Strongly Continuous Domains

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Introduction

In non-Hausdorff topology and domain theory, d-spaces, well-filtered spaces and sober spaces form three of the most important classes of T_0 spaces. In order to reveal finer links between well-filtered spaces and d-spaces, X. Xu and D. Zhao introduced another class of T_0 spaces–*strong* d*-spaces* which lie between d-spaces and T_1 spaces in [1].

1. A T_0 space X is a d-space iff X is a *directed complete poset* (*dcpo*, for short) and the topology on X is coarser than the Scott topology on X with respect to its *specialization order*.
2. In order to obtain some characterizations of strong d-spaces ([1, Proposition 3.22]), the authors [1] introduced the concept of *strong Scott topology*.

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[1] X. Xu, D. Zhao, *On topological Rudin's lemma, well-filtered spaces and sober spaces*, Topology and its Applications, 272 (2020) 107080.

Introduction

Definition 1.1 (see [1])

Let L be a dcpo. A subset $U \subseteq L$ is called *strongly Scott open* if (i) $U = \uparrow U$, and (ii) for any directed subset D of L and $x \in L$, $\bigcap_{d \in D} \uparrow d \cap \uparrow x \subseteq U$ (that is $\uparrow \bigvee D \cap \uparrow x \subseteq U$) implies $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$. Let $\sigma^s(L)$ denote the set of all strongly Scott open subsets of L.

Clearly, if $U, V \in \sigma^s(L)$, then $U \cap V \in \sigma^s(L)$. The topology generated by $\sigma^s(L)$ (as a base) is called the *strong Scott topology* on L and denote it by $\sigma_s(L)$. The space $(L, \sigma_s(L))$ is called the *strong Scott space* of L, and will be denoted by $\sum_s L$.

In order not to cause ambiguity, the elements in $\sigma_s(L)$ will be called *strong Scott topology open sets*.

Proposition 1.2 (see [1])

Let L be a poset, then $\upsilon(L) \subseteq \sigma^s(L) \subseteq \sigma_s(L) \subseteq \sigma(L)$.

Introduction

Strong Scott topology is a kind of new topology which is finer than the upper topology and coarser than the Scott topology.

Inspired by the topological characterizations of continuous domains (resp., hypercontinuous domains) that a dcpo P is a continuous domain (resp., a hypercontinuous domain) if and only if P is a C-space with respect to the Scott topology (resp., upper topology), we introduce the concept of *strongly continuous domains*. That is to say, a dcpo P is called a strongly continuous domain if P is a C-space with respect to the strong Scott topology.

We will investigate the properties of strongly continuous domains from the aspects of order structure and topology structure.

Strongly continuous domains

Definition 2.1

Let L be a dcpo. L is called a *strongly continuous domain* if $(L, \sigma_s(L))$ is a C-space, i.e., for any $U \in \sigma_s(L)$ and $x \in L$, $x \in U$ implies there exists a $u \in U$ such that $x \in \mathrm{int}_{\sigma_s(L)} \uparrow u \subseteq \uparrow u \subseteq U$.

Definition 2.2

Let L be a poset. A binary relation $\ll_s$ on L is defined as follows: $x \ll_s y$ if for any directed subset $D \subseteq L$ and $a \in L$, $\bigcap_{d \in D} \uparrow d \cap \uparrow a \subseteq \uparrow y$ implies $\uparrow d \cap \uparrow a \subseteq \uparrow x$ for some $d \in D$.

For $x \in L$, we write $\Downarrow_s x = \{y \in L : y \ll_s x\}$ and $\Uparrow_s x = \{y \in L : x \ll_s y\}$.

Remark 2.3

Let L be a dcpo and $u, x, y, z \in L$. We have the following.

1. $x \ll_s y$ *implies* $x \ll y$, *and* $x \ll_s y$ *is equivalent to* $x \ll y$ *if* L *is a sup semilattice.*
2. $u \leq x \ll_s y \leq z$ *implies* $u \ll_s z$.
3. $x \ll_s z$, $y \ll_s z$ *implies* $x \vee y \ll_s z$ *whenever* $x \vee y$ *exists in* L.
4. $0 \ll_s x$ *whenever* L *has a smallest element* 0.

Hence $\ll_s$ *is an auxiliary relation on* L.

Proposition 2.4

Let L be a dcpo. Suppose that there exists a directed set $D \subseteq \Downarrow_s x$ such that $\sup D = x$, then we have the following conditions.

(1) $\Downarrow_s x$ *is directed and* $\sup \Downarrow_s x = x$;

(2) *If* $y \ll_s x$ *in* $\downarrow x$, *then* $y \ll_s x$ *in* L.

Theorem 2.5

Let L be a dcpo. The following two conditions are equivalent:

(1) *L is a strongly continuous domain;*

(2) *For any $x \in L$, $\Downarrow_s x$ is directed, $x = \sup \Downarrow_s x$ and $\Uparrow_s x \in \sigma_s(L)$.*

Proposition 2.6

Let L be a strongly continuous domain. If $x \ll_s z$ and $z \leq \bigvee D$ for a directed set in L, then $x \ll_s d$ for some $d \in D$.

Proposition 2.7

In a strongly continuous domain L, $\ll_s$ satisfies the interpolation property, i.e., for all $x, y \in L$, $x \ll_s y$ implies that there exists a $z \in L$ such that $x \ll_s z \ll_s y$.

Lemma 2.8 (see [2])

Let L be a dcpo. Then the following conditions are equivalent:

(1) *L is hypercontinuous;*

(2) *L is continuous and $\ll = \prec$;*

(3) *L is continuous and $\upsilon(L) = \sigma(L)$.*

Theorem 2.9

Let L be a dcpo. Then the following conditions are equivalent:

(1) *L is hypercontinuous;*

(2) *L is strongly continuous and $\ll_s = \prec$;*

(3) *L is strongly continuous and $\upsilon(L) = \sigma_s(L)$.*

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[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encycl. Math. Appl., Vol. 93, Cambridge University Press, 2003.

Example 2.10

Let $L = \{0\} \cup \{a_1, a_2, \cdots, a_n, \cdots\}$ with the order generated by

(a) $a_n > 0$ for all $n \in \mathbb{N}$;

(b) There is no order relationship between a_i and a_j for all $i, j \in \mathbb{N}$ (Figure 1).

Then L is a strongly continuous domain, but not a hypercontinuous domain.

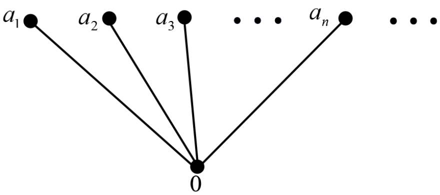

Figure 1: The poset L.

Theorem 2.11

Let L be a dcpo. Then the following conditions are equivalent:

(1) L *is strongly continuous;*

(2) L *is continuous,* $\ll_s = \ll$ *and* $\Uparrow_s x \in \sigma_s(L)$ *for all* $x \in L$*;*

(3) L *is continuous and* $\sigma_s(L) = \sigma(L)$*.*

The following example shows that a continuous domain need not be a strongly continuous domain.

Example 2.12 (see [1])

Let $C = \{a_1, a_2, \cdots, a_n, \cdots\} \cup \{\omega_0\}$ and $L = C \cup \{b\} \cup \{\omega_1, \omega_2, \cdots, \omega_n, \cdots\}$ with the order generated by

(a) $a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$;

(b) $a_n < \omega_0$ for all $n \in \mathbb{N}$;

(c) $b < \omega_n$ and $a_m < \omega_n$ for all $n, m \in \mathbb{N}$ with $m \leq n$ (Figure 2).

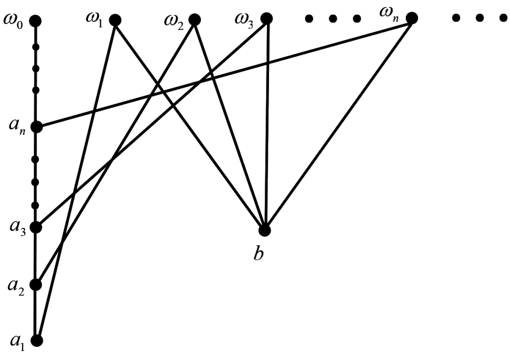

Figure 2: The poset L.

Topologies on strongly continuous domains

We study some properties of strongly continuous domains endowed with strong Scott topology and strong Lawson topology.

Proposition 3.1

Let L be a dcpo. Then we have the following conditions.

(1) $cl_{\sigma_s(L)}\{x\} = \downarrow x$ *for all* $x \in L$;

(2) $\sigma_s(L)$ *is a* T_0*-topology;*

(3) *If* $A = \uparrow A$, *then* $A = \bigcap\{U \in \sigma_s(L) : A \subseteq U\}$.

Proposition 3.2

Let L be a dcpo. Consider the following two conditions:

(1) $y \in int_{\sigma_s(L)} \uparrow x$;

(2) $x \ll_s y$.

Then $(1) \Rightarrow (2)$; *if L is strongly continuous, then* $(2) \Rightarrow (1)$, *and* (1) *and* (2) *are equivalent.*

Proposition 3.3

Let L be a strongly continuous domain. Then we have the following conditions.

(1) *If $U = \uparrow U$, then $U \in \sigma_s(L)$ iff there is a $u \in U$ such that $x \in \Uparrow_s u$ for all $x \in U$;*

(2) *$\{\Uparrow_s u : u \in L\}$ form a basis for the strong Scott topology;*

(3) *$int_{\sigma_s(L)} \uparrow x = \Uparrow_s x$ for all $x \in L$;*

(4) *For any subset $X \subseteq L$, $int_{\sigma_s(L)} X = \bigcup \{\Uparrow_s u : \Uparrow_s u \subseteq X\}$.*

Definition 3.4

Let L be a dcpo. The set of all strong Scott topology open filters of L is denoted by $\mathrm{SOFilt}(L)$, i.e., $U \in \mathrm{SOFilt}(L)$ iff $U \in \sigma_s(L)$ and U is a filter.

Proposition 3.5

Let L be a dcpo and $U \in \sigma_s(L)$.

(1) *U is a co-prime in $\sigma_s(L)$ iff $U \in SOFilt(L)$;*

(2) *$L \setminus \downarrow a \in PRIME(\sigma_s(L))$ for all $a \in L$; For each $U \in PRIME(\sigma_s(L)), U \neq L$, if L is strongly continuous, then there exists $a \in L$ such that $U = L \setminus \downarrow a$.*

Corollary 3.6

Let L be a strongly continuous domain. Then $(L, \sigma_s(L))$ is a locally compact sober space. If L has a smallest element, then $(L, \sigma_s(L))$ is compact.

Theorem 3.7

Let L be a dcpo. Then the following conditions are equivalent:

(1) *L is strongly continuous;*

(2) *For each $x \in L$, $\Uparrow_s x \in \sigma_s(L)$ and if $U \in \sigma_s(L)$, then $U = \bigcup\{\Uparrow_s u : u \in U\}$;*

(3) *$SOFilt(L)$ is a basis of $\sigma_s(L)$ and $\sigma_s(L)$ is a continuous lattice;*

(4) *$\sigma_s(L)$ has enough co-primes and is a continuous lattice;*

(5) *$\sigma_s(L)$ is completely distributive;*

(6) *Both $\sigma_s(L)$ and $\sigma_s(L)^{op}$ are continuous.*

If L is a complete semilattice, then these conditions are equivalent to

(7) *For each point $x \in L$, $\Uparrow_s x \in \sigma_s(L)$ and $x = sup\{infU : x \in U \in \sigma_s(L)\}$.*

Now we introduce the notation of strong Lawson topology and discuss the properties of strongly continuous domains endowed with the strong Lawson topology.

Definition 3.8

Let L be a dcpo. The common refinement $\sigma_s(L) \vee \omega(L)$ of the strong Scott topology and the lower topology is called the *strong Lawson topology* and is denoted by $\lambda_s(L)$.

Theorem 3.9

Let L be a complete semilattice. Then $\lambda_s(L)$ is a compact T_1 topology.

Theorem 3.10

Let L be a strongly continuous domain. Then $(L, \lambda_s(L))$ is a T_2 space.

Recall that the definition of Scott-continuous functions. For a function f from a dcpo P into a dcpo Q, if f is continuous with respect to the Scott topologies, that is, $f^{-1}(U) \in \sigma(P)$ for all $U \in \sigma(Q)$, then f is called a *Scott-continuous function*.

A function f is called *strong Scott-continuous* if it is continuous with respect to the strong Scott topologies. Next we discuss the properties of strong Scott-continuous functions.

Proposition 3.11

Let P be a strongly continuous domain. Both $f : (P, \sigma_s(P)) \to (Q, \sigma_s(Q))$ and $g : (Q, \sigma_s(Q)) \to (P, \sigma_s(P))$ are continuous functions, and $f \circ g = \mathbf{1}_Q$. Then Q is a strongly continuous domain.

Proposition 3.12

Let P, Q be dcpos and $f : P \to Q$. Consider the following conditions:

(1) $f : (P, \sigma_s(P)) \to (Q, \sigma_s(Q))$ *is continuous;*

(2) $f^{-1}(U) \in \sigma_s(P)$ *for all* $U \in \sigma^s(Q)$*;*

(3) $f^{-1}(U) \in \sigma^s(P)$ *for all* $U \in \sigma^s(Q)$*;*

(4) *For each directed set* $D \subseteq P$ *and* $x \in P$*,* $\uparrow f(\bigcap_{d \in D} \uparrow d \cap \uparrow x) = \bigcap_{d \in D} \uparrow f(d) \cap \uparrow f(x)$.

Then $(4) \Rightarrow (3) \Rightarrow (2) \Leftrightarrow (1)$*; if both* P *and* Q *are sup semilattices and* f *preserves finite sups, then* $(1) \Rightarrow (4)$*, and the four conditions are equivalent.*

Thank you for your attention.

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