The sheaf representation of residuated lattices

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Catalog

Residuated lattices, introduced by Ward and Dilworth [15], constitute the semantics of Höhle's Monoidal Logic [11]. Such algebras provide the fundamental framework for algebras of logics. Many familiar algebras, such as Boolean algebras, MV-algebras, BL-algebras, MTL-algebras, NM algebras (R_0-algebras) and Heyting algebras, are special types of residuated lattices.

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[15] M. Ward and R.P. Dilworth, Residuated lattice. 1939:335–354.
[11] U. Höhle and P. Klement, Non-Classical Logics and their Applications to Fuzzy Subsets. 1995.

In dealing with certain type of problems, sheaf representations of algebras often provide powerful tools as they convert the study of algebras to the study of stalks, a topological structure. Thus, in the past decades, sheaf spaces [14] have been constructed for various types of algebras to obtain their corresponding sheaf representations.

[14] S. Mac Lane and I. Moerdijk, Sheaves in Geometry and Logic. 1992.

- Ghilardi and Zawadowski constructed the Grothendieck-type duality and got the sheaf representation for Heyting algebras [9].
- Dubuc, Poveda, Ferraioli, Lettieri, Filipoiu and Georgescu studied the sheaf representations of MV-algebras [2,4-8].
- Di Nola and Leuştean defined the sheaf spaces of BL-algebras and obtained the sheaf representations of BL-algebras [3,13].

In [7], Gehrke and Gool dealt with sheaf representations of a $\mathcal{V}$-algebra. In their definition [7, Def 3.1], they required that the $\mathcal{V}$-algebra A is isomorphic to FY, where F is a sheaf, and FY is the algebra of global sections of F. In my talk, we loosen the isomorphism condition in [7] and the requirement on the stalks in [6] and further extend above results to the more general structures; namely, define the sheaf spaces of residuated lattices and obtain the sheaf representation of residuated lattices.

[7] M. Gehrke, S.J.v. Gool, Sheaves and duality. 2018:2164 - 2180.
[6] A. Filipoiu and G. Georgescu, Compact and Pierce representations of MV-algebras. 1995: 599-618.

Definition

A residuated lattice is an algebra $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that for all $x, y, z \in L$,
(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \otimes, 1)$ is a commutative monoid;
(3) for any $x, y, z \in L$, $x \otimes y \leq z$ iff $x \leq y \rightarrow z$.

Definition (see [10])

A nonempty subset F of a residuated lattice L is called a filter if
(1) $x \in F$ and $x \leq y$ imply $y \in F$;
(2) $x \in F$ and $y \in F$ imply $x \otimes y \in F$.

- A filter is *proper* if $F \neq L$.
- $\{1\}$ and L are always filters.
- With any filter F, we can obtain a congruence relation θ_F defined on L by $(x, y) \in \theta_F$ iff $x \to y \in F$ and $y \to x \in F$.
- Let L/F denote the set of the congruence classes of $\equiv_F$, i.e., $L/F = \{x/F | x \in L\}$, where $x/F := \{y \in L | y \equiv_F x\}$. Define operations on L/F as follows:
  $[x]_F \sqcap [y]_F = [x \wedge y]_F$,$[x]_F \sqcup [y]_F = [x \vee y]_F$,
  $[x]_F \odot [y]_F = [x \otimes y]_F$,$[x]_F \rightharpoonup [y]_F = [x \to y]_F$.
  We have that L/F is a residuated lattice.

Definition (see [10])

A proper filter P of L is called a prime filter if for any $x, y \in L$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

Definition

The set of all prime filters of L is called the *prime spectrum* of L and is denoted by $\mathrm{Spec}(L)$.

Definition (see [1])

Suppose that L and L' are residuated lattices. A residuated lattice morphism is a function $h : L \longrightarrow L'$ such that $h(a \wedge_L b) = h(a) \wedge_{L'} h(b), h(a \vee_L b) = h(a) \vee_{L'} h(b), h(a \otimes_L b) = h(a) \otimes_{L'} h(b), h(a \rightarrow_L b) = h(a) \rightarrow_{L'} h(b)$ and $h(0) = 0', h(1) = 1'$.

Definition

A topological space is a pair (X, τ), where X is a nonempty set and τ is a family of subsets of X, called the topology, such that (i) $\emptyset, X \in \tau$, (ii) a finite intersection of members of τ is in τ and (iii) an arbitrary union of members of τ is in τ.

Definition

The members of τ are called open sets of X and the elements of X are called points.

Definition

A neighbourhood of a point x in a topological space X is a subset $W \subseteq X$ such that there exists an open set U of X satisfying $x \in U \subseteq W$.

Definition

A base $\mathcal{B}$ for a topology space τ is a collection of open sets in τ such that every open set in τ is a union of some members of $\mathcal{B}$.

- A function $f : X \longrightarrow Y$ from a topological space (X, τ) to a topological space (Y, σ) is continuous at a point $x \in X$ if for any neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$. The function is called continuous if it is continuous everywhere.
- A function $f : X \longrightarrow Y$ between two topological spaces X and Y is an open function if for any open set U of X, $f(U)$ is an open set of Y.
- A bijective function $f : X \longrightarrow Y$ between two topological spaces is a homeomorphism if both f and f^{-1} are continuous.

Definition

A sheaf representation of a residuated lattice L will mean an injective residuated lattice morphism $\phi : L \longrightarrow \Gamma(X, E)$ from L to the residuated lattice $\Gamma(X, E)$ of a sheaf space of residuated lattice (E, p, X).

Definition

A sheaf space of residuated lattices is a triple (E, p, X) satisfying the following conditions:
(1) Both E and X are topological spaces.
(2) $p : E \longrightarrow X$ is a local homeomorphism from E onto X, i.e. for any $e \in E$, there are open neighbourhoods U and U' of e and $p(e)$ such that p maps U homeomorphically onto U'.
(3) For any $x \in X, p^{-1}(\{x\}) = E_x$ is a residuated lattice.
(4) The functions defined by $(a, b) \longmapsto a \wedge_x b, (a, b) \longmapsto a \vee_x b, (a, b) \longmapsto a \otimes_x b, (a, b) \longmapsto a \rightarrow_x b$ from the set $\{(a, b) \in E \times E | p(a) = p(b)\}$ into E are continuous, where $x = p(a) = p(b)$.
(5) The functions $\underline{0}, \underline{1} : X \longrightarrow E$ assigning to every x in X the 0_x and 1_x of E_x respectively, are continuous.

Definition

Let (E, p, X) be a sheaf space of residuated lattices. A function $\sigma : X \longrightarrow E$ is called a global section if it is continuous such that for any $x \in X, p(\sigma(x)) = x$.

Remark

If we define the operations pointwise on the set of all sections over X, it constitutes a residuated lattice. We denote it by $\Gamma(X, E)$.

The base space

Let $X = \mathrm{Spec}(L)$, for any $X \subseteq L$, we will write $D(X) = \{P \in \text{Spec}(L) | X \nsubseteq P\}$. For any $a \in L$, $D(\{a\})$ shall be denoted simply by $D(a)$.

Theorem

For any residuated lattice L, the family $\{D(X) | X \subseteq L\}$ is a topology on $\mathrm{Spec}(L)$, we call this topology the *Stone topology* on L.

Theorem

For any residuated lattice L, the family $\{D(a)\}_{a \in L}$ is a base for $\mathrm{Spec}(L)$.

Proposition

For any $P \in \mathrm{Spec}(L)$, $O(P)$ is a proper filter of L satisfying $O(P) \subseteq P$, where $O(P) = \{x \in L | a \vee x = 1$ for some $a \in L - P\}$.

The total space

Let E_L be the disjoint union of the set $\{L/O(P)\}_{P \in \mathrm{Spec}(L)}$ and $\pi : E_L \longrightarrow \mathrm{Spec}(L)$ the canonical projection.

Theorem

Let L be a residuated lattice. Then the family $\mathcal{B} = \{D(F, a) : F \in \mathcal{F}(L)$ and $a \in L\}$ is a base for a topology on E_L, where $D(F, a) = \{a_P : P \in D(F)\}$ and $a_P = a/O(P)$.

Remark

We use $\mathcal{T}(\mathcal{B})$ to denote the topology on E_L generated by $\mathcal{B}$.

Theorem

The assignment $\pi : E_L \longrightarrow \mathrm{Spec}(L)$ defined by $a_P \longmapsto P$ is a local homeomorphism of $(E_L, \mathcal{T}(\mathcal{B}))$ onto $\mathrm{Spec}(L)$.

Lemma

For any $P \in \mathrm{Spec}(L), \pi^{-1}(\{P\}) = L/O(P)$ is a residuated lattice.

Proposition

For any $P \in \mathrm{Spec}(L)$, the functions $(a_P, b_P) \longmapsto a_P \wedge_P b_P, (a_P, b_P) \longmapsto a_P \vee_P b_P, (a_P, b_P) \longmapsto a_P \otimes_P b_P, (a_P, b_P) \longmapsto a_P \rightarrow_P b_P$ from the set $\{(a_P, b_P) \in E_L \times E_L | \pi(a_P) = \pi(b_P)\}$ into E_L are continuous, where $P = \pi(a_P) = \pi(b_P)$.

Proposition

For any $a \in L$, the function $\hat{a} : \mathrm{Spec}(L) \longrightarrow E_L$ defined by $\hat{a}(P) = a_P$ is a global section of $(E_L, \pi, \mathrm{Spec}(L))$.

Corollary

The functions $\hat{0} : \mathrm{Spec}(L) \longrightarrow E_L$ and $\hat{1} : \mathrm{Spec}(L) \longrightarrow E_L$ are global sections of $(E_L, \pi, \mathrm{Spec}(L))$.

Theorem

For any residuated lattice L, $(E_L, \pi, \mathrm{Spec}(L))$ is a sheaf space of L.

Proposition

If L is a residuated lattice, then the family $\{O(P)\}_{P\in\mathrm{Spec}(L)}$ canonically determines a sheaf representation of L.

Proof. Define $\varphi : L \longrightarrow \Gamma(\mathrm{Spec}(L), E_L)$ by $\varphi(a) = \hat{a}$.

Problem

For what L, is the mapping φ surjective?

Conclusion

In this paper, we investigate the properties of the family of all the prime filters of residuated lattices. Based on this, we construct the sheaf space of residuated lattices and obtain a sheaf representation of residuated lattices.

Conclusion

In [6], Ferraioli and Lettieri took the primary ideals as the corresponding ideals of the prime ideals and proved that every MV-algebra and the MV-algebra of all global sections of its sheaf space are isomorphic. In [7,8], the scholars proved every MV-algebra A is isomorphic to the MV-algebra of global sections of a sheaf F of MV-algebras with stalks that are linear. In our future work, we will investigate when these results hold for a residuated lattice.

References

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Thank you!