On open well-filtered spaces and $\mathcal{U}_s$-admitting spaces

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Overview

- Heckmann introduced the well-filtered spaces, and asked whether every well-filtered Scott space of a dcpo is sober? [1];
- Kou provides a negative answer for Heckmann's question. Moreover, he proved that locally compact well-filtered space is sober [2];
- X. Jia in his PhD thesis asked whether every core-compact well-filtered space is sober? (usually called Jia-Jung problem) [3];
- Jia-Jung problem was first solved positively by J. Lawson, G. Wu and X. Xi [4].

Some more results on well-filtered results are shown as follows:

- core-compact + ω-well-filtered $\Rightarrow$ sober [5, 6];
- first-countable + well-filtered $\Rightarrow$ sober [5, 6];
- the category of well-filtered spaces is reflective in T_0 spaces [7, 8];

In this presentation, we introduce a new class of T_0 spaces, called **open well-filtered spaces**, and show some results on such spaces. In addition, we will show some new results on $\mathcal{U}_s$-**admitting spaces**.

Definition (sober space, well-filtered space, d-space)

- A nonempty subset C of a T_0 space X is called irreducible if for any closed sets C_1, C_2 in X, $C \subseteq C_1 \cup C_2$ implies $C \subseteq C_1$ or $C \subseteq C_2$. A T_0 space X is sober if for every irreducible closed set C, there is $x \in X$ such that $C = \mathrm{cl}(\{x\})$.
- A T_0 space X is called well-filtered if for any filtered family $\mathcal{F}$ of compact saturated sets and any open set U, $\bigcap \mathcal{F} \subseteq U$ implies that $K \subseteq U$ for some $K \in \mathcal{F}$.
- A T_0 space X is called a d-space if X is a dcpo and every open set is Scott open with respect to the specialization order.

Definition (locally compact, core-compact)

Let X be a T_0 space, $\mathcal{O}(X)$ be the set of all open subsets of X. Then, X is called

- locally compact, if for $x \in U \in \mathcal{O}(X)$, there is compact saturated set K such that $x \in K^o \subseteq K \subseteq U$.
- core-compact, if $(\mathcal{O}(X), \subseteq)$ is continuous.

Now the following relations are clear:

- locally compact $\Rightarrow$ core-compact;
- sober space $\Rightarrow$ well-filtered space $\Rightarrow$ d-space;
- core-compact $+$ well-filtered $\Rightarrow$ sober.

Definition (open well-filtered space)

Let X be a T_0 space, and $\mathcal{O}(X)$ be the family of all open subsets of X.

- For $U, V \in \mathcal{O}(X)$, use $U \ll V$ to denote that U is way-below V in poset $(\mathcal{O}(X), \subseteq)$;
- A family $\mathcal{F}$ of $\mathcal{O}(X)$ is called $\ll$-filtered if for any $U_1, U_2 \in \mathcal{F}$, there is $U \in \mathcal{F}$ such that $U \ll U_1$ and $U \ll U_2$;
- A T_0 space X is called open well-filtered if for each $\ll$-family $\mathcal{F} \subseteq \mathcal{O}(X)$ and each $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $V \subseteq U$ for some $V \in \mathcal{F}$.

Theorem (Shen-Xi-Xu-Zhao-2020)

Every core-compact open well-filtered space is sober.

Proof.

1. Take a closed irreducible set A, let
$$\mathcal{F} = \{U \in \mathcal{O}(X) : U \cap A \neq \emptyset\};$$
2. core-compact $\Rightarrow \mathcal{F}$ is $\ll$-filtered;
3. open well-filtered $\Rightarrow \exists x \in A \cap \bigcap \mathcal{F} \neq \emptyset$;
4. easy to show $A = \mathrm{cl}(\{x\})$.

Completing the proof. □

Proposition (Shen-Xi-Xu-Zhao-2020)

Every well-filtered space is open well-filtered.

Corollary

Every core-compact well-filtered space is sober.

Definition (OWF-set)

A nonempty subset A of a T_0 space X is called an open well-filtered set, or OWF-set for short, if there exists a $\ll$-filtered family $\mathcal{F} \subseteq \mathcal{O}(X)$ such that

$$\mathrm{cl}(A) \in \min\{C \in \mathcal{C}(X) : \forall U \in \mathcal{F}, C \cap U \neq \emptyset\}.$$

Theorem

Let X be a T_0 space. TFAE:

(1) *X is open well-filtered;*

(2) *for any closed OWF-set A, there exists a unique $x \in X$ such that $A = \mathrm{cl}(\{x\})$.*

Proposition

Every open subspace of an open well-filtered space is also open well-filtered.

Example

- **Neither** the cofinite topology **nor** the Alexandorff topology on the natural numbers is open well-filtered;
- The Scott space of the Johnstone's dcpo J is open well-filtered, but the maximal point space of J is not open well-filtered.

Corollary

The saturated subspace of an open well-filtered space need **not** *be open well-filtered.*

Example

Let J is the Johnstone's dcpo and $\mathbb{N}^+$ the positive integers with the usual order. Let $P = J \cup \mathbb{N}^+$, and for any $x, y \in P$, $x \leq y$ iff one of the following conditions holds:
(i) $x, y \in \mathbb{N}^+$ and $x \leq y$ in $\mathbb{N}^+$;
(ii) $x, y \in J$ and $x \leq y$ in J;
(iii) $x \in \mathbb{N}^+$ and $y \in J$ and $y = (x, \omega)$.

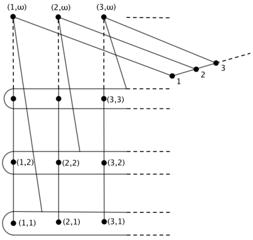


Figure 1: The poset P

Then, (1) the Scott space of P is open well-filtered, and clearly not a d-space; (2) the (closed) subspace $\mathbb{N}^+$ (homeomoprhic to the Alexandroff space of $\mathbb{N}^+$) is a retract of P.

Corollary

1. *An open well-filtered space need* **not** *be a d-space.*
2. *A retract or closed subspace of an open well-filtered space need* **not** *be open well-filtered.*

Proposition

For each T_0 space X, the product $X \times \Sigma J$ is open well-filterd.

The above result indicates that

- the product is open well-filtered $\not\Rightarrow$ the factor spaces are.

Theorem

If the product of finite T_0 spaces is open well-filtered, then one of its factor spaces is also open well-filtered.

The following question is still open:

- Whether the product of two open well-filtered spaces is also open well-filtered?

For any topological space X, we use $\mathcal{D}(X)$ to denote the set of all nonempty compact saturated subsets of X. The upper Vietoris topology on $\mathcal{D}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathcal{D}(X) : K \subseteq U\}$. The set $\mathcal{D}(X)$ equipped with the upper Vietoris topology is called the upper space or Smyth power space of X.

Theorem

For any open well-filtered space X, the upper space $\mathcal{D}(X)$ is open well-filtered.

The following question is open:

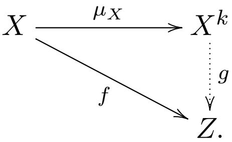
- Whether the open well-filteredness of $\mathcal{D}(X)$ implies that X is open well-filtered?

Definition (b-topology[1])

Let X be a T_0 space, the b-topology is generated by the family $\{\downarrow x \cap U : x \in U \in \mathcal{O}(X)\}$.

Definition (reflection)

A subcategory $\mathbf{K}$ of $\mathbf{Top_0}$ is called reflective, if $\forall X \in \mathbf{Top_0}$, $\exists X^k \in \mathbf{K}$ and a continuous map $\mu_X : X \longrightarrow X^k$ s.t. for any continuous map $f : X \longrightarrow Z \in \mathbf{K}$, $\exists$ a unique continuous map $g : X^k \longrightarrow Z$ such that $g \circ \mu_X = f$:



[1] L. Skula, On a reflective subcategory of the category of all topological spaces, Trans. Amer. Math. Soc. 142 (1969) 37–41.

In 2009, Keimel and Lawson [2] showed that a subcategory $\mathbf{K}$ of T_0 spaces is reflective in the category $\mathbf{Top_0}$ of all T_0 spaces if it satisfies the following **four conditions**:

(K1) $\mathbf{K}$ contains all sober spaces;

(K2) If $X \in \mathbf{K}$ and Y is homeomorphic to X, then $Y \in \mathbf{K}$;

(K3) If $\{X_i : i \in I\} \subseteq \mathbf{K}$ is a family of subspaces of a sober space, then the subspace $\bigcap_{i \in I} X_i \in \mathbf{K}$.

(K4) If $f : X \longrightarrow Y$ is a continuous map from a sober space X to a sober space Y, then for any subspace Z of Y, $Z \in \mathbf{K}$ implies that $f^{-1}(Z) \in \mathbf{K}$.

[2] K. Keimel, J.D. Lawson, D-completions and the d-topology, Ann. Pure Appl. Logic 159 (2009) 292–306.

Theorem

Let $\mathbf{K}$ *be a subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{T_1}$*. Then the following statements are equivalent:*

(1) $\mathbf{K}$ *is reflective in* $\mathbf{Top_0}$*;*

(2) $\mathbf{K}$ *satisfies conditions (K1)–(K4);*

(3) $\mathbf{K}$ *is productive and* b*-closed-hereditary;*

(4) $\mathbf{K}$ *is productive and has equalizers.*

Example

Let J be the Johnstone's dcpo, Then,

- ΣJ is open well-filtered;
- the maximal point space $\mathrm{Max}(J)$ is b-closed, which is not open well-filtered.

Therefore, the category of open well-filtered spaces is not b-closed-hereditary.

Corollary

The category of open well-filtered spaces is **not** *reflective in* $\mathbf{Top_0}$*.*

The $\mathcal{U}_s$-admitting spaces [3] were introduced by Heckmann to provide a novel and succinct upper power domain construction of the Scott space on dcpos, which is defined by means of **strongly compact** sets.

Definition (strongly compact)

Let X be a T_0 space. A subset S of X is strongly compact iff for all open sets O with $S \subseteq O$ there is a finite set F such that $S \subseteq \uparrow F \subseteq O$.

Definition ($\mathcal{U}_s$-admitting space)

A T_0 space is called *$\mathcal{U}_s$-admitting* if for any filtered family $\mathcal{F}$ of strongly compact saturated sets and any open set U in X, $\bigcap \mathcal{F} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{F}$.

[3] Heckmann, R.: An upper power domain construction in terms of strongly compact sets. In: Brookes, S., Main, M., Melton, A., Mislove, M., Schmidt, D. (eds.) Mathematical Foundations of Programming Semantics, pp. 272–293. Springer, Berlin (1992).

Definition ($\mathcal{U}_s$-set)

Let X be a T_0 space. A nonempty subset A of X is called an *$\mathcal{U}_s$-set*, if there is a filtered family $\mathcal{F}$ of stronlgy compact saturated sets such that (under the inclusion order)

$$\mathrm{cl}(A) \in \min\{C \in \mathcal{C}(X) : \forall K \in \mathcal{F}, C \cap K \neq \emptyset\}.$$

Theorem (Characterization for $\mathcal{U}_s$-admitting space)

Let X be a T_0 space. The following conditions are equivalent:

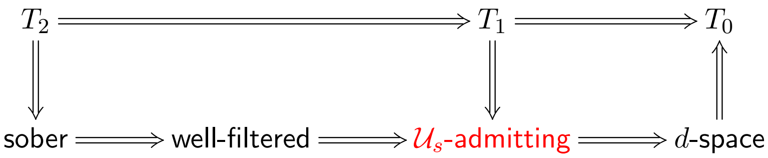
(1) *X is $\mathcal{U}_s$-admitting;*

(2) *for each closed $\mathcal{U}_s$-set $A \subseteq X$, there exists $x \in X$ such that $A = \mathrm{cl}(\{x\})$;*

(3) *for each closed $\mathcal{U}_s$-set $A \subseteq X$ and $U \in \mathcal{O}(X)$, $\bigcap_{a \in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$;*

- With "compact" replacing "strongly compact", we can obtain the notion of KF-set and the Characterization for well-filtered spaces.
- Directed set $\Rightarrow$ $\mathcal{U}_s$-set $\Rightarrow$ KF-set
- In T_1 space, strongly compact sets $\Leftrightarrow$ finite sets.

From the above, the following relations hold:



Denote **Us-Admitting** be the category of all $\mathcal{U}_s$-admitting spaces with continuous mappings. Using the characterization theorem of $\mathcal{U}_s$-admitting space (in terms of $\mathcal{U}_s$-set), we can prove that

Lemma

Us-Admitting *satisfies (K1)–(K4) in the sense of Keimel and Lawson.*

Corollary

Us-Admitting *is reflective in* $\mathbf{Top_0}$.

For any topological space X, we use $P_s(X)$ to denote the set of all nonempty **strongly compact** saturated subsets of X. The upper Vietoris topology on $P_s(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in P_s(X) : K \subseteq U\}$. The set $P_s(X)$ equipped with the upper Vietoris topology is called the upper space or Smyth power space of X.

Remark

Let X be a T_0 space. The following results hold trivially.

(1) *The specialization order of $P_s(X)$ is $\supseteq$.*

(2) *The mapping $\xi : X \longrightarrow P_s(X)$, $x \mapsto \uparrow x$ is continuous.*

(3) *Let $\mathcal{A} \subseteq P_s(X)$. Then $\bigcap \mathrm{cl}_{P_s(X)}(\mathcal{A}) = \bigcap \mathcal{A}$.*

(4) *If $\mathcal{K}$ is a strongly compact saturated set in $P_s(X)$, then so is $\bigcup \mathcal{K}$ in X.*

In the paper[4], Heckmann asked

- whether the upper space preserves $\mathcal{U}_s$-admitting property?

Theorem

Let X be a T_0 space. Then the following conditions are equivalent:

1. X *is* $\mathcal{U}_s$*-admitting;*
2. $P_s(X)$ *is* $\mathcal{U}_s$*-admitting;*
3. $P_s(X)$ *is a* d*-space;*
4. *for each (closed)* $\mathcal{U}_s$*-set* $\mathcal{A}$ *in* $P_s(X)$ *and each* $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ *implies* $K \subseteq U$ *for some* $K \in \mathcal{A}$.

[4] Heckmann, R.: An upper power domain construction in terms of strongly compact sets. In: Brookes, S., Main, M., Melton, A., Mislove, M., Schmidt, D. (eds.) Mathematical Foundations of Programming Semantics, pp. 272–293. Springer, Berlin (1992).

Definition (poset model)

A poset model of a topological space X is a poset P such that the set $\mathrm{Max}(P)$ of all maximal points of P with the relative Scott topology is homeomorphic to X.

Remark

Topological spaces that has a poset model must be T_1.

Theorem (Zhao-2009, Zhao-Xi-2018)

Every T_1 space X has a dcpo model, denoted by $D(X)$.

Procedures of the proof:

(i) Construct a bounded complete algebraic poset model $(P, \leq_P)$ of X;

(ii) From $(P, \leq_P)$, construct a dcpo $\widehat{P}$ as follows:

$$\widehat{P} = \{(x, e) : x \in P, e \in \mathrm{Max}(P) \text{ and } x \leq_P e\},$$

and $(x, e) \leq (y, d)$ in $\widehat{P}$ iff either $e = d$ and $x \leq_P y$, or $y = d$ and $x \leq_P d$.

(iii) Then, $\mathrm{Max}(\widehat{P}) = \{(e, e) : e \in \mathrm{Max}(P)\}$ and the mapping $h : \mathrm{Max}_\sigma(P) \longrightarrow \mathrm{Max}_\sigma(\widehat{P})$, where $h(e) = (e, e)$ for each $e \in \mathrm{Max}(P)$, is a homeomorphism.

- $D(X)$ is a weak domain, and X is sober (resp., well-filtered, Baire, Choquet complete, weak sober, Rudin space) iff the Scott space of $D(X)$ is so (see Shen-Wu-Zhao-2019, Zhao-Xi-2018, Xi-Zhao-2017, He-Xi-Zhao-2019, Chen-Li-2022).

Recently, Chen and Li proved that

- $D(\mathbb{N}_{cof})$ is not $\mathcal{U}_s$-admitting, where $\mathbb{N}_{cof}$ is the natural numbers with the cofinite topology.

Since T_1 implies $\mathcal{U}_s$-admitting, the result can be restated as

Proposition (Chen-Li-2022)

The Xi-Zhao dcpo model $D(X)$ need not be $\mathcal{U}_s$-admitting.

Then one can naturally ask:

- when $D(X)$ is $\mathcal{U}_s$-admitting?

Lemma

For a bounded complete algebraic poset P, $\Sigma\widehat{P}$ is $\mathcal{U}_s$-admitting iff it is well-filtered.

A sufficient and necessary condition for above question is as follows:

Theorem

For a T_1 space X, the following statements are equivalent:

1. *X is well-filtered;*
2. *$D(X)$ is well-filtered;*
3. *$D(X)$ is $\mathcal{U}_s$-admitting.*

Thanks !!!

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