On k-ranks of topological spaces

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d-spaces, *d*-completions and d-ranks

Well-filtered spaces, well-filterifications and *wf*-ranks

k-well-filtered spaces, *k*-well-filterifications and *k*-ranks

Applications

d-spaces, d-completions and d-ranks

A T_0 space X is called a *d-space* (i.e., *monotone convergence space*) if

- it is a *dcpo* with the specialization order;
- $\mathcal{O}(X) \subseteq \sigma(X)$.

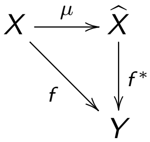
A description of d-spaces

A T_0 space X is a d-space iff for each *directed set* D with respect to the specialization order, there exists $x \in X$ such that $\mathrm{cl}(D) = \mathrm{cl}(\{x\})$.

For example, a *dcpo* with the Scott topology is a d-space.

d-spaces, *d*-completions and d-ranks

A ***d-completion*** of X is a pair $\langle \widehat{X}, \mu \rangle$ comprising a d-space $\widehat{X}$ and a continuous mapping $\mu \colon X \longrightarrow \widehat{X}$ satisfying that for any continuous mapping $f \colon X \longrightarrow Y$ to a d-space, there exists a unique continuous mapping $f^* \colon \widehat{X} \longrightarrow Y$ such that $f^* \circ \mu = f$, that is, the following diagram commutes.



We denote it by $\mathrm{H}_d(X)$ if the d-completion of X exists.

d-spaces, d-completions and d-ranks

As shown by Ershov,

- the d-completion of X, can be obtained by *adding the closures of directed sets*, as points, to X (and then repeating this process).
- he called it *d-rank* which is an ordinal that measures how many steps from a T_0 space to a d-space.
- for any given ordinal α, he proven that there exists a T_0 space X whose d-rank equals to α.

We denote the d-rank of X by $\mathrm{rank}_d(X)$.

Y. Ershov, On d-spaces, Theoretical Computer Science, 224 (1999) 59-72.

Y. Ershov, The d-rank of a topological space, Algebra and Logic, 56 (2017) 98-107.

d-spaces, d-completions and d-ranks

The topological construction in Ershov's paper is called *fibred sum*.

For topological spaces X and Y_x, $x \in X$, let

$$Z = \bigcup_{x \in X} Y_x \times \{x\},$$

$$\tau = \{U \subseteq Z \mid (U)_x \in \tau(Y_x) \text{ for any } x \in X \text{ and } (U)_X \in \tau(X)\}$$

where $U_x = \{y \in Y_x \mid (y, x) \in U\}$ for any $x \in X$ and $(U)_X = \{x \in X \mid (U)_x \neq \emptyset\}$.

The space (Z, τ) is also denoted by $\sum_X Y_x$.

d-spaces, d-completions and d-ranks

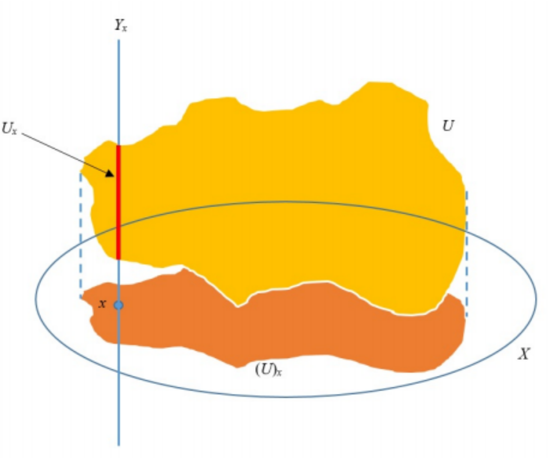


Figure: Fibred sum of $(Y_x)_{x\in X}$: $\sum\limits_{X} Y_x$

d-spaces, *d*-completions and d-ranks

The specialization order on $Z = \sum_X Y_x$ is described as follows.

Lemma (Ershov)

Let X be a T_0 space, Y_x an irreducible T_0 space for any $x \in X$, and $Z = \sum_X Y_x$. For all $(y_0, x_0), (y_1, x_1) \in Z$, we have $(y_0, x_0) \leq (y_1, x_1)$ if and only if the following two alternatives hold:

(1) $x_0 = x_1$ and $y_0 \leq_{Y_{x_0}} y_1$;

(2) $x_0 <_X x_1$ and $y_1 = \top_{x_1}$ is the greatest element in Y_{x_1}.

d-spaces, *d*-completions and d-ranks

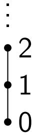
For $\alpha = 0$,

$Z = (\{\top\}, \{\varnothing, \{\top\}\})$

$\mathrm{H}_d(Z)$

$\mathrm{rank}_{\mathrm{d}}(Z) = 0$.

For $\alpha = 1$,



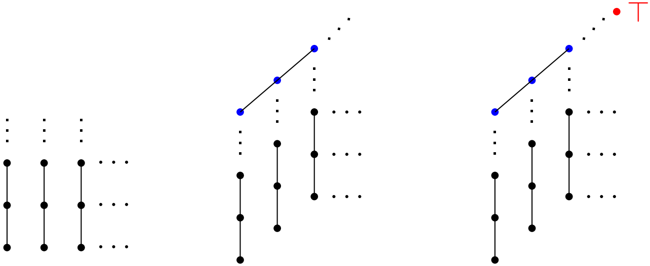
$Z = \mathbb{N}$

$\mathrm{H}_d(Z)$

$\mathrm{rank}_{\mathrm{d}}(Z) = 1$.

d-spaces, d-completions and d-ranks

For $\alpha = 2$,



$Z = \sum_{\mathbb{N}} \mathbb{N}$ step 1 step 2: $\mathrm{H}_d(Z)$

$\mathrm{rank_d(Z)} = 2$.

For $\alpha = 3$, $Z = \sum_{\mathbb{N}}(\sum_{\mathbb{N}} \mathbb{N}) \cdots \cdots$

d-spaces, d-completions and d-ranks

By induction on α,

Theorem (Ershov)

For any ordinal α, there exists an irreducible T_0 space X whose d-rank is equal to α.

Well-filtered spaces, well-filterifications and *wf*-ranks

A T_0 space X is called *well-filtered* if for any open subset U and any filtered family $\mathcal{K}$ in $Q(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

Definition (Shen, Xi, Xu and Zhao)

Let X be a T_0 space. A nonempty subset A is said to have *Rudin property* (i.e., KF property), if there exists $\mathcal{K} \subseteq_{filt} Q(X)$ such that A is a minimal closed set that intersects all members of $\mathcal{K}$.

We call such a set *Rudin* or *Rudin set*. The set of all Rudin sets of X is denoted by $\overline{KF}(X)$.

C. Shen, X. Xi, X. Xu, D. Zhao, On well-filtered reflections of T_0 spaces, Topology and its Applications, 267 (2019) 106869.

Well-filtered spaces, well-filterifications and wf-ranks

It turns out that

directed sets $\Rightarrow$ Rudin sets

A description of well-filtered spaces (Shen, Xi, Xu and Zhao)

Let X be a T_0 space. TFAE:

(1) X is a well-filtered space.

(2) for each *Rudin set* A, there exists $x \in X$ such that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.

Therefore,

well-filtered spaces $\Rightarrow$ d-spaces

Well-filtered spaces, well-filterifications and wf-ranks

A *well-filterification* or *well-filtered reflection* of X is a pair $\langle \widehat{X}, \mu \rangle$ comprising a well-filtered space $\widehat{X}$ and a continuous mapping $\mu\colon X \longrightarrow \widehat{X}$ satisfying that for any continuous mapping $f\colon X \longrightarrow Y$ to a well-filtered space, there exists a unique continuous mapping $f^*\colon \widehat{X} \longrightarrow Y$ such that $f^* \circ \mu = f$, that is, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ \mu\ } & \widehat{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^*} \\ & & Y \end{array}$$

We denote it by $\mathrm{H}_{wf}(X)$ if the well-filterification of X exists.

Well-filtered spaces, well-filterifications and wf-ranks

Recently, various researchers shown that the category of all well-filtered spaces is reflective in the category of all T_0 spaces.

- Wu, Xi, Xu and Zhao proven it by *Keimel-Lawson category*.
- Shen, Xi, Xu and Zhao obtained it by *adding the closures of Rudin sets*, as points, to a T_0 space (and then repeating this process).

G. Wu, X. Xi, X. Xu, D. Zhao, Existence of well-filterifications of T_0 topological spaces, Topology and its Applications, 270 (2019), 107044.
C. Shen, X. Xi, X. Xu, D. Zhao, On well-filtered reflections of T_0 spaces, Topology and its Applications, 267 (2019) 106869.

Well-filtered spaces, well-filterifications and *wf*-ranks

_wf_-rank (Liu, Li and Wu)

It is an ordinal that measures how far a T_0 space is from being a well-filtered space.

Theorem (Liu, Li and Ho)

For any given ordinal α, there exists a T_0 space whose *wf*-rank equals to α.

We denote the *wf*-rank of X by $\mathrm{rank}_{wf}(X)$.

B. Liu, Q. Li, G. Wu, Well-filterifications of topological spaces, Topology and its Applications, 279 (2020) 107245.
B. Liu, Q. Li, W. Ho, The *wf*-rank of topological spaces, accepted for publication.

Well-filtered spaces, well-filterifications and *wf*-ranks

For $\alpha = 0$,

$\top$

$Z = (\{\top\}, \{\varnothing, \{\top\}\})$

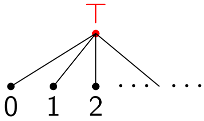
$\top$

$\mathrm{H}_{wf}(Z)$

$\mathrm{rank}_{\mathrm{wf}}(Z) = 0.$

For $\alpha = 1$,

$Z = X$

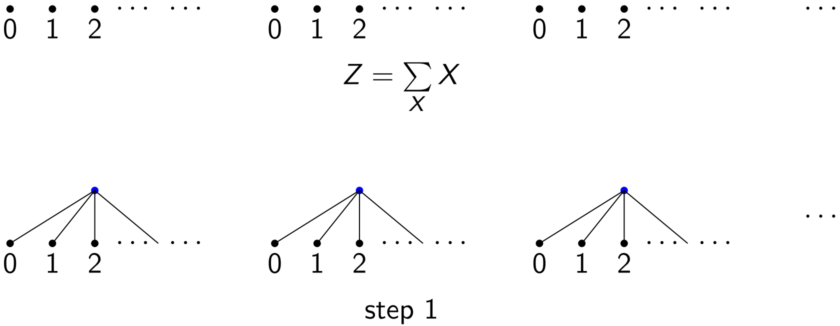


$\mathrm{H}_{wf}(Z)$

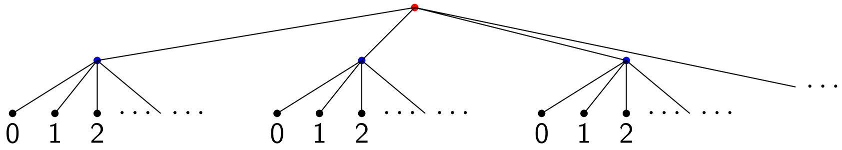
$\mathrm{rank}_{\mathrm{wf}}(Z) = 1.$

Well-filtered spaces, well-filterifications and *wf*-ranks

For $\alpha = 2$,



Well-filtered spaces, well-filterifications and *wf*-ranks



step 2: $\mathrm{H}_{wf}(Z)$

$\mathrm{rank}_{\mathrm{wf}}(Z) = 2.$

For $\alpha = 3$, $Z = \sum_{X}(\sum_{X} X) \cdots \cdots$

Well-filtered spaces, well-filterifications and wf-ranks

By induction on α,

Theorem (Liu, Li and Ho)

For any ordinal α, there exists an irreducible T_0 space X whose wf-rank is equal to α.

Questions

Question 1

Whether there is a uniform approach to d-spaces and well-filtered spaces and develop a general framework for dealing with all these spaces.

Answer: Yes! We call it *k-well-filtered space*.

Question 2

For any given ordinal α as to whether there exists a T_0 space X whose k-rank equals to α.

Answer: Yes!

k-well-filtered spaces, k-well-filterifications and k-ranks

Definition

$Q_k : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ is called a *C-subset system* if $S^u(X) \subseteq Q_k(X) \subseteq Q(X)$ for all $X \in ob(\mathbf{Top_0})$, where $S^u(X) = \{\uparrow x \mid x \in X\}$.

- A nonempty subset A is said to have *k-Rudin property*, if there exists $\mathcal{K} \subseteq_{filt} Q_k(X)$ such that A is a minimal closed set that intersects all members of $\mathcal{K}$. We call such a set *k-Rudin* or *k-Rudin set*. We denote it by $\mathrm{K}^R(X)$.
- $\overline{D}(X) \subseteq \mathrm{K}^R(X) \subseteq \overline{KF}(X)$, where $\overline{D}(X) = \{A \subseteq X\}$ that there exists a directed subset D such that $\overline{A} = \overline{D}$.

k-well-filtered spaces, k-well-filterifications and k-ranks

Definition

A C-subset system is called a *K-subset system* provided that for T_0 spaces X, Y and any continuous mapping $f : X \longrightarrow Y$, $f(A) \in \mathrm{K}^R(Y)$ for all $A \in \mathrm{K}^R(X)$.

Definition

Let $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be a K-subset system and X a T_0 space. X is called *k-well-filtered* if for any open set U and $\mathcal{K} \subseteq_{filt} Q_k(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

k-well-filtered spaces, k-well-filterifications and k-ranks

For a K-subset system $Q_k : \mathbf{Top_0} \longrightarrow \mathbf{Set}$,

A description of k-well-filtered spaces

Let X be a T_0 space. TFAE:

(1) X is a k-well-filtered space.

(2) for each *k-Rudin set* A, there exists $x \in X$ such that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.

Therefore,

well-filtered spaces $\Rightarrow$ k-well-filtered spaces $\Rightarrow$ d-spaces

We denote it by $\mathrm{H}_k(X)$ if the k-well-filterification of X exists.

k-well-filtered spaces, k-well-filterifications and k-ranks

We get that:

- the k-well-filterification of X, can be obtained by *the equivalent classes of k-Rudin subsets* (i.e., adding the closure of k-Rudin sets, as points, to X and repeat this process until it stabilizes).
- for k-ranks, given an ordinal α, the structure of $Q_k(X)$ for any T_0 space X is indeterminate, where $S^u(X) \subseteq Q_k(X) \subseteq Q(X)$. So the question is:

What conditions should the T_0 space satisfy such that its k-rank equals to α?

k-well-filtered spaces, k-well-filterifications and k-ranks

For $\alpha = 0$,

$$Z = (\{\top\}, \{\emptyset, \{\top\}\})$$

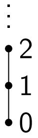
$$\mathrm{H}_{wf}(Z) = \mathrm{H}_{d}(Z)$$

- $\mathrm{rank}_{\mathrm{wf}}(Z) = \mathrm{rank}_{\mathrm{d}}(Z) = 0$.

Thus, $\mathrm{H}_{wf}(Z) = \mathrm{H}_{k}(Z) = \mathrm{H}_{d}(Z)$ and $\mathrm{rank}_{\mathrm{k}}(Z) = 0$.

k-well-filtered spaces, k-well-filterifications and k-ranks

For $\alpha = 1$,



$Z = \mathbb{N}$

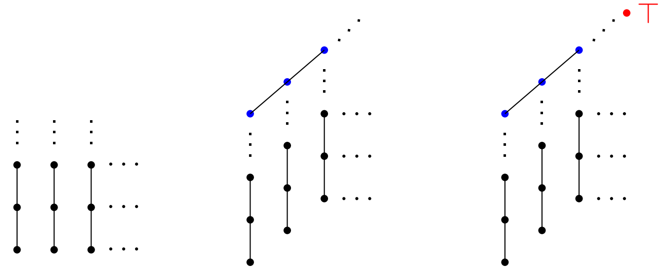
$\mathrm{H}_{wf}(Z) = \mathrm{H}_{d}(Z)$

- $\overline{D}(Z) = \overline{KF}(Z)$.
- $\mathrm{rank}_{\mathrm{wf}}(Z) = \mathrm{rank}_{\mathrm{d}}(Z) = 1$.

Thus, $\mathrm{H}_{wf}(Z) = \mathrm{H}_{k}(Z) = \mathrm{H}_{d}(Z)$ and $\mathrm{rank}_{\mathrm{k}}(Z) = 1$.

k-well-filtered spaces, k-well-filterifications and k-ranks

For $\alpha = 2$,



$Z = \sum_{\mathbb{N}} \mathbb{N}$ step 1:$K(Z)$ step 2: $\mathrm{H}_{wf}(Z) = \mathrm{H}_d(Z)$

- For Z, $\overline{D}(Z) = \overline{KF}(Z)$.
- For $K(Z)$, $\overline{D}(K(Z)) = \overline{KF}(K(Z))$.
- $\mathrm{rank_{wf}}(Z) = \mathrm{rank_d}(Z) = 2$

Thus, $\mathrm{H}_{wf}(Z) = \mathrm{H}_k(Z) = \mathrm{H}_d(Z)$ and $\mathrm{rank_k}(Z) = 2$.

k-well-filtered spaces, k-well-filterifications and k-ranks

For a given ordinal α, the T_0 space X satisfies:

(1) $\mathrm{rank}_{\mathrm{wf}}(X) = \mathrm{rank}_{\mathrm{d}}(X) = \alpha$.

(2) $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$ for $0 \leq m < \alpha$, where $K_m(X)$ denotes the topological space of X after m steps.

Theorem

For any ordinal α, there exists an irreducible T_0 space X whose k-rank is equal to α.

Applications

Let $Q_s(X)$ denote the set of all nonempty strongly compact saturated subsets of a topological space X.

Definition (Heckmann)

A T_0 space X is called *$\mathcal{U}_S$-admitting* if for any open subset U and $\mathcal{K} \subseteq_{filt} Q_s(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

$$\text{well-filtered spaces} \Rightarrow \mathcal{U}_S\text{-admitting spaces} \Rightarrow d\text{-spaces}$$

From our results, $\mathcal{U}_S$-admitting spaces is a special case of k-well-filtered spaces.

R. Heckmann, An upper power domain construction in terms of strongly compact sets, Lecture Notes in Comput. Sci., 598 (1992) 272-293.

Thank you !