed students understand statistical terms because of their common use; however, research into their understanding of specific statistical terms is scant. This report focuses on 58 Year 3 students' responses to the basic question, "What does the term 'data' mean?", and associated examples of data and data representations. The results indicate students are making progress in establishing meaning about data and their representations. Recommendations include more use of varying contexts within which students can explore data to enrich and enhance their learning about the practice of statistics.

Statistics in school curricula in Australia dates to the *National Statement for Mathematics in Australian Schools* (Australian Education Council, 1991) following the US National Council of Teachers of Mathematics' (NCTM) publication of its *Curriculum and Evaluation Standards* (1989). Neither of these documents, nor the later *Principles and Standards for School Mathematics* (NCTM, 2000) or the *GAISE Report* (Franklin et al., 2007), defines the term "data". Indeed, the focus in early childhood professional learning for teachers has been on representing data and not specifically on defining the term data (e.g., Schwartz & Whitin, 2006). Reflecting this background, in the early years children are often introduced to activities that involve collecting and representing data (e.g., Taylor, 1997), apparently with the assumption that by giving them many examples of data, they will eventually "understand" what data stand for and what the term means. Russell (2006) claims that "[t]o understand what data are and how to use them, students must themselves be engaged in developing questions about their world and creating data to shed light on those questions" (p. 17) but does not go so far as to define the word. She stresses the importance of *creating data* by noting the connections data allow and the reason for their existence: "Data are not the same as events in the real world, but they can help us understand phenomena in the real world" (p. 17). In the adult world, Moore and McCabe (1989) define statistics in relation to defining data: "Statistics is the science of collecting, organizing, and interpreting numerical facts, which we call *data*" (p. xvii). Cobb and Moore (1997) go further in claiming that "Statistics requires a different *kind* of thinking, because *data* are not just numbers, they are numbers with a context" (p. 801). This statement complements well Russell's (2006) linking data to events in the real world.

New Zealand was likely the first country to define data, in its *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992): "*Data* A set of facts, numbers, or information" (p. 211). In Australia, the development of the most recent *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019a) began in 2010 and currently includes a definition of data as, "*Data* is a general term for information (observations and/or measurements) collected during any type of systematic investigation." The inclusion of "systematic investigation" in this definition adds a third element, "context", to "information" and "collection," for making meaning of data. From Year 1 of the curriculum, the word "data" appears with the creation of representations with objects or drawings, along with descriptions of displays (Year 1, ACMSP263). By Year 2, students are gathering, checking, classifying, creating displays, and interpreting categorical data for a question (ACMSP048, ACMSP049, ACMSP50). In

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Year 3, the representations may be lists, tables, picture graphs or simple column graphs (ACMSP069, ACMSP070). Although the word "context" is not used in the content descriptors, the act of interpreting data implies that students will link data with the representations created and contexts within which the data were collected.

The other source of curriculum input on data comes from the Numeracy component of the *General Capabilities* section of the *Australian Curriculum* (ACARA, 2019a). As part of Numeracy in that document, "Interpreting statistical information" is one of six interrelated capabilities in the learning continuum. "This element involves students gaining familiarity with the way statistical information is represented. Students solve problems in authentic contexts that involve collecting, recording, displaying, comparing, and evaluating the effectiveness of data displays of various types" (ACARA, 2019a). In addition, although the Achievement Standard of the *Australian Curriculum* for Year 3 includes "conducting simple data investigations for categorical variables," the Numeracy Capability reduces the element of "Interpreting statistical information" to "interpret data displays" for all year levels. It is not until the end of Year 6 that students are expected to evaluate or analyse data representations. Prior to that, students are only expected to be able to collect, record, and display data.

Research into children's early understanding of data and their representation has focused on using data in contexts meaningful for children (e.g., Russell, 1990) without asking for a description or definition of the word itself. Similarly, Fitzallen (2012) analysed children's early appreciation of data in the context of graphing and analysis, without asking specifically about the word itself. An extensive search of the research literature found no instances where children were asked the meaning of "data." Given the importance of the term and its definition in the Glossary of the *Australian Curriculum* (ACARA, 2019a), it seems appropriate to ask this question of students.

The results reported in this paper are drawn from the beginning of a four-year teaching intervention related to studying the impact of making data the focus of learning activities, with the goal of enhancing the emerging STEM curriculum (Fitzallen & Watson, 2020). The student learning activities in the study were grounded in the concepts imbedded in the Practice of Statistics (Watson et al., 2018), which encapsulates all aspects of working with data: formulating questions, collecting data, analysing data, and interpreting results. At the beginning of the longitudinal project, Year 3 students were asked to respond to the item in Figure 1 as part of a pre-test of students' initial understanding related to the goals of the project. In retrospect, however, it also provided the opportunity to monitor the implementation of the *Australian Curriculum* definition of "data" and expectations for creating representations from data over the previous 3+ years of schooling (Foundation to Year 2 and half of Year 3). The research question hence becomes: *How well do Year 3 students understand data in relation to the Australian Curriculum's definition of "data", and its expectations related to data displays?*

Survey Questions about data
(a) What do you think "data" means?
(b) Give an example of some data you have seen or collected.
(c) Sketch a graph of the data.

Figure 1. Survey item for Year 3.

Method

For the research question asked in this report, a survey method using open-ended questions is appropriate to obtain the required data. Ballou (2008) suggests open-ended questions provide the opportunity to gain insights into how terms are understood, and ideas are developed. The three tasks in Figure 1 solicit qualitative data related to students' understanding of the topic of interest: a basic appreciation of the meaning of data and how they might be represented. The item was included in an eight-item survey. The other items focused on visual representations, sampling, and questioning.

Participants

Fifty-eight students from two Year 3 classes in a parochial K-10 school in an inner regional centre with a socio-economic status index ICSEA value of 1026 (mySchool.com.au; mean of 1000 and standard deviation of 100) were surveyed: 33 boys and 25 girls, 8-9 years of age. At the time students completed the survey, it had been two months since the NAPLAN testing for Year 3 had taken place. At the time of the survey, the researchers had no background on the teachers or the students. In terms of the results on NAPLAN testing nationally, the Year 3 cohort in this study was in the Average range for Reading, Writing, Grammar, and Numeracy, and in the Above Average range for Spelling (ACARA, 2019b). These results, and the fact that the teachers and students had no content interaction with the researchers prior to the survey, suggest that the sample can reasonably be assumed to be only marginally above average for Australian Year 3 children at this time in their education. The project had approval of the Tasmania Social Sciences Human Research Ethics Committee (H0015039).

Data Analysis

Due to the cognitive nature of mathematics learning, the method of analysing the data involved characterising similar responses with relation to a learning theory. The hierarchical model chosen was the *Structure of Observed Learning Outcomes* (SOLO) model (Biggs & Collis, 1982). The SOLO model has been used across the field of mathematics education for many years to analyse what respondents say or write (e.g., Watson, 2001) and continues to be useful in statistics education (e.g., Groth et al., 2019). For the survey questions in Figure 1, responses are expected to occur within the Concrete Symbolic (CS) mode, typical of students in the primary and middle years (ages 7 to 12 years). The levels are: Unistructural (Uni), where a single element or idea is presented; Multistructural (Multi), where responses include two or more elements presented in a serial fashion; and Relational (Rel), where responses describe links or relationships among the elements presented. Responses judged not to use elements involved in the task, including no response, are often labelled Pre-structural, but here they are examined from Groth et al.'s perspective, which considered in more detail the Ikonic mode (IK) for evidence of response compatibility (*c*) or incompatibility (*ic*) with the context of the task. Incompatible responses include superstitious, subjective, or deterministic beliefs, whereas compatible responses include personal experiences, imagery, or intuition related to the context of the task. It is hence of interest to observe responses considered to be in the IK mode for their compatibility as a step toward the CS mode. Using this structured analysis of students' responses, it is possible to suggest the degree to which a sample of children have had access to and taken on the goals of the curriculum in relation to "data", introduced by the middle of Year 3.

The coding scheme based on the SOLO model was designed to reflect the three components in the definition of "data" (information, collection, and context), the complexity of the example described, the representation created and its completeness, including the link between the representation and the context in the example. The elements that were considered appropriate for a definition of data (Figure 1, Part a) included a word interpreted as an appropriate synonym for "information" at the Year 3 level, a word related to the process of "collecting" information, and a word or phrase suggesting a meaningful context (systematic investigation) for information to be collected. Providing single elements was classified as Uni; putting two together as Multi; and combining all three in a meaningful sentence as Rel. For the example of data given (Figure 1, Part b), a single suggestion of a "variable" was considered Uni, whereas if it were connected with a second variable, it was considered Multi. This question did not lead to the expectation of a Rel response. With respect to Part (c), the representations were categorised according to the three representations noted in the content descriptor for Year 3 of the *Australian Curriculum* (ACMSP069): pictographs, tables, and column graphs. Within each type of graph there were increasing levels of combining the elements required to construct the representation. For pictographs, a picture without labels or categories was considered IK. Supplying categories but no variation in represented data was Uni, whereas displaying variation across categories was Multi. For both Tables and Column Graphs, an incomplete representation or no labels added was considered IK, whereas Uni or Multi representations included either one or two, respectively, of the essential components of the entity. For Tables the components were tallies and totals and for Column Graphs they were one or both axes meaningfully labelled including the column bars. Given the way that the questions were linked, if a complete pictograph, table, or column graph was labelled to reflect the context of the example suggested in Part (b), the response was considered Rel. Given the expectations of the content descriptors and the definition of "data" in the *Australian Curriculum* (ACARA, 2019a), Table 1 outlines the SOLO response levels for the three questions asked of the students. The coding was initially completed by the first author and repeated separately by an experienced research assistant. Agreement was 83% with discrepancies decided by negotiation.

Table 1
SOLO Levels of Response to the Three Parts of the Survey Item on Data

Level	Part (a)	Part (b)	Part (c) Sketch a graph		
	Defining data	Example data	Pictograph	Table	Column Graph
IK (*c*, *ic*)	Idiosyncratic; self-reference; specific case	Idiosyncratic; source; tallies with no categories	Icons without labels or categories	Table without information	Incomplete/no labels; unconventional
CS Uni	Single element mentioned	Single aspect of process or context	Icons in categories not showing variation	Labelled categories with <u>either</u> tallies or totals	Column graph with <u>one</u> dimension meaningfully labelled
CS Multi	Linking information either to the process or context	Clear summary of data or process with context	Icons in categories displaying variation	Labelled categories with <u>both</u> tallies and totals	Column graph with <u>both</u> dimensions meaningfully labelled
CS Rel	Linking information to both process and context		Complete pictograph related to context in Part (b)	Complete table related to context in Part (b)	Complete column graph related to context in Part (b)

Results

The results are considered with respect to the three parts of the item. Table 2 contains the total number and percentage of representations coded for the four SOLO levels for the question, "What do you think 'data' means?" (Figure 1, Part a). Also included are indicative examples of student responses for each level. Two IK responses were considered incompatible (*ic*) with the context, and three were compatible (*c*). The word "information" (or an abbreviation) was used 23 times across the levels but sometimes, the meaning was conveyed in general terms. The ability to construct a sentence that related the ideas of collecting, information, and context, which was needed to be coded at the Rel level, was not demonstrated by many students.

Table 2
SOLO Levels for Part (a) of the Survey Item on Data

Level	What do you think "data" means?	%
IK (c, ic)	Like Friday 7th August 2015. (c) (ID115) It means a graph? (c) (ID151) Something you do. (ic) (ID165)	17%* (n=10)
CS Uni	Tells you stuff. (ID105) Information. (ID108) Like a survey. (ID132)	28% (n=16)
CS Multi	Collecting information. (ID103) Calculating graphs and collecting information. (ID145) Data means what you know and you put it into a graph. (ID148)	45% (n=26)
CS Rel	It means collecting information about people or a person. (ID104) Information collected on a question like = what is your favourite colour? (ID122) Data means that you collect knowledge about something and put it in a graph. (ID142)	10% (n=6)

*This value includes five students who did not reply to the question.

Table 3 contains responses to the request for examples of data (Figure 1, Part b). The difference between Uni and Multi responses depended on the implied action of collecting information or asking questions related to the example provided. Two IK responses were *ic*.

Table 3
SOLO Levels for Part (b) of the Survey Item on Data

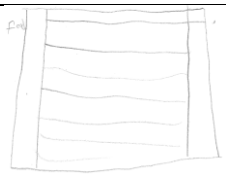
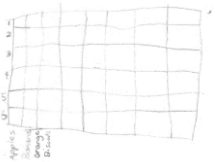
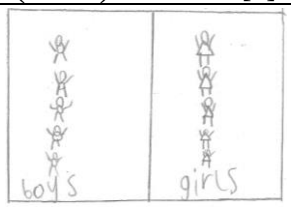
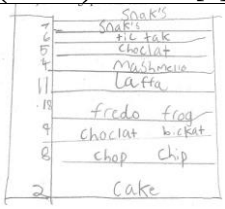
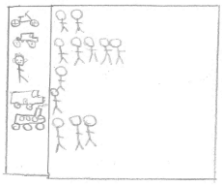
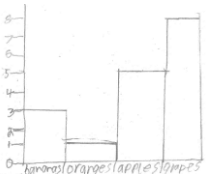
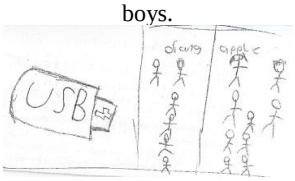
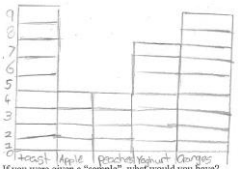
Level	Give an example of some data you have seen or collected.	%
IK (c, ic)	I have seen data with pictures in data. (c) (ID116) We/have done graph work. (c) (ID129) Making lunch? (ic) (ID164)	19%* (n=11)
CS Uni	How many boys or girls. (ID102) A food graph. (ID104) I have some data of the earth. (ID140)	26% (n=15)
CS Multi	How many people had cake for recess. (ID103) What is your favourite colour. (ID108) Who ate what fruit and veg. (ID119)	55% (n=32)

*This value includes six students who did not reply to the question.

With respect to Part (c) (Figure 1), the students produced three types of representation: Pictographs (n=5), Tables (n=13), and Column graphs representing frequency (n=37), as expected by Year 3 (ACARA, 2019a). Table 4 contains examples of each level of representation that was assessed for the three graph types. The numbers in square brackets

in each cell indicate the number of representations in that category, whereas the percentages in the right column represent the percentages across the three categories combined. For Pictographs, one variable is represented at the Uni level and two variables for Multi. At the Uni level for Tables, the list is supplemented by totals (could be tallies), whereas both are present at the Multi level. Similarly, for the Column graphs, the bars are accompanied by labels on either one (Uni) or two axes (Multi). At the Rel level, the response provided in Part (b) is included to demonstrate the connection made between the two questions by the student. The seven IK responses were considered compatible with the context.

Table 4
SOLO Levels for Part (c) of the Survey Item on Data

	Sketch a graph of the data					
Level	Pictograph		Table		Column graph		%
IK (*c*)							17%* (n=10)
	(ID164)	[1]	(ID108)	[2]	(ID155)	[4]	
CS Uni							28% (n=16)
	(ID103)	[1]	(ID120)	[4]	(ID131)	[11]	
CS Multi							12% (n=7)
	(ID102)	[1]	(ID143)	[1]	(ID121)	[5]	
CS Rel	7 boys liked oranges and 8 girls liked apples, so did two boys. 		Who ate fruit and veg. 		I collected data for a healthy breakfast. 		43% (n=25)
	(ID111)	[2]	(ID119)	[6]	(ID145)	[17]	

*This value includes three students who did not reply to the question.

Although blank responses are a concern, the presence of 17 IK responses across the three questions, with only four considered incompatible with the contexts, suggests that expectation of CS responses is reasonable in Year 3. Of particular interest is the association between the responses to Parts (a) and (c). Whereas 43% of students could produce a Relational level representation linked to the data in their examples, only 10% could provide a complete Relational definition of data. Fifteen students performed better on Part (a) than Part (c), whereas 26 did better on Part (c), with 14 consistent across the parts. An indicative Pearson's correlation coefficient (r=0.302, p<0.05) suggests significance but only about 9%

of shared variance. This is a sign that there is not a strong relationship between these two aspects of early learning about data.

Discussion and Conclusion

Interest in the question in the title of this paper arose at the beginning of a longitudinal project that was underpinned by the practice of statistics, fundamental to which are the data collected to answer a statistical question. Aware of the *Australian Curriculum*'s (ACARA, 2019a) definition of "data" but finding no published report of students' responses to the question prompted including the question as a survey item.

The official definition of data in the *Australian Curriculum* (ACARA, 2019a) elaborates on the word "information" in parenthesis with "observations and/or measurements", as well as with the reference to collecting data for "any type of systematic investigation." In the definitions provided by the students in this study, 40% mentioned a version of the word "information" but only one student mentioned "measuring"; none mentioned observations or observing. It may be that teachers are not making the distinction that information in the context of statistical investigations can be numerical or categorical, and measurable or observable in nature. It is possible closer attention to the definition and meaning of data will make the use of data more meaningful for students when answering statistical questions (Russell, 2006) and conducting systematic investigations (Watson et al., 2018).

It does, however, appear that young students are given the background to represent data in many ways. The students in this study utilised tallies, tables, pictographs, and column graphs, all of which are expectations of the curriculum at Year 3. This reflects appreciation of the quantifiable nature of data and the notion that data are plural in nature and collected from multiple sources as seen at all SOLO CS levels. That many responses to "What do you think data means?" described data in very general, non-quantifiable ways suggests a disconnect between how data are described and how they are represented (e.g., the correlation reported). Making explicit the connections between these two aspects of a statistical investigation in Year 3 may help students in posing questions that generate meaningful data that can be represented and analysed, part of the practice of statistics with which they have been shown to have difficulty (e.g., English et al., 2017; Wright et al., 2020).

Making meaning from data and creating data are emphasised in both the curriculum and the extant literature on student learning of statistical concepts. In terms of the contexts suggested in Parts (b) and/or (c) of the survey item, 44 students (76%) based their contexts around food, including food at recess, food for breakfast, and fruit choices. Although investigations about the contents of young students' lunch boxes provide convenient and legitimate data collection opportunities, they potentially limit exposure to contexts in which students can conduct a systematic investigation, learn about different data types, and explore how data explain and are influenced by the context of the investigation (Fitzallen & Watson, 2011; Russell, 2006). There are many resources available that provide engaging contexts for investigations that require observations and measurements to collect information (e.g., Fitzallen & Watson, 2020). It is recommended teachers embrace the learning opportunities made available when students' experiences with statistical concepts are positioned within investigations across the curriculum that explore issues related to a range of contexts.

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Preservice teachers' wellbeing balance when learning mathematics and numeracy

Philemon Chigeza
James Cook University
<philemon.chigeza@jcu.edu.au>

This presentation reports on teaching and learning practices in teacher education that address preservice teachers' wellbeing when learning about mathematics and numeracy. The participatory research study evolved through three phases. Data collected included a survey and focus group interviews with preservice teachers and open-ended interviews with teacher educators. Four themes that emerged from data analysis include the need to: (i) proactively address the emerging dynamic state of stable wellbeing; (ii) understand that lack of challenges can be detrimental to the emerging dynamic state of stable wellbeing; (iii) address the overlapping challenges that can exist for preservice teachers and educators that can negatively affect learning; and (iv) the need for guiding frameworks to help address the emerging challenges. The presentation discusses possible implications to the practice of teaching and learning in mathematics and numeracy classrooms.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, p. 417. Singapore: MERGA.

Pre-service teachers on the use of mobile apps for geometry

Shiyama Edirisinghe
University of Waikato
<mnse1@ waikato.ac.nz >

Nigel Calder
University of Waikato
<nigel.calder@waikato.ac.nz>

Sashi Sharma
University of Waikato
<sashi@waikato.ac.nz

The Sri Lankan curriculum stipulates the use of digital technologies in learning, but the practice is different for most teachers. Two case studies were conducted in two teacher education institutes in Sri Lanka to examine perceptions on the use of mobile applications in geometry after the block-teaching experience of pre-service teachers. The study followed the mixed method, explanatory sequential design. The findings of this study will contribute to the literature addressing new models relevant to the pedagogy perspectives of pre-service teachers' use of mobile applications for secondary geometry.

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Digital competencies of high school mathematics teachers in Pakistan: A pilot study to validate an online survey

Mairaj Jafri
University of Waikato
<mj115@students.waikato.ac.nz>

This presentation reports the results of a pilot study for a PhD research that investigates the digital competencies of high school mathematics teachers. The main objective was to identify appropriate survey questions in the context of mathematics teaching with digital resources in Pakistan. The pilot study was conducted with 42 participants. The response rate was 36 per cent. The results demonstrated that the study protocols are feasible. The changes made in the instrument included rewording, layout, structure, statements and flow of the survey items. As the research on digital competence continues, I believe the mathematics research community can use the survey in different contexts. With possible new frameworks that may emerge in future studies on digital competence, the survey can be further refined.

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Analysis of secondary school textbooks on trigonometric identities

Si Ying Lim
National Institute of Education, Nanyang Technological University
<nie184461@e.ntu.edu.sg>

Textbooks play a significant role in the teaching and learning of mathematics. They aid teachers in framing their lessons, and students in learning the subject content. In this research, I analysed three Singapore Ministry of Education (MOE) approved Additional Mathematics textbooks to examine the approaches used to develop concepts of trigonometric identities, as well as the cognitive demands of the exercises provided by the textbooks. It was discovered that different textbooks use different approaches for conceptual development and most of the exercise questions are at the basic or intermediate levels.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, p. 420. Singapore: MERGA.

Pedagogical and epistemic beliefs of pre-service secondary mathematics teachers: A pilot study

Margaret Marshman
University of the Sunshine Coast
<mmarshma@usc.edu.au>

Anne Bennison
University of the Sunshine Coast
<abenniso@usc.edu.au>

As problem solving and reasoning are best suited to constructivist teaching approaches, initial teacher education courses should promote beliefs aligned to these approaches. This pilot study investigated the beliefs of a small cohort of secondary mathematics pre-service teachers (PSTs) and their intended pedagogical practices. Data from an online survey were analysed using descriptive statistics and data from interviews were coded using content analysis to identify consistency between the PSTs' beliefs and their intended pedagogical practices. The PSTs' beliefs could be categorised as constructivist or developing towards constructivist and their pedagogical beliefs were aligned to their intended pedagogical practices.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, p. 421. Singapore: MERGA.

Teaching 21st Century skills in the mathematics classroom

Jennifer Missen
Radford College
<Jennifer.missen@radford.act.edu.au>

This presentation describes the necessity and difficulties faced by Mathematics teachers when working towards implementing 21st Century skills. While the experience and examples come from the Australian Context, the difficulties are faced across most countries and jurisdictions. The presentation introduces two approaches to incorporating these 21st Century skills but admits that much work still needs to be done to support teachers to do this successfully.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, p. 422. Singapore: MERGA.

Hands, Head and Heart (3H) framework: More evidence for self-similarity

Da Yang Tan
Singapore University of Technology and Design
<dayang_tan@sutd.edu.sg>

Eng Guan Tay
National Institute of Education, Nanyang Technological University
<engguan.tay@nie.edu.sg>

Kok Ming Teo
National Institute of Education, Nanyang Technological University
<kokming.teo@nie.edu.sg>

Paul Maurice Edmund Shutler
National Institute of Education, Nanyang Technological University
<paul.shutler@nie.edu.sg>

In this presentation, we report on more detailed evidence that shows self-similarity in the *Hands, Head and Heart* (3H) framework (Tan et al., 2021) for curriculum review. We describe two examples where the framework was validated: (a) pre-service teachers' surveys and (b) meeting transcripts between the committee members. The analysis of the data uncovered both the interactions and the self-similarity of 3H domains, both of which are the key features of the 3H framework.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, p. 423. Singapore: MERGA.

A decade of MERGA research papers in mathematics teacher education

Daya Weerasinghe
Federation University Australia
<d.weerangsinghe@federation.edu.au>

The purpose of this presentation was to examine MERGA research papers in Mathematics teacher education in the past decade. This presentation includes an analysis about the country and university the study was carried out, number of participants, nature of teacher education course, research methods and methodologies involved, commonly used seminal work, theories and theoretical frameworks, and study findings. Research papers on Mathematics teacher education from 2010 to 2019 were downloaded from MERGA website. Document analysis was used as the research method. Findings show that most of these research papers are focused on pre-service teachers' mathematical knowledge for teaching and there are several under-researched areas.

2021. In Y. H. Leong, B. Kaur, B. H. Choy, J. B. W. Yeo, & S. L. Chin (Eds.), *Excellence in Mathematics Education: Foundations and Pathways (Proceedings of the 43rd annual conference of the Mathematics Education Research Group of Australasia)*, p. 424. Singapore: MERGA.

Using metaphors to evaluate pre-service teachers' attitude change over first year mathematics unit

Jonathan Adams
University of Canberra
<jonathan.adams@canberra.edu.au>

Tracy Logan
University of Canberra
<tracy.logan@canberra.edu.au>

This study builds on Brady and Winn's (2014) use of metaphors to understand pre-service teachers' attitudes towards mathematics. 32 pre-service teachers (PSTs) across early-childhood, primary and secondary initial teacher education (ITE) courses completed metaphors on their attitudes towards mathematics in Week 1 of a compulsory mathematics content unit. They reflected on these metaphors in the final week of the unit. Many PSTs demonstrate a negative emotional disposition towards mathematics (Harper & Daane, 1998), and this was reflected in students' initial metaphor. Upon reflection, while PSTs' metaphors did not change dramatically, many commented on an increase in confidence towards mathematics. The study emphasises the value of a mathematics content unit for PSTs.

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The impact of the COVID19 induced primary school closures on the use of engaging mathematics pedagogies

Catherine Attard
Western Sydney University
<c.attard@westernsydney.edu.au>

Steve Murphy
La Trobe University
<steve.murphy@latrobe.edu.au>

Lena Danaia
Charles Sturt University
<ldanaia@csu.edu.au>

Kathryn Holmes
Western Sydney University
<k.holmes@westernsydney.edu.au>

Jacquie Tinkler
Charles Sturt University
<jtinkler@csu.edu.au>

Fiona Collins
Charles Sturt University
<fcollins@csu.edu.au>

The COVID19 school closures forced many primary school teachers to adopt relatively unfamiliar remote-teaching practices in mathematics, and other learning areas. Given primary school mathematics experiences significantly influence students' ongoing engagement with mathematics (Larkin & Jorgensen, 2016; McPhan et al., 2008), it is important to understand how this disruption impacted mathematics pedagogies. Drawing on data from semi-structured interviews with Australian primary teachers from two separate studies, we apply the *Framework for Engagement with Mathematics* (Attard, 2014) to examine the pedagogies employed during school closures. We identify challenges and opportunities revealed by these pandemic experiences that can be addressed to develop the engaging use of online pedagogies in primary school mathematics.

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Pre-schoolers' number sense strategies and patterns of strategy use during interactions with multi-touch technology

Chengxue Ge
University of Queensland
<chengxue.ge@uqconnect.edu.au>

Stephen I. Tucker
University of Queensland
<s.tucker@uq.edu.au>

Siyu Huan
University of Queensland
<siyu.huan@uqconnect.edu.au>

Early number sense, including subitizing and composing, is fundamental for mathematics (Clements & Sarama, 2007; Samara & Clements, 2009). Multi-touch digital technologies can afford foregrounding fingers and gesture in experiencing and developing number sense (Baccaglini-Frank et al., 2020). Researchers used iterative stages of analytic memoing, coding, and theming to qualitatively analyse weekly videos of 18 4-5-year-old pre-schoolers playing the multi-touch number sense digital game Fingu for five weeks. Initial findings include: (a) use of subitising, composing, and less commonly, counting strategies, with corresponding finger patterns; and (b) strategy use patterns, often evident when encountering challenges. Potential implications include relevance of gesture, quantification strategies and flexible strategy use in developing early number sense.

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What teachers notice about students' online mathematical thinking

Anita Green
Monash University
<anita.green1@monash.edu>

Teachers spend part of each day communicating with their students. During this time important aspects of students' mathematical thinking are noticed. Teachers then interpret these moments and make decisions on how to act (Jacobs & Empson, 2016). The online environment offers teachers a different setting in which to notice student's mathematical thinking. Previous research in this area focuses on what teachers notice and why these moments might be worthy of teachers' attention (Sherin et al., 2011). In this study, informal interviews were conducted with primary school teachers on two separate occasions to investigate what cues lead teachers to noticing moments of mathematical significance. In this short communication, I present findings from the first round of interviews that were conducted while teachers were engaged in online learning with their students. The online environment provided different opportunities and cues for teachers to notice students' mathematical thinking and dispositions.

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Professional learning using a peer learning circle

Vesife Hatisaru
University of Tasmania
<vesife.hatisaru@utas.edu.au>

Sharon Fraser
University of Tasmania
<sharon.fraser@utas.edu.au>

Carol Murphy
University of Tasmania
<carol.murphy@utas.edu.au>

Greg Oates
University of Tasmania
<greg.oates@utas.edu.au>

STEM capability is accepted as one of the key competences necessary for creative thinking and problem solving. Many countries consider the issue of competence in STEM as important and incorporate strategies for its development during schooling, at the highest policy level (e.g., the USA). Students, however, sometimes perceive individual STEM disciplines (e.g., mathematics, science) as irrelevant abstract subjects dominated by rules and formulae. Perceptions such as these may deter them from studying STEM subjects and negatively influence their facility in them. Consequently, teachers of STEM need to employ explicit teaching strategies to urge students to engage more in learning tasks.

Research in STEM education suggests that the development of STEM competency requires effective learning environments. One of the elements evident in effective learning environments is the use of varied representations (e.g., visual, symbolic) and opportunities for students to make connections between them (sometimes referred to as "representational competence"). Although such practices have been advocated in the teaching of mathematics and science for some time, recently there has been a renewed focus on the use of representations in the teaching and learning of STEM (e.g., Glancy & Moore, 2013).

Working with representations plays a critical role in helping students develop flexible thinking and problem solving, and provides multiple entry points and access to the study of individual STEM subjects. The ability to create effective learning environments, inclusive of explicit strategies to develop students' representational competence, is one element of teacher knowledge. Expertise in this area is key to achieving desirable STEM learning outcomes.

Developing teacher knowledge is a focus of professional learning initiatives. Here we report the progress of an interdisciplinary learning circle (*Using Multiple Representations in Mathematics and Science Teaching Practices*) that met regularly over the course of a school semester to explore the use of representations in teaching and learning of STEM (with a focus on mathematics and science). The group developed their own understandings of representational competence, culminating in the development of learning tasks aimed at improving representational competence of mathematics and science undergraduates.

Acknowledgements

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From modelling perspectives to analyse the mathematics grounding activities in classes: Take the game of adding and subtracting decimal numbers module as an example

Wei-Hung Huang
National Taiwan Normal University
<onelfisher@gmail.com>

Wan-Ching Tseng
Taipei Municipal Nanhu Elementary School
<cloudiatseng@nhps.tp.edu.tw>

Orchestrating interesting and informative math activities within the limited classroom time is a complex task. Teachers need to master the game and course content at the same time and encourage students to participate in the construction of the concept of decimals. Through interesting mathematics activities, students are motivated to participate in the classroom, so that they can experience and construct models of mathematical concepts. This article aims to use the perspective of modelling as an analytical structure to understand how the mathematics grounding activities (MGA) are used in class and further to know its teaching effectiveness. In this study, we applied a hybrid approach. Participants are 28 third-grade students from urban schools and a teacher who has been teaching for 20 years. Data collection included observation videos, semi-structured interviews in five classes, and a learning attitude scale.

The results revealed that : (i) The teaching of the teacher in a decimal unit combined with MGA (The Game of Adding and Subtracting Decimal Numbers) mostly corresponds to the modelling teaching stage such as model construction, model validation, and model application; (ii) The teacher often use students' problem-solving results as teaching materials and invited students to evaluate, compare, and explain their peers' answers; (iii) If necessary, the teacher will be given a question as scaffolding integrate students' responses and re-narrate to help students construct and validate models; (iv) Using MGA in the classroom can improve students' interest in mathematics learning and self-confidence. (v) Students hoped that the mathematics class can be conducted like MGA.

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Disrupting deficit discourses in mathematics education: Documenting the funds of knowledge of young diverse learners

Jodie Hunter
Massey University
<j.hunter1@massey.ac.nz>

Both in New Zealand and internationally, diverse groups of people including indigenous, migrant, and other minority communities are under-represented in mathematics with an accompanying narrative or "gap story" in relation to achievement within school systems (Faulkner et al., 2019; Martin, 2019). Arguably, the privileging of white middle-class ways of knowing and being in the mathematics classrooms has led to these ongoing deficit discourses in mathematics education (Adiredja & Louie, 2020). Within the context of New Zealand, Pāsifika and Māori communities have been positioned within a deficit framing and a subsequent outcome has been a lack of awareness of the rich mathematics within these cultural groups. One way to challenge and disrupt deficit discourses is to highlight the strengths and resources of marginalized communities through a focus on mathematical funds of knowledge. This presentation will focus on the stories of mathematics at home and in the community from Pāsifika and Māori students from New Zealand and Niue (a small Pacific nation) to highlight what we can learn from the voices of minority communities.

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How can novice STEM teachers develop integrated STEM materials: The first step from mathematics textbooks

Takashi Kawakami
Utsunomiya University
<t-kawakami@cc.utsunomiya-u.ac.jp>

Akihiko Saeki
Naruto University of Education
<asaeki@naruto-u.ac.jp>

One of the barriers for novice STEM teachers to implement integrated STEM education with mathematics at its core is the development of teaching materials (e.g., Anderson et al., 2020). To break through this, we focus on mathematics textbooks (Fujii, 2016) as an important resource for teachers to design and find appropriate materials. The aim of study is to explore the possibility of translating mathematics textbooks into STEM teaching materials. We will focus on the "paper helicopter material", which is a statistical material in a Japanese seventh-grade mathematics textbook (Okamoto et al., 2016), and analyse the concepts and ideas in STEM fields contained in the material. The implications for teachers and teacher educators of the transformation of mathematics textbooks into STEM materials will be discussed.

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Overcoming issues of status and creating pathways for learning mathematics in one primary school classroom

Generosa Leach
Massey University
<G.Leach@massey.ac.nz>

In this presentation, I report on the pedagogical actions of one primary school teacher to provide equitable opportunities for all students to learn mathematics. Data were collected in one New Zealand primary school mathematics classroom over a year-long investigation examining how classroom environments can be restructured and revisioned as a means of striving toward equity. Initial attempts by the teacher to create a reform-style collaborative learning environment were impeded by issues of status. Status issues arise when generalisations relating to notions of other's perceived intellectual ability, social advantage, or cultural difference are made by peers (Cohen & Lotan, 1995; Dunleavy, 2015; Featherstone et al., 2011; Shah & Crespo, 2018). These generalisations create status hierarchies, which in turn affect student engagement in learning mathematics (Cohen, 1997; Langer-Osuna, 2016). In class, four students afforded themselves high status during mathematics lessons and dominated classroom discussions. The imbalance in status impeded all students' access to learning mathematics. Through critical reflection and enactment of specific pedagogical actions, the teacher mitigated these status issues, and pathways to learning mathematics for all students were created.

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Primary school mathematics teachers' exploration of integration strategies within a community of practice

Tarryn Lovemore
Rhodes University
<tarrynlove07@gmail.com>

Sally-Ann Robertson
Rhodes University
<s.a.robertson@ru.ac.za>

We report on initial stages of a study where a group of eight primary school mathematics teachers, guided by the first author, work within a *Community of Practice* (Wenger, 1998) to explore ways of integrating music note values into their teaching of fractions to learners in Years 4 to 6. The teachers trial, reflect on, and adapt strategies to exploit opportunities deriving from synergies between mathematics and music, and, in so doing, pursue the dual curriculum goal of deepening young learners' conceptual understanding of fractions while simultaneously helping them recognise the beauty and elegance of mathematics as a human activity.

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Investigating the disconnect of theory and practice: Differentiating instruction in secondary mathematics

Andrew Marks
Southern Cross University
<andrew.marks@lism.catholic.edu.au>

Geoff Woolcott
Southern Cross University
<geoff.woolcott@scu.edu.au>

Christos Markopoulos
Southern Cross University
<christos.markopoulos@scu.edu.au>

Lisa Jacka
Southern Cross University
<lisa.jacka@scu.edu.au>

A teacher's ability to effectively differentiate instruction in the classroom is crucial in catering for student individuality and diversity, especially in the context of inclusive learning. Tomlinson (2004) defines differentiated instruction as a pedagogical approach where teachers modify curriculum content, proactively develop a variety of teaching strategies, and continually revisit the desired product of learning. The goal of the teacher in a differentiated classroom, therefore, is to allow students to make connections with their prior learning and build upon their knowledge quickly and efficiently.

In the secondary mathematics classroom, however, the most common approach used to address diverse learning needs is to place the students into homogenous ability groupings ("streaming" or "tracking"). A flexible-grouping alternative, heterogeneous grouping, assembles a mixture of abilities in the same classroom, aiming to create a well–rounded blend of all levels that allows higher achieving students to mentor their peers in a supportive and cohesive environment, promoting the concept of inclusive education.

Differentiated instruction offered in heterogeneous groupings could foster positive learning environments in the Australian secondary mathematics classroom. Any potential advantages, however, such as embracing diversity in a way that provides for individual growth in learning (based on a student's ability, interest and readiness levels) have not been fully investigated. Therefore, the present study focuses on the ability and motivation of mathematics teachers to implement differentiated instruction effectively and sustainably and to thereby provide a new model of learner engagement.

This presentation outlines a prototype practice framework for mathematics teachers designed to transform mathematics education by leveraging recent progress in adapting theory to practice. Implementation of the framework should enable mathematics teachers, regardless of teaching experience, to progress on a continuum of practice, leading to differentiated instruction that is integral to their teaching. The framework engages collaboration and co-creation using a design-based implementation approach (Woolcott et al., 2019), in conjunction with a strategic focus on the guiding questions that form the basis of generative dialogue (Adams et al., 2019).

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Reflecting upon mathematical competency: An appreciative inquiry

Catherine McCluskey
University of South Australia
<catherine.mccluskey@unisa.edu.au>

A key principle underlying the Reggio Emilia approach is the recognition of children's existing capabilities and competencies (Infant-Toddler Centres and Preschools, 2010). Therefore, this presentation reports an appreciative inquiry into mathematical competency situated within a Reggio Emilia inspired primary school in South Australia (McCluskey & Moyse, 2020). This appreciative inquiry aimed to uncover teachers' use of language to describe children's mathematical competencies alongside identifying characteristics of effective practice (Gaffney & Faragher, 2010) to illuminate a sense of reciprocity between learning and teaching. This involved an iterative process of reflecting upon documented stories of mathematical learning and practice. Pedagogical themes emerging from the inquiry are identified and areas for further research are identified.

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Implementing a Spatial Reasoning Mathematics Program (SRMP) in Grades 3 through 4

Joanne Mulligan
Macquarie University
<joanne.mulligan@mq.edu.au>

Geoff Woolcott
Southern Cross University
<geoff.woolcott@scu.edu.au>

Michael Mitchelmore
Macquarie University
<mike.mitchelmore@y7mail.com>

Susan Busatto
Macquarie University
<susan.busatto@mq.edu.au>

Jennifer Lai
Macquarie University
<Jennifer.lai@mq.edu.au>

Brent Davis
University of Calgary
<abdavi@ucalgary.ca>

This presentation provides an overview of the development and trialling of a *Spatial Reasoning Mathematics Program* (SRMP) in one cohort of 30 students over an 18-month period in Grades 3 through 4. Integral to a larger study, *Connecting Spatial Reasoning with Mathematics Learning**, the SRMP embedded transformation skills in learning sequences comprising repeating and growing patterns, 2D and 3D relationships, structuring area and perimeter, directionality and perspective taking. There were significantly better gains by the experimental group on the *Pattern and Structure Assessment-2* (PASA-2) measure of awareness of pattern and structure, and on the PASA-Sp assessment of spatial ability at post-SRMP. There were no significant differences found between groups on the PATMaths4 test of mathematics achievement. Qualitative analyses indicated that students developed complex spatial concepts that supported their mathematical reasoning, well beyond curriculum expectations.

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Connecting calculation strategies through grounding metaphors

Carol Murphy
University of Tasmania
<carol.murphy@utas.edu.au>

Fluency is regarded as a key proficiency in learning mathematics (Sullivan, 2011). Such fluency requires strategic flexibility (Threfall, 2009) underpinned by rich connections and adaptive expertise (Baroody & Dowker, 2003). Whilst Australia has recently introduced numeracy learning progressions (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2000), there is little evidence of guidance for teachers to help students make rich connections. I draw on Lakoff and Núñez's (2000) theory of grounding metaphors to explore how the two source domains (an object collection domain and motion domain) may be used to underpin connections in calculation strategies in additive and multiplicative thinking.

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A scoping review of research into mathematics classroom practices and affect

Steve Murphy
La Trobe University
<steve.murphy@latrobe.edu.au>

Naomi Ingram
University of Otago
<naomi.ingram@otago.ac.nz>

We argue that affective research in mathematics is compelling when affect is explored within the mathematics classroom and in relation to classroom practices. We therefore used a systematic scoping review methodology (Peters et al., 2020) to identify a notably small data corpus of approximately 250 papers relating to mathematical classroom practice and student affect. Initial analysis described a range of classroom practices employed in mainly upper-primary and secondary school. Classroom practices were described to varying depths, including use of technology, teacher interactions, and collaborative group work, and were related to a range of, often poorly-defined, affective constructs.

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Supporting pre-service teachers of mathematics to 'notice'

Lisa O'Keeffe
University of South Australia
<Lisa.OKeeffe@unisa.edu.au>

Bruce White
University of South Australia
<Bruce.White@unisa.edu.au>

Video recordings have been shown to be effective in supporting pre-service teachers to better understand their own practice (Balzaretti et al., 2019, Clark et al., 2018). In our current research we explore the ways in which 360degree video can extend this, by creating an immersive experience for pre-service teachers to review their own practice from multiple perspectives. We present examples of pre-service teachers of mathematics' emerging understandings of their own practice and discuss ways in which their ability to 'notice' becomes a key element of their development.

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Collaborative problem-solving: An initial analysis of the role of prompts to support online learners in mathematics

Lisa O'Keeffe
University of South Australia
<Lisa.OKeeffe@unisa.edu.au >

Bruce White
University of South Australia
<Bruce.White@unisa.edu.au >

Amie Albrecht
University of South Australia
<Amie.Albrecht@unisa.edu.au>

Chelsea Cutting
University of South Australia
<Chelsea.Cutting@unisa.edu.au>

Bec Neil
University of South Australia
<Bec.Neil@unisa.edu.au >

Collaboration has been shown to be beneficial when problem solving in mathematics (Retnowati et al., 2017); however, it is difficult to achieve this collaboration in an online teaching and learning environment. As part of a project focused on exploring the potential of 360degree video to support and develop online learners' collaborative problem-solving experiences, the authors have video recorded groups of university mathematics students undertaking group problem solving. In the initial analysis of this 360degree video data, the theme of external (to the group) and internal (to the group) prompts emerged. We will present two examples of the ways prompts supported students to persist with working on their problem.

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Spatial and numeracy skills at the beginning of preschool: A large-scale, nationally representative study

Ilyse Resnick
University of Canberra
<ilyse.resnick@canberra.edu.au>

Tom Lowrie
University of Canberra
<thomas.lowrie@canberra.edu.au>

Early numeracy skills and spatial reasoning skills are both key predictors of later mathematics learning (e.g., Nguyen et al., 2016; Verdine et al., 2017), highlighting the critical role of preschool mathematics education in supporting mathematics achievement through the primary and secondary years. The current observational study engaged a nationally representative sample of 1,770 preschool children at the beginning of the academic year using a game-based digital activity to capture their patterning, spatial language, perspective-taking (a kind of spatial reasoning skill), and a range of numeracy skills. This talk will present on the findings, which informs preschool mathematics education.

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Assessment-related affect in mathematics: Results from a quasi-experimental study

Kaitlin Riegel
University of Auckland
<krie235@aucklanduni.ac.nz>

Tanya Evans
University of Auckland
<t.evans@auckland.ac.nz>

Jason Stephens
University of Auckland
<jm.stephens@auckland.ac.nz>

Student affect is interwoven with cognition and achievement (Zan et al., 2006), so it is vital to understand how affective constructs develop and change. We conducted a quasi-experimental study in a university mathematics course to test the effects of an intervention (Riegel & Evans, in press) on promoting positive assessment-related affect in students (N = 379). Preliminary results from cross-sectional analysis of Time 1 (baseline) data indicate that students' exam-related self-efficacy is predicted by their prior achievement, gender, stress mindset, and emotions. In the analyses to be presented, we will focus on how students' assessment-related affect changed during the semester and its relationship with their academic performance.

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Tuning-in to non-linguistic resources during collective problem-solving in a second language context

Sally-Ann Robertson
Rhodes University
<s.a.robertson@ru.ac.za>

Mellony Graven
Rhodes University
<m.graven@ru.ac.za>

Drawing on the claim that the *zone of proximal development* (Vygotsky, 1978) is multi-directional (Abtahi et al., 2017), we describe collaboration between an English-medium after-school mathematics club facilitator and four Year 3 learners in solving the sharing of 24 candy-sticks equally among five people. We show how, in the course of the interaction, the "more knowledgeable other" role shifted between participants, and how, despite the children's lack of English proficiency, the facilitator's prompting, in combination with the children's use of whiteboards to diagrammatically represent and share their thinking, and the physical presence of the candy-sticks, generated productive learning engagement towards the solution.

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From deficiency to strengths: Prospective teachers' shifting frames in noticing student mathematical thinking

Thorsten Scheiner
Institute for Learning Sciences and Teacher Education, Australian Catholic University
<thorsten.scheiner@acu.edu.au>

In recent years, mathematics education scholars have shown an increasing interest in teachers' professional noticing (e.g., Choy & Dindyal, 2020; Kaiser & König, 2019; Philipp et al., 2014; Scheiner, 2016). This presentation explores shifts in prospective mathematics teachers' frames in noticing students' mathematical thinking over time. The focus is on what prospective teachers attend to students' mathematical thinking, how they talk about what they notice, and in what ways both what they notice and how they talk about it changes over time.

In particular, two changes in prospective teachers' noticing are discussed in detail. First, prospective teachers changed what they have paid attention to students' mathematical understandings. Initially, teachers attended to missing aspects of students' mathematical thinking, and later, they attended to productive aspects of students' mathematical thinking that serve as resources for students' further learning. Second, prospective teachers changed with regard to the ways they have interpreted students' mathematical thinking. Initially, teachers interpreted students' mathematical understandings as faulty and deficient compared to the canonical understanding of mathematics. Later, they interpreted the same understandings as productive and valuable in their own right.

Analyses of data of prospective teachers' written responses to students' mathematical work are presented, and two framings of teacher noticing are discussed that resulted from these analyses: a deficit-based framing and a strength-based framing. These two framings are considered fundamental in accounting for the changes in teachers' noticing of students' mathematical thinking.

The presentation concludes with the outline of a model of teacher noticing that suggests that noticing is directed by teachers' framing (see Scheiner, 2021). More important, perhaps, this model suggests that perception and cognition reinforce each other, and that the teacher is an integral part of the world of classroom events. Implications of framing theory for the notion of teacher noticing are discussed, and its consequences for the study and development of teacher noticing are outlined.

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Accounting for embodiment via gestural number sense

Stephen I. Tucker
The University of Queensland
<s.tucker@uq.edu.au>

Traditional approaches to number sense (e.g., Clements et al., 2019) insufficiently accommodate embodiment, including gestures, which can support learning when conceptually congruent (Segal et al., 2014). This study examined weekly video recordings of 66 4-6 year old students regularly interacting with the multi-touch number sense app Fingu for 3-5 weeks. Iterative qualitative analyses included microgenetic learning analysis, analytic memoing, and eclectic coding. Four main types of gestural number sense emerged from this context: gestural subitising, gestural estimating, gestural composing, and gestural counting, plus subtypes and combinations. These can be considered conceptually congruent, embodied versions of number sense, but can also support reconceptualizing number sense to account for embodiment.

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Comparing mathematics curricula across countries: What do they tell us?

Jodie Hunter
Massey University
<J.Hunter1@massey.ac.nz>

Ban Heng Choy
National Institute of Education
Nanyang Technological University
<banheng.choy@nie.edu.sg>

A well thought-through mathematics curriculum is central to any efforts aimed at improving mathematics teaching and learning. Seeing curriculum as a collection of learning experiences—"the interaction between the learner and the external conditions in the environment to which he can react" (Tyler, 1949, p. 63)—it is crucial that curriculum documents are clear about the mathematics content to be taught and more importantly, how students are taught these ideas. International benchmark assessments, such as TIMSS, have provided opportunities for mathematics educators to "pursue questions about what makes a difference in those countries for students' learning of mathematics and science" (National Research Council, 1996, p. 2). Therefore, it is not surprising that countries have begun to examine the mathematics curricula of other nations to fine-tune their own curricular. However, we should be cautious about using observations from these comparisons to determine which curriculum is better, or even attempting to synthesise features from different curricula to fuse into our own. Instead, these observations can, at best, "point to questions for further investigation about educational practices and what they may imply for students' learning" (National Research Council, 1996, p. 3). But what can we learn from comparing mathematics curricula across countries? Given that each country has its own unique historical, cultural, political, and social contexts, it is very challenging to pose relevant questions and find answers that will help to refine the current mathematics curriculum. In this roundtable discussion, we will use examples from two contrasting economies—New Zealand and Singapore—to discuss the kind of questions and insights we can derive from such comparisons. This will have important implications for what and how we can learn from comparing mathematics curricula across countries.

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