Quantaloidal Completions of Order-enriched Categories and Their Applications

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Outline

Definition 1.1 ([Martin1991])

An *order-enriched category* is a locally small category $\mathcal{A}$ such that:

(1) for $a, b \in \mathcal{A}_0$, the hom-set $\mathcal{A}(a, b)$ is a poset,

(2) composition of morphisms of $\mathcal{A}$ preserves order in both variables.

[Martin1991] Martin C. E., C. A. R. Hoare, and J. F. He, *Pre-adjunctions in order enriced categories*, Math. Struct. in Comp. Science **1** (1991) 141–158

Introduction

In 1979, M. Wand ([Wand1979]) studied fixed-point constructions in order-enriched categories, which extended Scott's result based on continuous lattices.

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In 1982, M. Smyth and G. Plotkin considered solving recursive domain equations in this framework ([Smyth1982]). In 2007, this ideal was further extended to the framework of bicategories([Cattani2007]).

[Wand1979] Wand M., *Fixed-point constructions in order-enriched categories*, Theoretical Computer Science **8** (1979) 13–30

[Smyth1982] Smyth M., and G. Plotkin, *The category-theoretic solution of recursive domain equations*, SIAM Journal of Computing **11** (1982) 761–783

[Cattani2007] Cattani G. L., and M. P. Fiore, *The Bicategory-Theoretic Solution of Recursive Domain Equations*, Electronic Notes in Theoretical Computer Science **172** (2007) 203–222

Introduction

In 1991, C. E. Martin, C. A. R. Hoare and He Jifeng studied pre-adjunctions in order enriched-categories ([Martin1991]). In that paper, the concepts of lax functors, natural transformations and pre-adjunctions are studied with the purpose to explain their understanding of programming languages. We also note that an order-enriched category in the sense of ([Martin1991]) means a category with hom-sets preordered.

[Martin1991] Martin C. E., C. A. R. Hoare, and J. F. He, *Pre-adjunctions in order enriced categories*, Math. Struct. in Comp. Science **1** (1991) 141–158

A *quantaloid* $\mathcal{Q}$ is a category enriched in the symmetric monoidal closed category **Sup** of complete lattices and morphisms that preserve arbitrary sups. In elementary terms:

Definition 1.2 ([Rosenthal1991])

A *quantaloid* is a locally small category $\mathcal{Q}$ such that:
(1) for $a, b \in \mathcal{Q}_0$, the hom-set $\mathcal{Q}(a, b)$ is a complete lattice,
(2) composition of morphisms of $\mathcal{Q}$ presevers sups in both variables.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

Just as every complete lattice is a special partially ordered set, every quantaloid is a special order-enriched categories. A quantaloid with only one object is a quantale [Rosenthal1990], thus quantaloids are naturally viewed as quantales with many objects.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

[Rosenthal1990] Rosenthal K. I., "Quantales and Their Applications," Pitman Research Notes in Mathematics Series, **234**, Longman, Essex, 1990

Introduction

Quantaloids were studied by Pitts [Pitts1988] in investigating distributive categories of relations and topos theory under the name of sup-lattice enriched categories. In [Abramsky1993] quantaloids are studied in order to include a notion of type on the processes. Quantaloids and their applications were further developed in the monograph [Rosenthal1996] . In recent years, Quantaloid-enriched categories received considerable attention.

[Pitts1988] Pitts A., *Applications of sup-lattice enriched category theory to sheaf theory*, Proceedings of the London Mathematical Society **57** (1988) 433–480

[Abramsky1993] Abramsky S., and S. Vickers, *Quantales, observational logic and process semantics*, Mathematical Structures in Computer Science **3** (1993) 161–227

[Rosenthal1996] Rosenthal K. I., "The Theory of Quantaloids," Pitman Research Notes in Mathematics Series, Vol.348, Longman, Essex, 1996

What's the problem?

Relationships between order-enriched categories and quantaloids have not received enough attention, though they have similar backgrounds and close relations.

- What relations exist between order-enriched categories and quantaloids?

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- What relations exist between order-enriched categories and quantaloids?
- How to establish relations between them?

What's the way?

The process of completion is a classic approach to study ordered structures. Various completion methods for ordered structures are developed with different characteristics. Inspired by research on completion methods for ordered semigroups and their applications [Han2008,Lambek2012,Rosenthal1991], this paper is devoted to study quantaloidal completions of order-enriched categories and their applications.

[Han2008] Han S. W., and B. Zhao, *The quantale completion of ordered semigroup*, Acta Mathematica Sinica, Chinese Series **51** (2008) 1081-1088

[Lambek2012] Lambek J., M. Barr, J. F. Kennison, and R. Raphael, *Injective hulls of partially ordered monoids*. Theory Appl. Categ. **26** (2012) 338–348

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

Definition 2.1 ([Zhang2019])

Let $\mathcal{C}$, $\mathcal{D}$ be order-enriched categories. A *lax semifunctor* $F : \mathcal{C} \to \mathcal{D}$ is given by functions $F : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_{a,b} : \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ for all $a, b \in \mathcal{C}_0$ such that $F_{a,b}$ is order-preserving and $(Fg) \circ (Ff) \leq F(g \circ f)$ for all $a, b, c \in \mathcal{C}_0$, $f \in \mathcal{C}(a,b), g \in \mathcal{C}(b,c)$. A *lax functor* $F : \mathcal{C} \to \mathcal{D}$ is a lax semifunctor such that $1_{Fa} \leq F(1_a)$ for all $a \in \mathcal{C}_0$. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor such that

$$F_{a,b} : \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$$

is order-preserving for all $a, b \in \mathcal{C}_0$.

[Zhang2019] Zhang X., and J. Paseka, *On injective constructions of S-semigroups*, Fuzzy Sets and Systems 373 (2019) 78–93

Definition 2.2 ([Rosenthal1991])

Let $\mathcal{Q}$, $\mathcal{S}$ be quantaloids. A *quantaloidal homomorphism* $F : \mathcal{Q} \to \mathcal{S}$ is a functor such that

$$F : \mathcal{Q}(X, Y) \to \mathcal{S}(FX, FY)$$

is sup-preserving for all $X, Y \in \mathcal{Q}_0$.

A *quantaloidal isomorphism* is a quantaloidal homomorphism such that it is bijective on objects and hom-sets.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

Example 2.3

Let $\mathcal{A}$ be an order-enriched category.

(1) $\mathcal{P}(\mathcal{A})$ is a quantaloid ([Rosenthal1991]). The objects of $\mathcal{P}(\mathcal{A})$ are those of $\mathcal{A}$. For $a, b \in \mathcal{A}$, the hom-set $\mathcal{P}(\mathcal{A})(a,b) = \mathcal{P}(\mathcal{A}(a,b))$, the power set of the hom-set $\mathcal{A}(a,b)$. For $S \in \mathcal{P}(\mathcal{A})(a,b), T \in \mathcal{P}(\mathcal{A})(b,c)$, $T \circ S = \{g \circ f \mid g \in T, f \in S\}$.

(2) $\mathcal{D}(\mathcal{A})$ is a quantaloid. The objects of $\mathcal{D}(\mathcal{A})$ are those of $\mathcal{A}$. For $a, b \in \mathcal{A}$, the hom-set $\mathcal{D}(\mathcal{A})(a,b) = \mathcal{D}(\mathcal{A}(a,b))$, the set of down sets[a] of the hom-set $\mathcal{A}(a,b)$. For $S \in \mathcal{D}(\mathcal{A})(a,b), T \in \mathcal{D}(\mathcal{A})(b,c)$, $T \circ S = \downarrow \{g \circ f \mid g \in T, f \in S\}$. We note that $\downarrow 1_a \in \mathcal{D}(\mathcal{A}(a,a))$ is the identity morphism.

[a] A set D in a poset P is a down set, if $D = \downarrow D$, where $\downarrow D = \{x \mid \exists d \in D, \text{ s. t. } x \leq d\}$.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

Definition 2.4 ([Rosenthal1991])

Let $\mathcal{Q}$ be a quantaloid. A *quantaloidal nucleus* is a lax functor $j : \mathcal{Q} \to \mathcal{Q}$, which is the identity on the objects of $\mathcal{Q}$ and such that the maps

$j_{a,b} : \mathcal{Q}(a,b) \to \mathcal{Q}(a,b)$ satisfy:

(1) $f \leq j_{a,b}(f)$ for all $f \in \mathcal{Q}(a,b)$,

(2) $j_{a,b}(j_{a,b}(f)) = j_{a,b}(f)$ for all $f \in \mathcal{Q}(a,b)$,

(3) $j_{b,c}(g) \circ j_{a,b}(f) \leq j_{a,c}(g \circ f)$ for all $g \in \mathcal{Q}(b,c)$ $f \in \mathcal{Q}(a,b)$.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

For a quantaloidal nucleus j on a quantaloid $\mathcal{Q}$, let $\mathcal{Q}_j$ be the bicategory with the same objects as $\mathcal{Q}$ and $\mathcal{Q}_j(a,b) = \{f \in \mathcal{Q}(a,b) \mid j_{a,b}(f) = f\}$ for $a, b \in (\mathcal{Q}_j)_0$. Composition in $\mathcal{Q}_j$ is defined as follows: $g \circ_j f = j_{a,c}(g \circ f)$ for $f \in \mathcal{Q}_j(a,b)$, $g \in \mathcal{Q}_j(b,c)$.

Lemma 2.5 ([Rosenthal1991])

If j is a quantaloidal nucleus on a quantaloid $\mathcal{Q}$, then $\mathcal{Q}_j$ is a quantaloid and $j : \mathcal{Q} \to \mathcal{Q}_j$ is a quantaloidal homomorphism.

Lemma 2.6 ([Rosenthal1991])

Let $\mathcal{S}$ be a subcategory of a quantaloid $\mathcal{Q}$, which contains all the objects of $\mathcal{Q}$. Then, $\mathcal{S}$ is a quotient quantaloid of the form $\mathcal{Q}_j$ for some quantaloidal nucleus j iff

(1) each hom-set $\mathcal{S}(a,b)$ is closed under infs,

(2) if $f \in \mathcal{S}(a,c)$, then $g \rightarrow_l f \in \mathcal{S}(b,c)$ for all $g \in \mathcal{Q}(a,b)$ and $h \rightarrow_r g \in \mathcal{S}(a,b)$ for all $h \in \mathcal{Q}(b,c)$.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

Definition 3.1

Let $\mathcal{A}$ be an order-enriched category, $j : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ a quantaloidal nucleus. j is said to be *compatible* if for $a, b \in \mathcal{A}_0$, $f \in \mathcal{A}(a, b)$, we have $j_{a,b}(\{f\}) = \downarrow f$.

Definition 3.2

Let $\mathcal{A}$ be an order-enriched category, $\mathcal{Q}$ a quantaloid, $F : \mathcal{A} \to \mathcal{Q}$ a 2-functor. The pair $(F, \mathcal{Q})$ is said to be a *quantaloidal completion* of $\mathcal{A}$, if the following conditions are satisfied:

(1) $F : \mathcal{A}_0 \to \mathcal{Q}_0$ is bijective,

(2) $F_{a,b} : \mathcal{A}(a, b) \to \mathcal{Q}(Fa, Fb)$ is an order embedding for all $a, b \in \mathcal{A}_0$,

(3) for every $a, b \in \mathcal{A}_0$ and $f \in \mathcal{Q}(Fa, Fb)$, there exists $U_f \subseteq \mathcal{A}(a, b)$ such that $f = \bigvee F(U_f)$.

[HanZhao2008] Han S. W., and B. Zhao, *The quantale completion of ordered semigroup*, Acta Mathematica Sinica, Chinese Series **51** (2008) 1081–1088

Theorem 3.3

If j is a compatible nucleus on an order-enriched category $\mathcal{A}$, then $(F_j, \mathcal{P}(\mathcal{A})_j)$ is a quantaloidal completion of $\mathcal{A}$, where $F_j : \mathcal{A} \to \mathcal{P}(\mathcal{A})_j$ is defined as follows:

(1) $F_j : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A})_j)_0$ is the identity map,

(2) $F_j(f) = \downarrow f$ for every $f \in \mathcal{A}(a,b)$, $a, b \in \mathcal{A}_0$.

Corresponding to several classical completion methods of posets and ordered semigroups, we can obtain a series of compatible nucleus.

Example 3.4

(Down-set completion) Let $\mathcal{A}$ be an order-enriched category. Define a lax functor $\downarrow: \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ as follows:

(1) $\downarrow: \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is the identity map,

(2) $\downarrow_{a,b}(S) = \downarrow S$ for $S \in \mathcal{P}(\mathcal{A})(a,b)$, $a, b \in \mathcal{P}(\mathcal{A})_0$.

Then $\downarrow$ is a compatible nucleus. The quotient corresponding to $\downarrow$ is $\mathcal{D}(\mathcal{A})$.

Example 3.5

(MacNeille completion) Let $\mathcal{A}$ be an order-enriched category. Define a lax functor $\mathrm{cl} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ as follows:

(1) $\mathrm{cl} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is the identity map,

(2) $\mathrm{cl}_{a,b}(S) = \{f \in \mathcal{P}(\mathcal{A})(a,b) \mid \forall g \in \mathcal{P}(\mathcal{A})(a',a), h \in \mathcal{P}(\mathcal{A})(b,b'), k \in \mathcal{P}(\mathcal{A})(a,b), h \circ S \circ g \subseteq\downarrow k \text{ implies } h \circ f \circ g \leq k\}$ for $S \in \mathcal{P}(\mathcal{A})(a,b)$, $a, b \in \mathcal{P}(\mathcal{A})_0$.

Then cl is a compatible nucleus.

Example 3.6

(Equivariant completion) Let $\mathcal{A}$ be an order-enriched category. Suppose $S \subseteq \mathcal{P}(\mathcal{A})(a,b)$. If the join of S exists and is preserved by composition, i.e., $f \circ (\bigvee S) = \bigvee(f \circ S)$, $(\bigvee S) \circ g = \bigvee(S \circ g)$ whenever the composition is well-defined, then $\bigvee S$ is said to be an *equivariant join* with respect to S. For $S \subseteq \mathcal{P}(\mathcal{A})(a,b)$, let $S^{EJ} = \{f \in \mathcal{P}(\mathcal{A})(a,b) \mid \exists T \subseteq S, \text{s.t. } f = \bigvee T$ is an equivariant join with respect to $T\}$.

Let $EJ(\mathcal{A})$ be the subcategory of $\mathcal{A}$, which contains all the objects of $\mathcal{A}$. The hom-sets

$$EJ(\mathcal{A})(a,b) = \{S \in \mathcal{D}(\mathcal{A})(a,b) \mid S = S^{EJ}\}.$$

Then $EJ(\mathcal{A})$ is a quotient of $\mathcal{A}$ such that $\downarrow f \in EJ(\mathcal{A})(a,b)$ for every $f \in \mathcal{P}(\mathcal{A})(a,b)$. Consequently, the corresponding quantaloidal nucleus is compatible.

For an order-enriched category $\mathcal{A}$, $CN(\mathcal{A})$ denotes the class of all compatible nuclei on $\mathcal{P}(\mathcal{A})$, $QC(\mathcal{A})$ denotes the set of all quantaloidal completions of $\mathcal{A}$.

Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Define $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ as follows:

(1) $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is the identity map,

(2) $j_{(F,\mathcal{Q})}(S) = \{f \in \mathcal{A}(a, b) \mid F(f) \leq \bigvee_{g \in S} F(g)\}$ for every $S \in \mathcal{P}(\mathcal{A})(a, b), a, b \in \mathcal{A}_0$.

Lemma 3.7

Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Then $j_{(F,\mathcal{Q})}$ is a compatible nucleus on $\mathcal{P}(\mathcal{A})$.

Lemma 3.7

Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Then $j_{(F,\mathcal{Q})}$ is a compatible nucleus on $\mathcal{P}(\mathcal{A})$.

Theorem 3.8

Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Then $\mathcal{Q}$ is quantaloidal isomorphism to $\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}$.

As a combination of the above results, we obtain that quantaloidal completions of an order-enriched category $\mathcal{A}$ are completely determined by compatible quantaloidal nuclei on $\mathcal{P}(\mathcal{A})$.

Theorem 3.9

Let $\mathcal{A}$ be an order-enriched category. Then $(F, \mathcal{Q})$ is a quantaloidal completion of $\mathcal{A}$ if and only if there is a compatible nucleus j on $\mathcal{P}(\mathcal{A})$ such that $\mathcal{Q}$ is quantaloidal isomorphism to $\mathcal{P}(\mathcal{A})_j$.

In this section, we shall give two kinds of applications for the quantaloidal completions of order-enriched categories.

Injective constructs of order-enriched categories

Let $\mathbf{O\text{-}Cat}_l$ be the category of order-enriched categories and lax semifunctors. Let $\mathcal{E}$ be the class of all lax semifunctors in $\mathbf{O\text{-}Cat}_l$ satisfy the following conditions:

(1) $F : \mathcal{C}_0 \to \mathcal{D}_0$ is bijective;

(2) $F(f_1) \circ F(f_2) \cdots F(f_n) \leq F(f)$ implies $f_1 \circ f_2 \cdots f_n \leq f$ for $f_1 \circ f_2 \cdots f_n, f \in C(a, b)$, $a, b \in \mathcal{C}_0$.

Lemma 4.1

In the category $\mathbf{O\text{-}Cat}_l$, every retract of a quantaloid is a quantaloid.

Theorem 4.2

Let $\mathcal{A}$ be an order-enriched category. Then $\mathcal{A}$ is $\mathcal{E}$-injective in $\mathbf{O\text{-}Cat}_l$ if and only if $\mathcal{A}$ is a quantaloid.

Let $\mathcal{A}$ be an order-enriched category. Define $\eta : \mathcal{A} \to \mathcal{P}(\mathcal{A})_{\rm cl}$ as follows:

(1) $\eta : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A})_{\rm cl})_0$ is the identity map;

(2) $\eta(f) = \downarrow f$ for $f \in \mathcal{A}(a, b)$, $a, b \in \mathcal{A}_0$.

Then it is routine to check that η is a lax semifunctor and it is $\mathcal{E}$-essential.

Theorem 4.3

Let $\mathcal{A}$ be an order-enriched category. Then $\mathcal{P}(\mathcal{A})_{\rm cl}$ is an $\mathcal{E}$-injective hull of $\mathcal{A}$ in $\mathbf{O}\text{-}\mathbf{Cat}_l$.

Free quantaloids

Let $\mathbf{O\text{-}Cat}$ be the category of order-enriched categories and 2-functors.

Let $\mathbf{Qtlds}$ be the category of quantaloids and quantaloidal homomorphisms.

Theorem 4.4

The functor $\mathcal{D} : \mathbf{O\text{-}Cat} \to \mathbf{Qtlds}$ *is left adjoint to the forgetful functor* $\mathbf{Qtlds} \to \mathbf{O\text{-}Cat}$.

Let $\mathbf{LocSm}$ be the category of locally small categories and functors between them.

Every locally small category can be viewed as an order-enriched category with the discrete order on hom-sets. We know $\mathcal{D}(\mathcal{A}) = \mathcal{P}(\mathcal{A})$ for every locally small category with discrete order on hom-sets. Thus, we can recover the following results ([Rosenthal1991]).

Corollary 4.5

The functor $\mathcal{P} : \mathbf{LocSm} \to \mathbf{Qtlds}$ *is left adjoint to the forgetful functor* $\mathbf{Qtlds} \to \mathbf{LocSm}$.

[Rosenthal1991] Rosenthal K. I., *Free quantaloids*, Journal of Pure and Applied Algebra **72** (1991) 67–82

Conclusion and some further work

We only considered quantaloidal completions for order-enriched categories.

- Other types of completions for order-enriched categories.
- Order-enriched categories with certain type of completeness.
- Categories not only order-enriched but also with ordered structure on objects.

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Thank you!