The d^*-space

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Well-filtered space

A T_0 space is called well-filtered if for any open subset U and any filtered family $\mathcal{K}$ in $Q(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

In [1], Jia, Jung and Li give an equivalent condition on well-filtered dcpo L for coherence is the compactness of $\uparrow x \cap \uparrow y$ for any $x, y \in L$.

[1] X.Jia, A.Jung, Q.Li, *A note on coherence of dcpos*, Topology and its Applications, **209** (2016) 235-238.

The Johnstone space $\mathbb{J}$

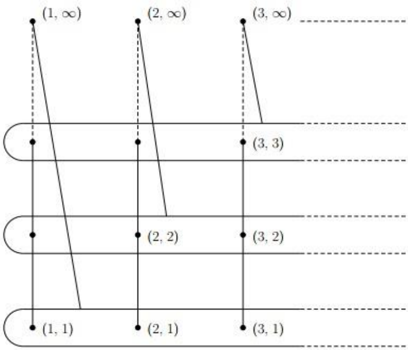


Figure: The Johnstone space $\mathbb{J}$

Weak well-filtered space

Recently, Lu and Li find that all open subsets except the empty subset are true for the definition of well-filtered in the johnstone space. Then they introduced the concept of weak well-filtered space and prove that the result of Jia, Jung and Li is also true on weak well-filtered posets.

Definition

A topological space (X, τ) is called *weak well-filtered* if for any nonempty open subset U and any filtered family $\mathcal{K}$ in $Q(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

[2] X.Lu Q.Li, *Weak well-filtered spaces and coherence*, Topology and its Applications, **230** (2017) 273-380.

Strong d-space

In[3], Xu and Zhao introduce the concept of strong d-space when they study the relation of well-filtered and sober.

Definition

A T_0 space X is called a *strong d-space* if for any directed subset D of X, x in X and any open subset U, $\bigcap_{d \in D} \uparrow d \cap \uparrow x \subseteq U$ implies $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$.

[3] X.Xu, D.Zhao, *On Topological Rudin's Lemma, well-filtered Spaces and sober Spaces*, Topology and its Applications, **272** (2020) 107080.

The following picture is given by Xu and Zhao.

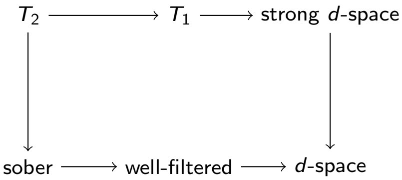


Figure: Relations of some spaces lying between d-spaces and T_2 space

Well-filtered space $\underset{\not\longleftarrow}{\longrightarrow}$ Weak well-filtered space

strong d-space $\underset{\not\longleftarrow}{\longrightarrow}$?

Naturally, whether we can propose a space weaker than a strong d space, and then study whether some properties in a strong d-space can be preserved and what properties of a weak space have?

We also find that all open sets except the empty set are true for the definition of the strong d-space in the Johnstone space.

d^*-space

Based on the idea of Lu and Li, we give the concept of the *d^*-space*.

Definition

A T_0 space X is called a *d^*-space* if for any directed subset D of X, x in X and any nonempty open subset U, $\bigcap_{d \in D} \uparrow d \cap \uparrow x \subseteq U$ implies $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$.

Example

$(N, \sigma(N))$ is not a strong d-space, but it is a d^*-space.

Remark

Every coherent weak well-filtered space is a d^*-space.

The strong d-space with respect to the specialization order is a dcpo. But the d^*-space with respect to the specialization order may not be a dcpo. We find the d^*-space with respect to the specialization order is a consistent dcpo.

Definition

A poset L is called a *consistent dcpo* if for any directed subset D of L with $\bigcap_{d \in D} \uparrow d \neq \varnothing$, the sup of D exists in L.

Proposition

Let (X, τ) be a d^*-space. Then $\Omega(X) = (X, \leq_\tau)$ is a consistent dcpo and $\tau \subseteq \sigma(\Omega(X))$.

Dcpo with the scott topology may not be a strong d-space. The Johnstone space is not a strong d-space.

A consistent dcpo with the scott topology may not be a d^*-space. $N \cup \{a\}$ with the order as shown in the picture is a consistent dcpo, but it with the scott topology is not a d^*-space.

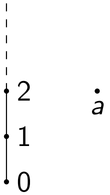


Figure: $N \cup \{a\}$

In[4], Xu and Zhao find that a poset with the scott topology is a well-filtered space while it is a stong d-space.

In this paper we consider that a dcpo with the scott topology is a weak well-filtered space while it is a d^*-space.

proposition

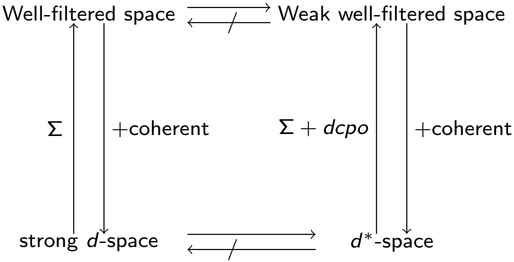
Let L be a dcpo. If $(L, \sigma(L))$ is a d^*-space, then $(L, \sigma(L))$ is weak well-filtered.

There is a question whether a poset with the scott topology is a weak well-filtered space while it is a d^*-space.

Problem

In the above theorem, if dcpo is replaced by poset, does the conclusion still holds?

[4] X.Xu, D.Zhao, *Some open problems on well-filtered spaces and sober spaces*, Topology and its Applications, **301** (2021) 107540.


Well-filtered space
Weak well-filtered space
Σ
+coherent
Σ + dcpo
+coherent
strong d-space
d*-space

Some properties of the d^*-space

	d-space	strong d-space	d^*-space
the saturated subspace	$\surd$	$\surd$ in[4]	$\surd$
the closed subspace	$\surd$	$\surd$ in[4]	$\surd$
the product space	$\surd$	$\times$ in[5]	$\times$
the retract	$\surd$	$\surd$ in[4]	$\surd$
the funtion space	$\surd$	? in[4]	$\times$
the upper space	$\surd$	?	$\times$

[5] Q.Li, M.Jin, H.Miao, S.Chen, *On some results related to sober spaces*, (preprint).

ΣN and $\Sigma 2$ (the Sierpinski space) are d^*-spaces. But the product space $\Sigma N \times \Sigma 2$ is not a d^*-space.

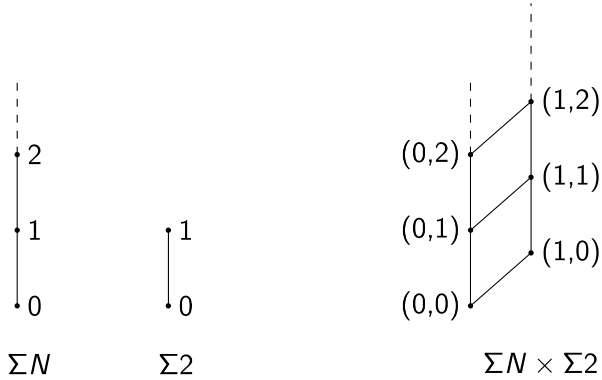


Figure: $\Sigma N \times \Sigma 2$ is not a d^*-space

$[X, Y]$ denotes that the set of all continuous functions from X to Y with the Isbell topology.

It is known that a topological space Y is a retract of the function space $[X, Y]$. And if $[X, Y]$ is a d-space (resp. strong d-space), then Y is a d-space (resp. strong d-space).

Similarly, we have:

Corollary

If $[X, Y]$ is a d^*-space, then Y is a d^*-space.

ΣN and the $\Sigma 2$ in the above picture are d^*-spaces. But $[\Sigma 2, \Sigma N]$ is not a d^*-space.

$Q_{\upsilon}(X)$ denotes the set of all nonempty compact saturated subsets of X with the *upper Vietoris topology*.

Let X be a topological space. If $Q_{\upsilon}(X)$ is a d-space, then X is a d-space.

Theorem

Let X be a topological space. If $Q_{\upsilon}(X)$ is a d^*-space, then X is a d^*-space.

Example

Natural number N with the co-finite topology $\tau_1 = \{U \mid N \backslash U \text{ is finite}\} \cup \{\varnothing\}$ and a single point set $\{a\}$ with the discrete topology $\tau_2 = \{\varnothing, \{a\}\}$. The topology on $N \cup \{a\}$ is generated by the refinement of $\tau_1 \vee \tau_2$.

Obviously, it is a T_1 space because every single point set is the closed set. Thus it is a d^* space.

But $Q(X) = 2^{N \cup \{a\}} \backslash \{\varnothing\}$ with the *upper Vietoris topology* is not a d^*-space.

Thank you!