Probabilistic computation and valuations monads on DCPO
some progress and many open problems

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The Jung-Tix Problem

The Troublesome Probabilistic Powerdomain

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Abstract

In [12] it is shown that the probabilistic powerdomain of a continuous domain is again continuous. The category of continuous domains, however, is not cartesian closed, and one has to look at subcategories such as **RB**, the retracts of bifinite domains. [8] offers a proof that the probabilistic powerdomain construction can be restricted to **RB**.

In this paper, we give a counterexample to Graham's proof and describe our own attempts at proving a closure result for the probabilistic powerdomain construction. We have positive results for finite trees and finite reversed trees. These illustrate the difficulties we face, rather than being a satisfying answer to the question of whether the probabilistic powerdomain and function spaces can be reconciled.

We are more successful with coherent or Lawson-compact domains. These form a category with many pleasing properties but they fall short of supporting function spaces.

Along the way, we give a new proof of Jones' Splitting Lemma.

Key words: Probabilistic powerdomain, FS-domains.

Achim Jung

Regina Tix

[Jung and Tix 1998] The Jung-Tix Problem asks for a nice category of semantic domains that can be used to model higher-order programming languages with probabilistic nondeterminism.

What do we mean by a "nice" category?

"NICE" $\begin{cases} \text{1. semantic domains should be nice mathematical objects} \\ \text{2. support higher-order programming languages} \\ \text{3. support probabilistic nondeterminism} \\ \text{4. versatile enough to support other computation features} \end{cases}$

How do we interpret these requirements in domain theory?

1. Nice mathematical objects

In domain theory, algebraic/continuous dcpos (algebraic domains/domains) are usually regarded as nice mathematical objects, on which a directed-complete partial order is imposed. And,

notions of approximation (the way below relation)

and

different (Scott, Lawson) topologies

can be defined naturally $\cdots$

Takeaway 1: semantic categories consist of algebraic/continuous domains.

2. Support higher-order programming languages

In early days, it became clear that the ability to define higher-order types in programming has its counterpart in the formation of function spaces of semantics domains, more precisely, in the requirement that the semantic category of domains be

Cartesian closed.

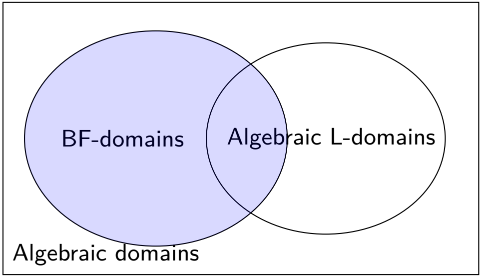
Takeaway 2: a nice semantic category should be Cartesian closed.

Takeaway 1 & 2

A nice semantic category should be

a Cartesian closed category consisting of algebraic/continuous domains.

Cartesian closedness of algebraic/continuous domains



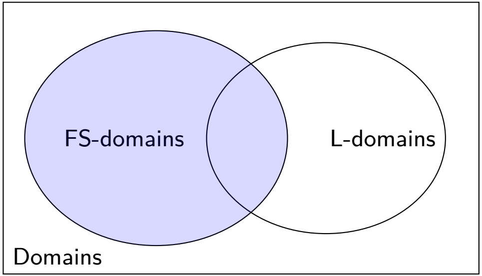
Michael B. Smyth

Achim Jung

Theorem [Michael B. Smyth 1983 and Achim Jung 1989]

Any Cartesian closed category of (pointed) algebraic domains and Scott-continuous functions consists of bi-finite domains or algebraic L-domains.

Cartesian closedness of algebraic/continuous domains



Theorem [Achim Jung 1990]

Any Cartesian closed category of (pointed) domains and Scott-continuous functions consists of FS-domains or L-domains.

Takeaway 1 & 2 updated

A nice semantic category should be

a Cartesian closed category consisting of BF-domains/FS-domains/(algebraic) L-domains.

3. Support probabilistic nondeterminism

- Traditional ways to model probabilistic phenomena: probability distributions and measures.
- Eugenio Moggi suggested that computational effects in computation correspond to monads on semantic categories. [Moggi 91]
- Claire Jones and Gordon Plotkin proposed the monad of continuous valuations as domain theoretical version of probability distributions/measures, and it has become one of the most dominant models for probabilistic computation in denotational semantics.

Eugenio Moggi

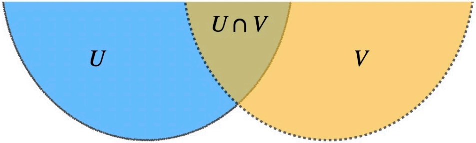
Claire Jones

Gordon Plotkin

3. Support probabilistic nondeterminism – continuous valuations

A continuous valuations on a topological space X is a Scott-continuous map $\mu : \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ that is:

1. strict - $\mu(\emptyset) = 0$;
2. monotone - $V \subseteq U$ implies $\mu(V) \leq \mu(U)$;
3. modular - $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$.



$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$$

3. Support probabilistic nondeterminism – examples:

- Dirac valuations: $\delta_x, x \in X$, $\delta_x(U) = 1$ if $x \in U$, and $\delta_x(U) = 0$ if $x \notin U$.
- Linear combinations of continuous valuations.
- Simple valuations: $\sum_{i=1}^{n} r_i \delta_{x_i}$, where r_i are nonnegative real numbers.
- Push-forward images of valuations along continuous maps f: $f_*(\nu) = \lambda O.\nu(f^{-1}(O))$.

3. Support probabilistic nondeterminism – continuous valuations

The set $\mathcal{V}X$ of all continuous valuations on X, is a dcpo in the stochastic order :

$$\mu_1 \leq \mu_2 \text{ if and only if } \mu_1(U) \leq \mu_2(U) \text{ for all } U \in \mathcal{O}X,$$

and the suprema of directed family of continuous valuations are computed pointwise:

$$(\sup_{i\in I} \mu_i)(U) \stackrel{\text{def}}{=} \sup_{i\in I}(\mu_i(U)).$$

For a dcpo L, $\mathcal{V}L$ is defined as $\mathcal{V}(L, \sigma L)$, and is called the probabilistic powerdomain of L.

3. Support probabilistic nondeterminism – Integration theory in non-Hausdorff setting

Integral of lower semicontinuous maps against continuous valuations can be defined as: For a lower semicontinuous map $f\colon X \to \overline{\mathbb{R}}_+$, and a continuous valuation ν on X;

$$\int_{x\in X} f(x)d\nu \stackrel{\text{def}}{=} \int_0^\infty \nu(f^{-1}((t,\infty])dt$$

Theorem [Claire Jones 1990]

$\int_X \_d\_$ is linear and Scott-continuous both in the integrand and the valuation component.

3. Support probabilistic nondeterminism – $\mathcal{V}$ is a monad on $\mathbf{DCPO}$

Monadicity: $\mathcal{V}$ defines a monad on $\mathbf{DCPO}$.

- the unit η_L at L is: $x \in L \mapsto \delta_x$;
- for $f\colon L \to \mathcal{V}M$, the Kleisli extension $f^\dagger\colon \mathcal{V}L \to \mathcal{V}M$ is

$$\mu \mapsto (\lambda U. \int_{x \in L} f(x)(U) d\mu);$$

- (multiplication) $m_L \stackrel{\text{def}}{=} \mathrm{id}^\dagger_{\mathcal{V}L}\colon \varpi \mapsto \lambda O. \int_{\nu \in \mathcal{V}L} \nu(O) d\varpi$.

Functoriality:

- On objects: $L \mapsto \mathcal{V}L$;
- On morphisms: $(f\colon L \to M) \mapsto \mathcal{V}f \stackrel{\text{def}}{=} f_* = \lambda O \in \sigma M. \nu(f^{-1}(O))$.

3. Support probabilistic nondeterminism

Theorem [Claire Jones 1990]

The construction $\mathcal{V}$ can be restricted on the category of domains and continuous domains, and bounded continuous valuations on domains could be uniquely extended to a Borel measure generated by Scott opens.

3. Support probabilistic nondeterminism

Takeaway 3: There is a nice theory of the valuations monad $\mathcal{V}$ and one could use it to denote probabilistic nondeterminism.

3. Support higher-order and probabilistic nondeterminism

Takeaway 1 & 2 & 3: A nice semantic category should be

a Cartesian closed category
consisting of BF-domains/FS-domains/(algebraic) L-domains,
on which the valuations monad $\mathcal{V}$ can be defined.

Additional properties might also be preferred, for example, being co-complete, for solving domain equations (denoting recursive types).

Sad facts

- Probabilistic powerdomain $\mathcal{V}(L)$ are never algebraic.
- Probabilistic powerdomain destroys lattice-like structure.

	cart.closed	closed under $\mathcal{V}$.
BF	$\checkmark$	$\times$
Algebraic L	$\checkmark$	$\times$
L	$\checkmark$	$\times$
FS	$\checkmark$	Jung-Tix problem

Some existing work in domain theory

	cart.closed	closed under $\mathcal{V}$	
BF	$\checkmark$	$\times$	
Algebraic L	$\checkmark$	$\times$	
L	$\checkmark$	$\times$	[Jones 1990]
FS	$\checkmark$	Jung-Tix problem	[Jung and Tix 1998]
RB	$\checkmark$	Jung-Tix problem	[Jung and Tix 1998]
LawsonCom.DOM	$\times$	$\checkmark$	[Jung and Tix 1998]
DOM	$\times$	$\checkmark$	[Jones 1990]
quasi.DOM	$\times$	$\checkmark$	[Goubault-Larrecq 2021]
QFS	$\times$	$\checkmark$	[Goubault-Larrecq and Jung 2014]
SCS	$\times$	$\checkmark$	[Alvarez-Manilla, Jung and Keimel 2004]
DCPO	$\checkmark$	$\checkmark$	[Jones 1990]

Claire Jones

Achim Jung

Regina Tix

Jean Goubault-Larrecq

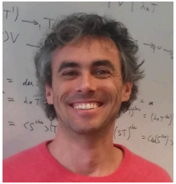
Klaus Keimel

Mauricio Alvarez-Manilla

Some existing work out the realm of (pure) domain theory (1)

[Thomas Ehrhard, Michele Pagani, and Christine Tasson. POPL2018]:

Thomas Ehrhard



Michele Pagani

Christine Tasson

- The category of probabilistic coherence spaces and matrices (in a probabilistic choice built-in manner), with a Linear Logic resource comonad !_.
- A denotational (equationally full abstract) semantics for probabilistic PCF. (probability is introduced by a fair coin and countably many biased coins)

Some existing work out the realm of (pure) domain theory (2)

[Ingo Battenfeld and Alex Simpson 2006]:

Ingo Battenfeld

Alex Simpson

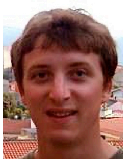
- The (Cartesian closed) category of topological domains.
- Two probabilistic choice monads.

Some existing work out the realm of (pure) domain theory (3)

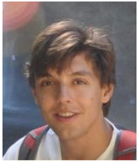
[Heunen, Kammar, Staton,Yang LICS17; Vákár, Kammar, Staton POPL19]:


Chris Heunen


Ohad Kammar


Sam Staton


Hongseok Yang


Matthijs Vákár

- The (Cartesian closed) category of quasi-Borel spaces/domains.
- A commutative probabilistic choice monad T.
- A adequate denotational semantics for statistical FPC. (probability is introduced by sampling in the unit interval)

Some existing work in (pure) domain theory

[Jean Goubault-Larrecq LICS2019]: No need to leave domain theory – by using Pual Levy's call-by-push-value paradigm.

Jean Goubault-Larrecq

Paul Levy

- J. G-L used two categories for the denotational semantics for the language **CBPV**(D,P).
- Probability effect is introduced by tossing a fair coin, and modelled by $\mathcal{V}$.
- Category of coherent continuous domains for types of values (where distributions are inhabited).
- Category of bc-domains for types of computation (where function space is accommodated).
- Inequational full abstraction with "parallel if" and statistical termination testers.

Some existing work in domain theory

	cart.closed	closed under $\mathcal{V}$	
BF	✓	×	
Algebraic L	✓	×	
L	✓	×	[Jones 1990]
FS	✓	Jung-Tix problem	[Jung and Tix 1998]
RB	✓	Jung-Tix problem	[Jung and Tix 1998]
LawsonCom.DOM	×	✓	[Jung and Tix 1998]
DOM	×	✓	[Jones 1990]
quasi.DOM	×	✓	[Goubault-Larrecq 2021]
QFS	×	✓	[Goubault-Larrecq and Jung 2014]
SCS	×	✓	[Alvarez-Manilla, Jung and Keimel 2004]
DCPO	✓	✓	[Jones 1990]

Some existing work in domain theory

	cart.closed	closed under $\mathcal{V}$	
BF	✓	×	
Algebraic L	✓	×	
L	✓	×	[Jones 1990]
FS	✓	Jung-Tix problem	[Jung and Tix 1998]
RB	✓	Jung-Tix problem	[Jung and Tix 1998]
LawsonCom.DOM	×	✓	[Jung and Tix 1998]
DOM	×	✓	[Jones 1990]
quasi.DOM	×	✓	[Goubault-Larrecq 2021]
QFS	×	✓	[Goubault-Larrecq and Jung 2014]
SCS	×	✓	[Alvarez-Manilla, Jung and Keimel 2004]
DCPO	✓	✓	[Jones 1990]

The category **DCPO**

The category of dcpos and Scott-continuous maps is:

- Cartesian closed;
- closed under $\mathcal{V}$;
- complete and co-complete.

Question: Why don't we use **DCPO** $+ \mathcal{V}$ for denotational semantics?

The category **DCPO**

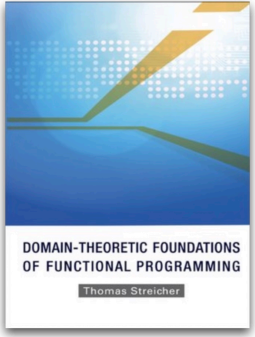
Question: Why don't we use **DCPO** $+ \mathcal{V}$ for denotational semantics?

- For general dcpos, very few mathematical tools are available: dcpos might not be sober in the Scott topology, for instance.
- The category **DCPO** is too large, hence contains many functions that is not denotable (a comment from G. Plotkin).
- $\mathcal{V}$ does not behave nicely enough on DCPO: it is unknown whether $\mathcal{V}$ is a commutative monad on **DCPO**.

The category **DCPO**

A quote from Jean Goubault-Larrecq:

"Continuity is not required to prove, say, soundness and adequacy theorems (for PCF, no probabilistic effects involved) using logical relations, as one realizes by reading the relevant parts of Thomas Streicher's 'Domain-Theoretic Foundations of Functional Programming'."



The category **DCPO** & $\mathcal{V}$

To model PCF with probabilistic effects using **DCPO** $+ \mathcal{V}$, the real issue is that $\mathcal{V}$ is unknown to be commutative on **DCPO**.

Commutativity of $\mathcal{V}$

The commutativity of $\mathcal{V}$ on **DCPO** amounts to a Fubini-Tonelli type equation:

For dcpo's A and B, $f\colon A \times B \to [0,1]$ a Scott-continuous map, and valuations ν_A on A, ν_B on B, whether it is true that

$$\int_{x\in A}\int_{y\in B} f(x,y)d\nu_B d\nu_A = \int_{y\in B}\int_{x\in A} f(x,y)d\nu_A d\nu_B? \tag{1}$$

The monad $\mathcal{V}$ on **DCPO** is commutative iff the Equation (1) holds for all A, B and ν_A, ν_B.

This is unknown since [Jones 1990].

Commutativity of $\mathcal{V}$

Why is this important?

$$\text{let } y = N_1 \text{ in let } x = N_2 \text{ in } M \Longleftrightarrow \text{let } x = N_2 \text{ in let } y = N_1 \text{ in } M$$

To validate this equation in semantics, we want:

$$[\![\text{let } y = N_1 \text{ in let } x = N_2 \text{ in } M]\!] = [\![\text{let } x = N_2 \text{ in let } y = N_1 \text{ in } M]\!]$$

Nested integrals appear in both sides of the equation, so we will need commutativity of $\mathcal{V}$.

Jones' Theorem

Good news:

Theorem [Claire Jones 1990]

Let A and B be dcpos, and ν_A and ν_B be continuous valuations on A and B, respectively. If $f\colon \Sigma A \times \Sigma B \to [0,1]$ is continuous, then

$$\int_{x\in A}\int_{y\in B} f(x,y)d\nu_B d\nu_A = \int_{y\in B}\int_{x\in A} f(x,y)d\nu_A d\nu_B.$$

Bad news:
Commutativity of $\mathcal{V}$ requires f to be of the type $\Sigma(A\times B)\to[0,1]$. In general,

$$\Sigma(A\times B)\neq \Sigma A\times\Sigma B.$$

Restriction to subcategories of $\mathbf{DCPO}$

Restriction to those sober dcpo's X, Y such that $\sigma(X \times Y) = \sigma(X) \times \sigma(Y)$?

Theorem II-4.13 in the "The Red Bible"

Let X be a dcpo. For arbitrary dcpo Y, $\sigma(X \times Y) = \sigma(X) \times \sigma(Y)$ iff X is core-compact.

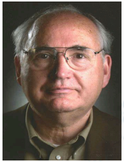
Gerhard Gierz

Karl Hofmann

Klaus Keimel

Jimmie Lawson

Michael Mislove

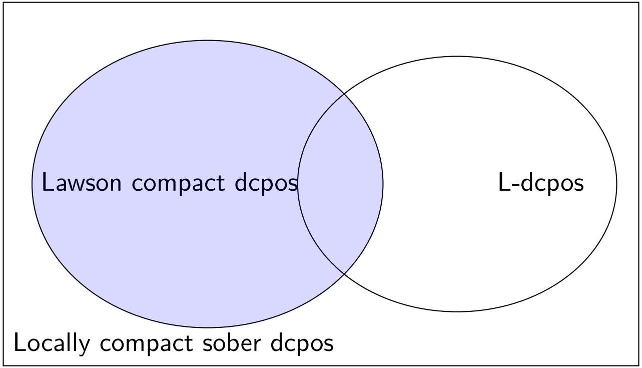


Dana Scott

Minimal requirement: sobriety and core-compactness

Minimal requirement: sobriety and *local compactness*

Restriction to locally compact sober dcpo's



Theorem [J., Achim Jung and Qingguo Li 2019]

Any Cartesian closed full subcategory of locally compact sober dcpo's is contained in the category of Lawson compact dcpo's or that of L-dcpo's.

What can we say about Cartesian closed full subcategories of stably compact dcpo's ?

Restriction to locally compact sober dcpo's

Commutativity for $\mathscr{V}$

Restricting to LCS

Category of stably compact dcpos

?

Change $\mathcal{V}$?

Commutativity for $\mathscr{V}$

Restricting to LCS

Category of stably compact dcpos

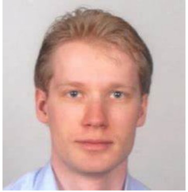
?

Can we change $\mathscr{V}$?

Change $\mathcal{V}$

[J., Lindenhovius, Mislove and Zamdzhiev 2021] & [Goubault-Larrecq, J. and Théron 2021]

Michael Mislove



Bert Lindenhovius

Vladimir Zamdzhiev

Jean Goubault-Larrecq

Clément Théron

Our solution: we revise $\mathcal{V}$ to obtain a commutative valuations submonad of $\mathcal{V}$ on $\mathbf{DCPO}$.

Idea: Start with the essential valuations, and include only those valuations that are really needed to guarantee a monad structure.

An easy observation

For fixed $\mu \in \mathcal{V}X$, the Fubini-type equation

$$\int_Y \int_X f(x,y) d\mu d\nu = \int_X \int_Y f(x,y) d\nu d\mu$$

holds for all simple valuations ν on Y, i.e., $\nu \in \mathcal{S}X$.

When $\nu = \sum_{i=1}^n r_i \delta_{y_i}$, both sides are $\sum_{i=1}^n r_i \int_X f(x, y_i) d\mu$.

Observation

1. $\mathcal{S}X$ is *not* not a dcpo, in general.
2. $\int_X \_d\_$ is Scott-continuous in the valuation component.
3. The Fubini-type equation holds when ν is a simple valuation, a directed supremum of simple valuations, a directed supremum of directed suprema of simple valuations, and so forth...

The monad $\mathcal{M}$

For each dcpo X, we take the smallest subdcpo of $\mathcal{V}X$ that contains $\mathcal{S}X$, which we denote as $\mathcal{M}X$.

$$\mathcal{S}X \subseteq \mathcal{M}X \subseteq \mathcal{V}X.$$

Theorem [J., Lindenhovius, Mislove and Zamdzhiev 2021, Goubault-Larrecq, J. and Théron 2021]

$\mathcal{M}$ restricts to a commutative monad on $\mathbf{DCPO}$.

- The unit at X is: $(x \in X \mapsto \delta_x \in \mathcal{M}X)$;
- for $f \colon X \to \mathcal{M}X$, $f^\dagger$ is given by the same formula:

$$f^\dagger \colon \mu \mapsto (U \mapsto \int_X f(x)(U) d\mu).$$

- the strength at (X, Y) is: $(x, \nu) \to (U \mapsto \int_Y \chi_U(x, y) d\nu)$

The restricted Fubini-Tonelli Theorem

Theorem [J., Lindenhovius, Mislove and Zamdzhiev 2021, Goubault-Larrecq, J. and Théron 2021]

Let A and B be dcpos, and $\nu_A \in \mathcal{M}A$ and $\nu_B \in \mathcal{M}B$, respectively. For every Scott-continuous $f\colon A \times B \to [0,1]$, we have

$$\int_{x \in A} \int_{y \in B} f(x,y) d\nu_B d\nu_A = \int_{y \in B} \int_{x \in A} f(x,y) d\nu_A d\nu_B.$$

Proof.

- Valuations in $\mathcal{M}A$ and $\mathcal{M}B$ are transfinite suprema of simple valuations.
- The Fubini-type equation holds for simple valuations.
- Integrals are Scott-continuous in the valuations' component.

A more general Fubini-type equation

- $\mathcal{V}A$ is a sober topological cone in the weak topology, which is generated by set of the form

$$[U > r] = \{\mu \in \mathcal{V}A \mid \mu(U) > r\}.$$

- $\mathcal{M}A$ is a monotone convergence space in the weak topology.
- $\mathcal{M}A$ is the monotone convergence space completion of $\mathcal{S}A$.

There is no reason why we cannot go further...once we realize that there are other completions.

A more general Fubini-type equation

We consider the sobrification of $\mathcal{S}A$ with the weak topology. It consists of Heckmann's point-continuous valuations [Heckmann 96].

$$\mathcal{P}A = \{\text{all point-continuous valuations on } A\}.$$

Defn. A continuous valuation μ is point-continuous: if for any open U with $\mu(U) > r$, there exists a finite set $F \subseteq U$ such that for any open V containing F, $\mu(V) > r$.

Reinhold Heckmann

A more general Fubini-type equation

[J., Lindenhovius, Mislove and Zamdzhiev 2021]

- $\mathcal{P}$ defines a monad on **DCPO**.
- The unit and multiplication of $\mathcal{P}$ are in the same form of $\mathcal{V}$.
- $\mathcal{S}A \subseteq \mathcal{M}A \subseteq \mathcal{P}A \subseteq \mathcal{V}A$.
- Moreover,

$$\int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu$$

  if either μ or ν is point-continuous.

A more general Fubini-type equation

Let $f\colon A \times B \to \overline{\mathbb{R}}_+$ be Scott-continuous map, μ be point-continuous, ν be any continuous valuation. Then,

$$\int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu.$$

A wrong proof.

- Consider $F(\mu) = \int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu$ and $G(\mu) = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu$.
- F and G are Scott-continuous.
- F and G agree on simple valuations.
- They agree on the sobrification of simple valuations.

A more general Fubini-type equation

Let $f\colon A \times B \to \overline{\mathbb{R}}_+$ be Scott-continuous map, μ be point-continuous, ν be any continuous valuation. Then,

$$\int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu$$

A wrong proof.

- Consider $F(\mu) = \int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu$ and $G(\mu) = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu$.
- F and G are Scott-continuous. We need that F and G are continuous w.r.t. the weak topology, which is coarser than the Scott topology.
- F and G agree on simple valuations.
- They agree on the sobrification of simple valuations.

A more general Fubini-type equation

Let $f\colon A \times B \to \overline{\mathbb{R}}_+$ be Scott-continuous map, μ be point-continuous, ν be any continuous valuation. Then,

$$\int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu$$

Proof.

- Consider $F(\nu) = \int_{x\in A}\int_{y\in B} f(x,y)d\mu d\nu$ and $G(\nu) = \int_{y\in B}\int_{x\in A} f(x,y)d\nu d\mu$.
- F is continuous for free, G is continuous because μ is point-continuous, Both w.r.t. the weak topology.
- F, G are continuous linear functionals from $\mathcal{V}_w A$ to $\overline{\mathbb{R}}_+$.
- F, G agree on simple valuations.
- F, G agree on $\mathcal{V}_w A$ – An application of the Schröder-Simpson Theorem.

The Schröder-Simpson Theorem

Theorem [Schröder and Simpson 2005 , Goubault-Larrecq 2014]

Any continuous linear functional F on $\mathcal{V}_w A$ is of the form $\mu \mapsto \int_{x\in A} F(\delta_x) d\mu$.

Matthias Schröder

Alex Simpson

This means continuous linear functional F on $\mathcal{V}_w A$ is determined by its actions on Dirac measures. $F(\nu) = \int_{x\in A}\int_{y\in B} f(x,y) d\mu d\nu$ and $G(\nu) = \int_{y\in B}\int_{x\in A} f(x,y) d\nu d\mu$ agree on simple valuations.

A question

Theorem [Goubault-Larrecq and J. 2021]

In general, on dcpos A, we have $\mathcal{S}A \subset \mathcal{M}A \subset \mathcal{P}A \subset \mathcal{V}A$. The inclusions are strict.

Do we restrict too much? Can we use $\mathcal{M}$ or $\mathcal{P}$ to denote probabilistic effects?

Adequate semantic for PFPC

Theorem [J., Lindenhovius, Mislove and Zamdzhiev LICS2021]

The category $\mathbf{DCPO}$ and monad $\mathcal{M}$ serves an adequate model for probabilistic FPC.

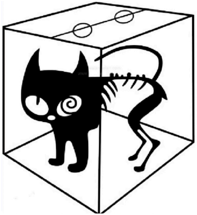
- PFPC supports higher-order type, type recursion (term recursion), probabilistic choice.
- Probabilistic choices are made by tossing a fair coin.
- (Strong Adequacy). For any term $\cdot \vdash M : A$,

$$[\![M]\!] = \sum_{V \in \mathrm{VAL}(M)} p[M \to_* V]\, [\![V]\!] \, .$$

Adequate semantic for a Variational Quantum Programming Language

Theorem [J., Konell, Lindenhovius, Mislove and Zamdzhiev POPL2022]

The category $\mathbf{DCPO}$ and monad $\mathcal{M}$ serves an adequate model for probabilistic effects introduced by quantum measurements.



Adequate semantic for ISPCF

Theorem [Goubault-Larrecq, J. and Théron 2021, submitted for publication]

The category $\mathbf{DCPO}$ and monad $\mathcal{M}$ serves an adequate model for Interval Statistical PCF.

- ISPCF supports higher-order type, probabilistic choice.
- Probabilistic choices are made by sampling a continuous valuation on the interval domain of R, which is determined by a measure μ on R.

Questions about $\mathcal{M}$

The monad $\mathcal{M}$ is simple, but there are unknowns about it.

The monad $\mathcal{M}$ is quasi-simple :-)

Questions about $\mathcal{M}$ – 1

What are the Eilenberg-Moore algebras of $\mathcal{M}$ over **DCPO**?

Theorem [J. and Kornell, in preparation]

Dcpo cones with natural norms and enough linear maps are Eilenberg-Moore algebras of $\mathcal{M}$.

Andre Kornell

Questions about $\mathcal{M}$ – 2

Minimal valuations in $\mathcal{M}$ and Borel measures.

[Alvarez-Manilla, Edalat and Saheb-Djahromi, 1998]

(minimal) Valuations that are directed suprema of simple valuations extend to Borel measures. The extension is unique if the directed set is bounded above.

Mauricio Alvarez-Manilla

Abbas Edalat

Nasser Saheb-Djahromi

Conjecture. This also is true for minimal valuations.

Conclusion

Instead solving the Jung-Tix Problem directly,
we walk around the Jung-Tix Problem,
by considering commutative submonads of $\mathcal{V}$,
and together with **DCPO**, they serve as adequate models for different higher-order probabilistic programming languages,
where probabilistic effects are introduced either by tossing coins, or sampling measures on reals.

Conclusion

A Domain-Theoretic Approach to Statistical Programming Languages

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Commutative Monads for Probabilistic Programming Languages

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Semantics for Variational Quantum Programming

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We consider a programming language that can manipulate both classical and quantum information. Our language is type-safe and designed for variational quantum programming, which is a hybrid classical-quantum computational paradigm. The classical subsystem of the language is the Probabilistic FixPoint Calculus (PFPC), which is a lambda calculus with mixed-variance recursive types, term recursion and probabilistic choice. The quantum subsystem is a first-order linear type system that can manipulate quantum information. The two subsystems are related by mixed classical/quantum terms that specify how classical probabilistic effects are induced by quantum measurements, and conversely, how classical (probabilistic) programs can influence the

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PAPER

Separating minimal valuations, point-continuous valuations, and continuous valuations

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谢谢! Thank you! Danke! Merci!

Dík! ありがとう!

ممنون! 감사! . . .