Completions, $\mathbb{K}$-categories and Commutative Probability Monads

To DANA SCOTT *on his 90th Birthday*

Michael Mislove

Tulane University

International Symposium on Domain Theory

Singapore and AWOE, July 4 - 6, 2022

Joint work with Xiaodong Jia, Bert Lindenhovius and Vladimir Zamdzhiev

Supported by US AFOSR

Some History

1972 *Continuous lattices* by D. S. Scott, in which the injective T_0-spaces X are characterized as retracts of $\mathbb{S}^{\mathcal{O}(X)}$ under maps preserving directed suprema, where $\mathbb{S}$ denotes *Sierpinski space*.

- Implies continuous lattices are *sober spaces* in the Scott topology.

Some History

1972 *Continuous lattices* by D. S. Scott, in which the injective T_0-spaces X are characterized as retracts of $\mathbb{S}^{\mathcal{O}(X)}$ under maps preserving directed suprema, where $\mathbb{S}$ denotes *Sierpinski space*.

1989 (M-) The *sobrification* of an algebraic poset is an algebraic domain which forms the DCPO-completion of the underlying poset.

1992 (H. Zhang) Same result holds for *continuous posets*.

Some History

1972 *Continuous lattices* by D. S. Scott, in which the injective T_0-spaces X are characterized as retracts of $\mathbb{S}^{\mathcal{O}(X)}$ under maps preserving directed suprema, where $\mathbb{S}$ denotes *Sierpinski space*.

1989 (M-) The *sobrification* of an algebraic poset is an algebraic domain which forms the DCPO-completion of the underlying poset.

1992 (H. Zhang) Same result holds for *continuous posets*.

2008 (*DCPO-completion of posets*, Zhao & Fan) Described a DCPO-completion of a poset that is finer than the sobrification.

2009 (*D-completions and the d-topology*, Keimel and Lawson) Gave topological account of Zhao & Fan's construction, and introduced $\mathtt{K}$-categories, which provide further examples.

In this talk, we'll apply these results to produce new *commutative* probabilistic monads over DCPO.

We begin by explaining the problem that inspired our work.

What's the Problem?

- The application of interest is *statistical programming:* functional languages that sample from probability distributions.

  These are viewed as the simply-typed lambda calculus extended with a *probability monad*.

What's the Problem?

- The application of interest is *statistical programming:* functional languages that sample from probability distributions.

  These are viewed as the simply-typed lambda calculus extended with a *probability monad*.

- *Natural model:* Sub-probabilistic power domain $\mathcal{V}_{\leq 1}(P)$ over a dcpo P, the family of *Scott-continuous valuations:* maps $\mu\colon \sigma(P) \to [0, 1]$ satisfying:
  - $\mu(\emptyset) = 0$
  - $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
  - $\mu(\bigcup_{U \in D} U) = \sup_{U \in D} \mu(U)$, if $D \subseteq \sigma(P)$ is a directed family.

What's the Problem?

- The application of interest is *statistical programming:* functional languages that sample from probability distributions.

  These are viewed as the simply-typed lambda calculus extended with a *probability monad*.

- *Natural model:* Sub-probabilistic power domain $\mathcal{V}_{\leq 1}(P)$ over a dcpo P, the family of *Scott-continuous valuations:* maps $\mu\colon \sigma(P) \to [0,1]$

- $\mathcal{V}_{\leq 1}$ defines a monad over DOM – category of domains and Scott continuous maps. In fact, $\mathcal{V}_{\leq 1}$ is *commutative*; i.e., the Fubini-like equation

$$\int_{x\in P}\int_{y\in Q}\chi_U(x,y)d\mu d\nu = \int_{y\in Q}\int_{x\in P}\chi_U(x,y)d\nu d\mu,$$

  holds, for domains P, Q, valuations $\mu \in \mathcal{V}_{\leq 1}(P), \nu \in \mathcal{V}_{\leq 1}(Q)$ and $U \subseteq P \times Q$ Scott open. *But,* DOM is not Cartesian closed.

  - *Jung-Tix Problem:* There is no known Cartesian closed category of domains on which $\mathcal{V}_{\leq 1}$ defines a monad.

What's the Problem?

- The application of interest is *statistical programming:* functional languages that sample from probability distributions.

  These are viewed as the simply-typed lambda calculus extended with a *probability monad*.
- *Natural model:* Sub-probabilistic power domain $\mathcal{V}_{\leq 1}(P)$ over a dcpo P, the family of *Scott-continuous valuations:* maps $\mu\colon \sigma(P) \to [0, 1]$
- However, $\mathcal{V}_{\leq 1}$ also defines a monad on DCPO, the category of dcpos and Scott continuous maps, which is Cartesian closed.
- *But:* $\mathcal{V}_{\leq 1}(P)$ is not known to be commutative over DCPO, so there is no proof that Fubini holds.
- So we search for *submonads of* $\mathcal{V}_{\leq 1}$ that are commutative over DCPO.
  - Xiaodong will describe some of the monads we found in his talk on Wednesday.
  - I want to outline the mathematical results that underpin such monads.

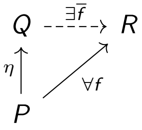
The DCPO-completion of a Poset

- *Zhao & Fan:* Defined the *d-topology* on a poset and the associated *d-completion*.[1]
  - A subset $A \subseteq P$ is *d-closed* if A is closed under existing suprema of directed sets: i.e., if $D \subseteq A$ is directed and $\sup D \in P$ exists, then $\sup D \in A$.
  - This defines the closed sets of the *d-topology:* the union of finitely many d-closed sets is d-closed (by short argument), and any intersection of d-closed sets is d-closed.
  - Scott-closed subsets are d-closed, so the d-topology refines the Scott topology.
  - The d-closed subsets of a dcpo are exactly the sub-dcpos.
  - Any lower set is d-open, so $\downarrow x$ is d-clopen for each $x \in P$.

[1] Zhao & Fan use the notation D-topology, etc., but that clashes with further results we discuss next.

The DCPO-completion of a Poset

- *Zhao & Fan:* Defined the *d-topology* on a poset and the associated *d-completion*.
- A dcpo Q is a *d-completion* of a poset P if there is $\eta\colon P \to Q$ Scott continuous satisfying



  where R is a dcpo and $f, \overline{f}$ are Scott continuous.
  Any two d-completions of P are isomorphic. Denote this by $\overline{P}$.

- Can be formed as follows:
  1) Embed $(P, \sigma(P))$ in a dcpo Q. E.g., take $Q = \Gamma(P)$.

  2) Take the intersection of all sub-dcpos of Q containing P. This is $\overline{P}$.

  Theorem: $\overline{P}$ is the smallest dcpo satisfying $P \hookrightarrow \overline{P}$ is an embedding in the Scott topology.

Monotone Convergence Spaces

2009 *D-completions and the d-topology*, Keimel and Lawson.

The d-topology is not order-theoretic: $Id\colon \mathbb{S} \to \{0,1\}^\flat$ is d-continuous.

A *monotone convergence space* is a T_0-space $(X, \mathcal{O}(X))$ in which each directed subset $D \subseteq X$ in the specialization order ($x \leq_s y$ iff $x \in \overline{\{y\}}$) converges to its supremum, $\sup D$.

Equivalently, $\Omega(X) = (X, \leq_s)$ is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X, \leq_s)$.

Examples: 1) Any sober space.

2) $\Sigma(P) = (P, \sigma(P))$ – Scott space of dcpo P.

Initially studied as *d-spaces* by Wyler (1981) and later by Ershov (1999).

D - category of monotone convergence spaces and continuous maps.

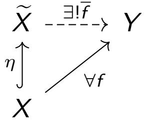
$$\mathrm{DCPO} \underset{\Omega}{\overset{\Sigma}{\underset{\bot}{\rightleftarrows}}} \mathrm{D}$$

Monotone Convergence Spaces

2009 *D-completions and the d-topology*, Keimel and Lawson.

D-completion of a T_0-space X: a d-dense embedding $X \hookrightarrow \widetilde{X}$ into a monotone convergence space $\widetilde{X}$. They exist because sober spaces are monotone convergence spaces. (Topological version of Zhao & Fan's d-completion.)

- *D-completions are universal:* for every $\eta \colon X \hookrightarrow \widetilde{X}$, for every $f : X \to Y (\in \mathrm{D})$:



This implies any two D-completions are homeomorphic. D_c denotes the D-completion.

Monotone Convergence Spaces

2009 *D-completions and the d-topology*, Keimel and Lawson.

- *D-completions are universal:* for every $\eta \colon X \hookrightarrow \widetilde{X}$, for every $f : X \to Y (\in \mathrm{D})$:

$$\begin{array}{ccc} \widetilde{X} & \overset{\exists! \overline{f}}{\dashrightarrow} & Y \\ {\scriptstyle \eta}\big\uparrow & \nearrow_{\forall f} & \\ X & & \end{array}$$

  This implies any two D-completions are homeomorphic. D_c denotes the D-completion.

- D is a full reflective subcategory of TOP_0 – T_0-spaces and continuous maps:

$$\mathrm{TOP}_0 \underset{incl}{\overset{\mathrm{D}_c}{\rightleftarrows}} \mathrm{D} \qquad (\mathrm{D}_c \dashv incl,\ \perp)$$

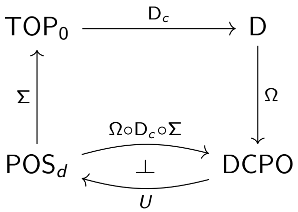
  D_c defines an idempotent monad on TOP_0:

$$f : X \to Y \quad \mapsto \quad \mathrm{D}_c(f) = \overline{\eta_Y \circ f} \colon \mathrm{D}_c(X) \to \mathrm{D}_c(Y)$$

Monotone Convergence Spaces

2009 *D-completions and the d-topology*, Keimel and Lawson.

- The d-completion of a poset in D; POS_d - posets and Scott continuous maps.



So, $\Omega \circ \mathrm{D}_c \circ \Sigma(P)$ is the d-completion of a poset P in the Scott topology.

K-categories

2009 *D-completions and the d-topology*, Keimel and Lawson.

- K-category: a subcategory K of TOP_0 (objects are called "K spaces") satisfying
    - SOB is a subcategory of K
    - K is closed under homeomorphic images
    - If X is a sober space, the intersection of any family of K-subspaces of X is a K-space
    - If $f : X \to Y$ in SOB, then f is K-continuous:
      i.e., if $Z \subseteq Y$ is a K-subspace, then $f^{-1}(Z)$ is a K-subspace of X;
      equivalently, if $W \subseteq X$ is a subspace, then $f(cl_K(W)) \subseteq cl_K(f(W))$.

Example: D is a K-category.

K-categories

2009 *D-completions and the d-topology*, Keimel and Lawson.

- K-category: a subcategory K of TOP_0 (objects are called "K spaces") satisfying
  - SOB is a subcategory of K
  - K is closed under homeomorphic images
  - If X is a sober space, the intersection of any family of K-subspaces of X is a K-space
  - If $f\colon X \to Y$ in SOB, then f is K-continuous:
    i.e., if $Z \subseteq Y$ is a K-subspace, then $f^{-1}(Z)$ is a K-subspace of X;
    equivalently, if $W \subseteq X$ is a subspace, then $f(cl_K(W)) \subseteq cl_K(f(W))$.

Example: D is a K-category.

We can form a K-completion $\mathrm{K}_c(X)$ of any T_0-space: use embedding $\eta\colon X \hookrightarrow X^s$ into its sobrification and then take

$$\eta_X\colon X \hookrightarrow \mathrm{K}_c(X) = \bigcap\{X' \subseteq X^s \mid X \subseteq X' \text{ a K space}\}.$$

The conditions assure that $\mathrm{K}_c(X)$ is a K-space and any continuous $f\colon X \to Y$ into a K-space admits a unique $\overline{f}\colon \mathrm{K}_c(X) \to Y$ with $\overline{f} \circ \eta_X = f$

$\mathtt{K}$-categories

2009 *D-completions and the d-topology*, Keimel and Lawson.

Theorem: Each $\mathtt{K}$-category is a full reflective subcategory of TOP_0:

$$\mathrm{TOP}_0 \underset{I}{\overset{\mathrm{K}_c}{\underset{\perp}{\rightleftarrows}}} \mathrm{K}$$

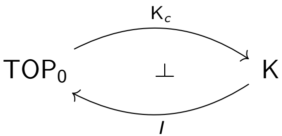
where K_c is the K-completion functor; this defines an idempotent monad on TOP_0:

$$f\colon X \to Y \quad \mapsto \quad \mathrm{K}_c(f) = \overline{\eta_Y \circ f}\colon \mathrm{K}_c(X) \to \mathrm{K}_c(Y)$$

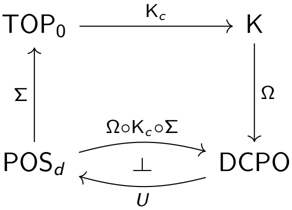
$\mathbb{K}$-categories

2009 *D-completions and the d-topology*, Keimel and Lawson.

Theorem: Each $\mathbb{K}$-category is a full reflective subcategory of TOP_0:



- If $\mathsf{K} \subseteq \mathsf{D}$, then $\Omega \circ \mathsf{K}_c \circ \Sigma(P)$ defines the K-completion of a poset P in the Scott topology:



Topological View of Valuations

Definition The *(extended) probabilistic power domain* $\mathcal{V}(X)$ over a topological space $(X, \mathcal{O}(X))$, is the family of maps $\mu\colon \mathcal{O}(X) \to \overline{\mathbb{R}_+}$ satisfying:

- $\mu(\emptyset) = 0$
- $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
- $\mu(\bigcup_{U\in D} U) = \sup_{U\in D} \mu(U)$, if $D \subseteq \mathcal{O}(X)$ is directed.

We endow $\mathcal{V}(X)$ with the *stochastic order:* $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ $(\forall U \in \mathcal{O}(X))$.

- $\mathcal{V}(X)$ is a DCPO – in fact, $\mathcal{V}\colon \mathsf{TOP}_0 \to \mathsf{DCPO}$ is a functor, where

$$f \in \mathsf{TOP}_0(X, Y) \quad \mapsto \quad \mu \mapsto \mathcal{V}f(\mu) = \mu \circ f^{-1} \in \mathsf{DCPO}(\mathcal{V}(X), \mathcal{V}(Y)).$$

(Jones) $\mathcal{V} \circ \Sigma\colon \mathsf{DCPO} \to \mathsf{DCPO}$ defines a strong monad;

$(\mathcal{V} \circ \Sigma)|_{\mathsf{DOM}}\colon \mathsf{DOM} \to \mathsf{DOM}$ is a commutative monad.

Topological View of Valuations

Definition The *(extended) probabilistic power domain* $\mathcal{V}(X)$ over a topological space $(X, \mathcal{O}(X))$, is the family of maps $\mu\colon \mathcal{O}(X) \to \overline{\mathbb{R}_+}$ satisfying:

- $\mu(\emptyset) = 0$
- $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
- $\mu(\bigcup_{U \in D} U) = \sup_{U \in D} \mu(U)$, if $D \subseteq \mathcal{O}(X)$ is directed.

We endow $\mathcal{V}(X)$ with the *stochastic order:* $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ $(\forall U \in \mathcal{O}(X))$.

$\mathcal{V}_w X$ denotes $\mathcal{V}X$ in the *weak topology:* $[U > r] = \{\mu \mid \mu(U) > r\} = \{\mu \mid \mathit{eval}(\mu, U) > r\}$ forms a sub-basis, where $\mathit{eval}\colon \mathcal{V}X \times \mathcal{O}(X) \to \mathbb{R}_+$. Heckmann showed $\mathcal{V}_w(X)$ is sober.

Topological View of Valuations

Definition The *(extended) probabilistic power domain* $\mathcal{V}(X)$ over a topological space $(X, \mathcal{O}(X))$, is the family of maps $\mu\colon \mathcal{O}(X) \to \overline{\mathbb{R}_+}$ satisfying:

- $\mu(\emptyset) = 0$
- $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
- $\mu(\bigcup_{U\in D} U) = \sup_{U\in D} \mu(U)$, if $D \subseteq \mathcal{O}(X)$ is directed.

(Goubault-Larrecq & Jia) $\mathcal{V}_w$ defines a monad on TOP_0:
$\mathcal{V}_w f\colon \mathcal{V}_w X \to \mathcal{V}_w Y$ is $\mathcal{V}_w f(\mu) = \mu \circ f^{-1}$ is continuous, for $f\colon X \to Y$.

The *unit* is the point valuation $x \mapsto \delta_x = \lambda U. \begin{cases} 1 & x \in U, \\ 0 & \text{otherwise.} \end{cases}$

We can define the Choquet-like integral $\int_X h(x) d\mu = \int_0^\infty \mu(h^{-1}(r, \infty])dr$ (the Riemann integral) for $h\colon X \to \overline{\mathbb{R}_+}$ lower semicontinuous and any space X.

Then the *multiplication* $m\colon \mathcal{V}_w^2 X \to \mathcal{V}_w X$ is defined as

$$m(\varpi) = \lambda U. \int_{\mathcal{V}_w X} \mu(U) d\varpi = \lambda U. \int_0^\infty \varpi(eval(-, U)^{-1}(r, \infty])dr = \lambda U. \int_0^\infty \varpi([U > r])dr.$$

Topological View of Valuations

Definition The *(extended) probabilistic power domain* $\mathcal{V}(X)$ over a topological space $(X, \mathcal{O}(X))$, is the family of maps $\mu\colon \mathcal{O}(X) \to \overline{\mathbb{R}_+}$ satisfying:

- $\mu(\emptyset) = 0$
- $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$
- $\mu(\bigcup_{U \in D} U) = \sup_{U \in D} \mu(U)$, if $D \subseteq \mathcal{O}(X)$ is directed.

Theorem For any space X, $\mathcal{V}_w X$ has a canonical cone structure making it a locally linear topological cone. Moreover, $\mathcal{V}_w f\colon \mathcal{V}_w X \to \mathcal{V}_w Y$ is continuous and linear, for each $f\colon X \to Y$.

$+\colon \mathcal{V}_w X \times \mathcal{V}_w X \to \mathcal{V}_w X$ is $(\mu + \nu)(U) = \mu(U) + \nu(U)$ for each open set $U \subseteq X$;
$r \cdot -\colon \mathcal{V}_w X \to \mathcal{V}_w X$ is $(r \cdot \mu)(U) = r \cdot \mu(U)$ for each $r \in \mathbb{R}_+$; and
$U \mapsto 0 \in \overline{\mathbb{R}_+}$ is the zero valuation.
The sub-basis $[U > r]$ consists of (open) half-spaces (i.e., convex subsets whose complement also is convex), which is what locally linear means. The last claim is straightforward from the definitions of $\mathcal{V}_w f(\mu) = \mu \circ f^{-1}$ and of the sub-bases $[U > r]$ for $\mathcal{V}_w X$ and $\mathcal{V}_w Y$.

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathbb{R}_+ \ \&\ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

$\mathcal{V}_s$ is a submonad of $\mathcal{V}_w$

- If $f \colon X \to Y$, then $\mathcal{V}_s f(\sum_{x \in F} r_x \cdot \delta_x) = \sum_{x \in F} r_x \cdot \delta_{f(x)}$.
- The unit $x \mapsto \delta_x : X \to \mathcal{V}_w X$ is simple.
- The multiplication $m \colon \mathcal{V}_w^2 X \to \mathcal{V}_w X$ restricts to $\mathcal{V}_s X^2 \to \mathcal{V}_s X$ via

$$m(\sum_{x \in F} r_x (\sum_{y \in G_x} s_{x,y} \delta_{x,y})) = \sum_{x,y} r_x s_{x,y} \delta_{x,y}.$$

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x\in F} r_x \delta_x \mid r_x \in \mathbb{R}_+ \ \& \ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

$\mathcal{V}_s$ is a submonad of $\mathcal{V}_w$

In particular, there is an embedding $\eta_{\mathcal{V}_s X} \colon \mathcal{V}_s X \hookrightarrow \mathcal{V}_w X$, which is sober.

If K is a $\mathtt{K}$-category, it follows that the K-completion $\mathcal{V}_{\mathsf{K}} X \stackrel{\text{def}}{=} \mathsf{K}_c(\mathcal{V}_s X)$ is a sub-dcpo of $\mathcal{V}_w X$.

Theorem For each $\mathtt{K}$-category K, $\mathcal{V}_{\mathsf{K}}$ is a monad on TOP_0.

Proof: $\mathcal{V}_{\mathsf{K}} X$ is a subcone of $\mathcal{V}_w X$, so it is a locally linear topological cone, and each continuous linear map $f \colon \mathcal{V}_s X \to \mathcal{V}_s Y$ satisfies $\eta_{\mathcal{V}_{\mathsf{K}}} \circ f(\mathcal{V}_s X) \subseteq \mathcal{V}_{\mathsf{K}} X$, so $\mathsf{K}_c(f) \colon \mathcal{V}_{\mathsf{K}} X \to \mathcal{V}_{\mathsf{K}} Y$. The linearity of $\mathsf{K}_c(f)$ follows from the density of $\mathcal{V}_s X$ in $\mathcal{V}_{\mathsf{K}} X$ and the continuity of addition on $\mathcal{V}_{\mathsf{K}} Y$. This shows $\mathcal{V}_{\mathsf{K}}$ is an endofunctor on TOP_0.

The unit is the point valuation $x \mapsto \delta_x$, and if $f \colon X \to \mathcal{V}_{\mathsf{K}} Y$ is continuous, then

$$\mathsf{K}_c(f) = \mu \mapsto (U \mapsto \int_X f(x)(U) d\mu) \colon \mathcal{V}_{\mathsf{K}} X \to \mathcal{V}_{\mathsf{K}} Y$$

is well-defined and continuous. This shows $\mathcal{V}_K$ defines a submonad of $\mathcal{V}_w$.

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathbb{R}_+ \ \& \ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

$\mathcal{V}_s$ is a submonad of $\mathcal{V}_w$

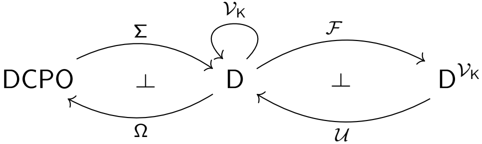
In particular, there is an embedding $\eta_{\mathcal{V}_s X} \colon \mathcal{V}_s X \hookrightarrow \mathcal{V}_w X$, which is sober.

If K is a $\mathtt{K}$-category, it follows that the K-completion $\mathcal{V}_{\mathsf{K}} X \stackrel{\text{def}}{=} \mathsf{K}_c(\mathcal{V}_s X)$ is a sub-dcpo of $\mathcal{V}_w X$.

Theorem For each $\mathtt{K}$-category K, $\mathcal{V}_{\mathsf{K}}$ is a monad on TOP_0.

Theorem For each $\mathtt{K}$-category $\mathsf{K} \subseteq \mathsf{D}$, $\mathcal{V}_{\mathsf{K},\leq} \stackrel{\text{def}}{=} \Omega \circ \mathcal{V}_K \circ \Sigma$ is a monad on DCPO.

Proof: $\mathcal{V}_{\mathsf{K}}$ is a monad on TOP_0, and since K is a full subcategory of D, it follows that $\mathcal{V}_{\mathsf{K}}$ restricts to a monad on D. Writing $\mathcal{V}_{\mathsf{K}} = \mathcal{U} \circ \mathcal{F}$, we have $\Omega \circ \mathcal{V}_{\mathsf{K}} \circ \Sigma = (\Omega \circ \mathcal{U}) \circ (\mathcal{F} \circ \Sigma)$, and adjoints compose:



It's straightforward to see that the unit and multiplication of $\mathcal{V}_{\mathsf{K}}$ are transported to DCPO.

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathbb{R}_+ \ \&\ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

$\mathcal{V}_s$ is a submonad of $\mathcal{V}_w$

Theorem For each $\mathbb{K}$-category K $\subseteq$ D, $\mathcal{V}_{\mathrm{K},\leq} \stackrel{\mathrm{def}}{=} \Omega \circ \mathcal{V}_K \circ \Sigma$ is a monad on DCPO.

In particular, this applies to the D-completion $\mathcal{V}_{\mathrm{D}} X$.

Theorem The monad $\mathcal{M} \stackrel{\mathrm{def}}{=} \mathcal{V}_{\mathrm{D},\leq 1}$ is commutative on DCPO.

Proof: If $\mu = \sum_{x \in F} r_x \cdot \delta_x \in \mathcal{M}X$ and $\nu \in \mathcal{M}Y$, then

$$\int_X \int_Y \chi_U(x,y) d\mu d\nu = \sum_{x \in F} r_x \cdot \int_Y \chi_U(x,y) d\nu = \int_Y \sum_{x \in F} r_x \cdot \chi_U(x,y) d\nu = \int_Y \int_X \chi_U(x,y) d\mu d\nu,$$

and since $\mathcal{V}_s X$ is dense in $\mathcal{M}X$ and integration is continuous, the equation holds for all $\mu \in \mathcal{M}X$.

K-completions of Simple Valuations

Simple valuations over X: $\mathcal{V}_s X = \{\sum_{x\in F} r_x\delta_x \mid r_x \in \mathbb{R}_+ \ \&\ F \subseteq X \text{ finite}\} \subseteq \mathcal{V}_w X$.

$\mathcal{V}_s$ is a submonad of $\mathcal{V}_w$ In particular, this applies to the D-completion $\mathcal{V}_{\mathrm{D}}X$.

Theorem The monad $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{V}_{\mathrm{D},\leq 1}$ is commutative on DCPO.

This also applies to any full $\mathtt{K}$-subcategory K of D; in fact, for such a K, we have

$\mathcal{V}_{s,\leq 1} \subseteq \mathcal{M} \subseteq \mathcal{V}_{\mathrm{K},\leq} \subseteq \mathcal{P}$ (but $\mathcal{V}_{s,\leq 1}$ isn't a subcategory of DCPO), where $\mathcal{P} = \mathcal{V}_{\mathrm{SOB},\leq 1}$.

Heckmann showed that $\mathcal{P}$ consists of *point continuous valuations*; hence the name.

Another example we know of is the full $\mathtt{K}$-subcategory W of D consisting of *well-filtered spaces*. So the commutative monads we know so far are:

$\mathcal{V}_{s,\leq 1} \subseteq \mathcal{M} \subseteq \mathcal{W} \subseteq \mathcal{P} \subseteq \mathcal{Z}$

The last – $\mathcal{Z}$ – denotes the category of *central valuations:* those for which integration satisfies the Fubini equation with any valuation in $\mathcal{V}$ for the other component. This category exists by abstract reasoning, and doesn't rely on $\mathcal{V}_s$ being a dense subcategory relative to some completion operation. It also is the only one we know that contains the pushforward of Lebesgue measure by a lower semicontinuous map to a DCPO.

Some References

- Completing Simple Valuations in K-categories, X. Jia and M. Mislove, Topology and Its Applications, *in press*, Available online at https://arxiv.org/abs/2002.01865v2
- Commutative Monads for Probabilistic Programming Languages, X. Jia, B, Lindenhovius, M. Mislove and V. Zamdzhiev, in: LICS '21: Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science (2021)
  DOI: https://doi.org/10.1109/LICS52264.2021.9470611
  Also available at https://arxiv.org/pdf/2102.00510.pdf
- The Central Valuations Monad (Early Ideas), X. Jia, M. Mislove and V. Zamdzhiev, 9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021), Leibniz International Proceedings in Informatics (LIPIcs),
  DOI: 10.4230/LIPIcs.CALCO.2021.18
  Also available at: https://arxiv.org/abs/2111.10873