Weakly meet s_Z-continuity and δ_Z-continuity

Singapore

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Contents

Background

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Definition

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For any $x, y \in P$, we say that x is way-below y, in symbols $x \ll y$, iff for all directed subset D of P for which $\sup D$ exists, the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$.

Definition

P is called continuous if for all $x \in P$, the set $\twoheaddownarrow x = \{u \in P : u \ll x\}$ is directed and $x = \sup \twoheaddownarrow x$.

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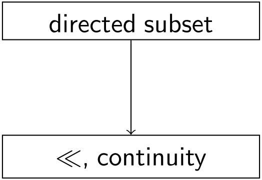

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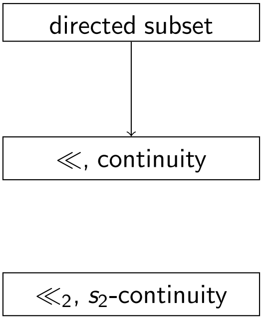

Background

Let P be a poset.

Definition [Erné]

For any $x, y \in P$, we say that $x \ll_2 y$, iff for all directed subset D of P , the relation $y \in D^\delta$ always implies the existence of a $d \in D$ with $x \leq d$.

Definition[Erné]

P is called s_2-continuous if for all $x \in P$, the set $\Downarrow_2 x = \{u \in P : u \ll_2 x\}$ is directed and $x = \sup \Downarrow_2 x$.

M. Erné, **Scott convergence and Scott topology in partially ordered sets II,** *Continuous lattices. Springer, Berlin, Heidelberg*, (1981).

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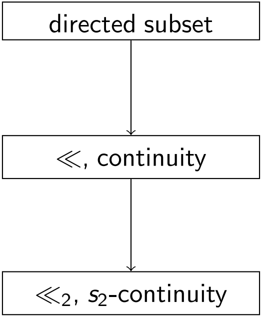

Background

Definition [Wright]

A subset system on **POSET** is a function Z which assigns to each poset, a set $Z(P)$ of subsets of P such that

(1) $\{x\} \in Z(P)$ for any $x \in P$;

(2) if $f : P \to Q$ in **POSET** and $S \in Z(P)$, then $f(S) \in Z(Q)$.

J.B. Wright, E.G. Wagner, J.W. Thatcher, **A uniform approach to inductive posets and inductive closure,** *Theoretical Computer Science*, 7 (1978).

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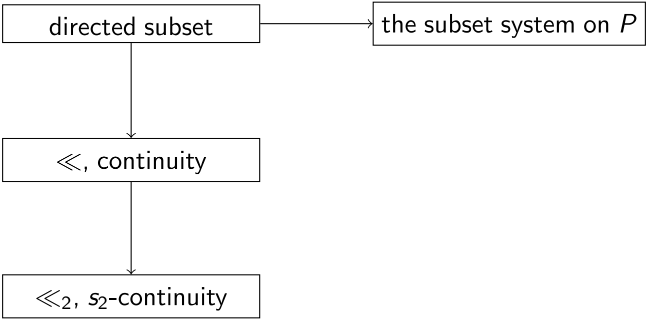

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Let P be a poset.

Definition [Baranga]

For any $x, y \in P$, we say $x \ll_Z y$ if for any $S \in Z(P)$ such that $\sup S$ exists, the relation $y \leq \sup S$ implies the existence of an element $s \in S$ with $x \leq s$.

Definition [Baranga]

P is called *Z-continuous* if for every $p \in P$, the set $\downarrow_Z p = \{x \in P, x \ll_Z p\} \in Z(P)$ and $p = \sup \downarrow_Z p$.

A. Baranga, **Z-continuous posets,** *Journal of Pure and Applied Algebra*, 30 (1983): 219-226.

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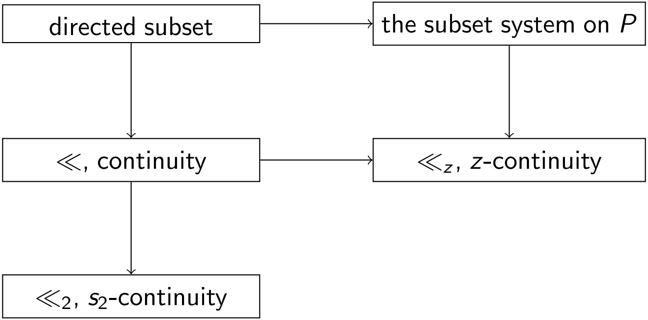

Background

Let P be a poset.

Definition [Ruan, Xu]

For any $A, B \subseteq P$, A is called Z-way below B, denoted by $A \ll_Z B$, if for any $S \in Z(P)$, $S^\delta \cap \uparrow B \neq \emptyset$ implies $S \cap \uparrow A \neq \emptyset$.

Specially, x is called Z-way below y, if for any $S \in Z(P)$, $y \in S^\delta$ implies that $x \in \downarrow S$, and we write $\Downarrow_z x = \{y \in P, y \ll_z x\}$.

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Definition [Ruan, Xu]

P is called a *s_Z-continuous* poset, if for any $x \in P$, $\Downarrow_Z x \in I_Z(P) = \{\downarrow S : S \in Z(P)\}$ and $x \in (\Downarrow_z x)^\delta$.

X. Ruan, X. Xu, **s_Z-quasicontinuous posets and meet s_Z-continuous posets,** *Topology and its applications*, 230 (2017): 295-307.

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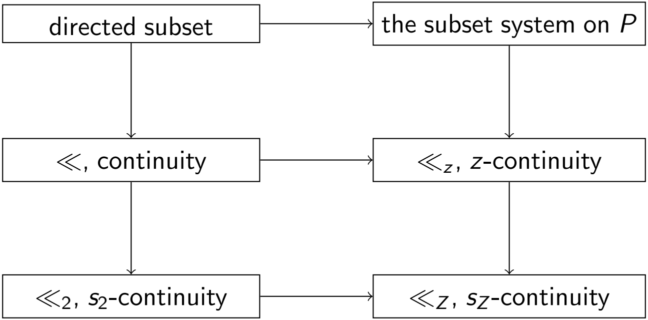

Motivation

Let P be a poset.

Definition[Ruan, Xu]

P is called meet s_Z-continuous if for all $x \in P$ and all $D \in Z(P)$ with $x \in D^\delta$, $x \in cl_{\sigma_{Z(P)}}(\downarrow x \cap \downarrow D)$.

Where $\sigma_Z(P)$ is the Z-Scott topology on P generated by the subbasic open subsets

$$\sigma^Z(P) = \{U \subseteq P : \text{for all } S \in Z(P), S^\delta \cap U \neq \emptyset \Rightarrow S \cap U \neq \emptyset\}.$$

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Definition[Ruan, Xu]

P is called an s_Z-quasicontinuous poset, if for all $p \in P$, $\{\uparrow F : F \in \omega_Z(P)\} \in Z(\mathbf{Fin}P)$ and $\uparrow p = \bigcap\{\uparrow F : F \in \omega_Z(P)\}$.

Motivation

Let P be a poset.

Theorem[Ruan, Xu]

If Z is a Rudin subset system which possesses the finite family union property and M property and $\sigma^Z(P) = \sigma_Z(P)$, $\twoheaddownarrow_Z x \in I_Z(P)$ for all $x \in P$. Then the following conditions are equivalent:

(1) P is s_Z-continuous.

(2) P is meet s_Z-continuous and s_Z-quasicontinuous.

Weak meet s_Z-continuous posets

Let P be a poset.

Definition[Ruan, Xu]

P is called weakly meet s_Z-continuous if for all $x \in P$ and all $D \in Z(P)$ with $x \in D^\delta$, we have $x \in cl_{\sigma^Z(P)}(\downarrow x \cap \downarrow D)$.

Proposition

Let P be a weakly meet s_Z-continuous poset. If for any $x, y \in P$, $x \not\leq y$, there are $U \in \sigma^Z(P)$, $V \in \omega(P)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$, then P is weak s_Z-continuous.

Weak meet s_Z-continuous posets

Lemma[Ruan, Xu]

If P is an s_Z-continuous poset, then P is weakly meet s_Z-continuous.

Lemma[Xu]

Let Z be a Rudin subset system which possesses M property. If P is an s_Z-continuous poset, then P is s_Z-quasicontinuous.

X. Xu, M. Luo, Y. Huang, **Quasi Z-continuous domains and Z-meet continuous domains,** *Acta Mathematica Sinica. Chinese Series*, 48 (2005): 221-234.

Weak meet s_Z-continuous posets

Theorem

Let P be a poset and Z a Rudin subset system which possesses the finite family union property and M property. If $\twoheaddownarrow_Z x \in I_Z(P)$ for each $x \in P$, then the following conditions are equivalent:

(1) P is s_Z-continuous;

(2) P is weakly meet s_Z-continuous and s_Z-quasicontinuous;

(3) P is weakly meet s_Z-continuous, and for any $x \nleq y$ in P, there are $U \in \sigma^Z(P)$, $V \in \omega(P)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

A characterization of the s_Z-continuity

$\Gamma_Z(P)$ denotes the family consists of all closed subsets of the Z-Scott topology on P, generated by the subbasic closed subsets

$$\Gamma^Z(P) = \{A \subseteq P : \text{for all } S \in Z(P), S \subseteq A \Rightarrow S^\delta \subseteq A\}.$$

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Definition

Let P be a poset. The Z-Scott topology on P is called lower hereditary if for each closed subbasis A of P, the Z-Scott topology of poset A is precisely generated by the subbasic closed subsets of the form $B \cap A$, where $B \in \Gamma^Z(P)$, that is, $\Gamma^Z(A) = \{B \cap A : B \in \Gamma^Z(P)\}$.

A characterization of the s_Z-continuity

Lemma

Let P be a poset and Z a subset hereditary subset system. Consider the following conditions:

(1) The Z-Scott topology on P is lower hereditary.

(2) For any $x \in P$ and $D \in Z(\downarrow x)$, $D^\delta \mid_x = D^\delta$.

(3) For any $D \in Z(P)$, D^u is filtered.

Then we have (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2).

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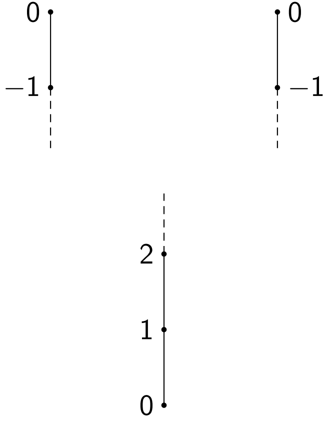
Corollary

Every zcpo P has a lower hereditary Z-Scott topology.

A characterization of the s_Z-continuity

Note: The above three conditions are not equivalent.

Example:



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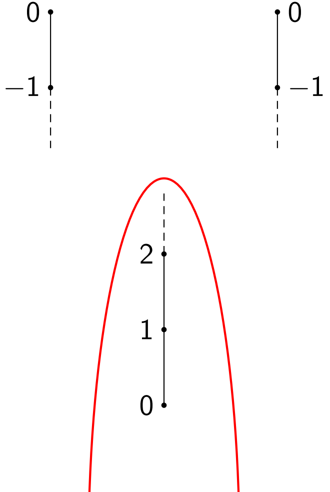

A characterization of the s_Z-continuity

Theorem

Let P be a poset with a lower hereditary Z-Scott topology and Z be subset hereditary. Then the following conditions are equivalent:

(1) P is s_Z-continuous and $\Downarrow_Z^x y \in Z(\downarrow x)$ for any $x \in P$ and $y \in \downarrow x$;

(2) $\downarrow x$ is s_Z-continuous and $\Downarrow_Z x \in Z(P)$ for any $x \in P$.

Background

Let $\mathbf{POS_d}$ denote the category which consists of posets and Scott continuous mappings and $\Gamma(P)$ be the set of all Scott closed subsets of the poset P.

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Problem

Which subcategories of $\mathbf{POS_d}$ are Γ-faithful?

Background

Theorem [Ho, Zhao]

Let P and Q be complete semilattices. The following statements are equivalent:

(i) $P \cong Q$.

(ii) $\Gamma(P) \cong \Gamma(Q)$.

W. Ho, D. Zhao, **Lattices of Scott-closed sets,** *Commentationes Mathematicae Universitatis Carolinae*, 50 (2009).

Background

Definition [Ho, Zhao]

Let P be a poset and $x, y \in P$.

(1) We say that x is beneath y, denoted by $x \prec y$, if for every nonempty Scott closed set $C \subseteq P$ for which $\sup C$ exists, $y \leq \sup C$ implies that $x \in C$.

(2) P is said to be C-prealgebraic if for each $a \in P$,

$$a = \sup\{x \in k(P), x \leq a\},$$

where $k(P) = \{x \in P, x \prec x\}$.

Background

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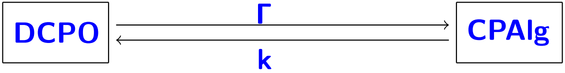

POSET$_\delta$

Definition

Let P, Q be two posets. A mapping $f : P \to Q$ is called σ^Z-continuous if for any $A \in \Gamma^Z(Q)$, $f^{-1}(A) \in \Gamma^Z(P)$.

$\mathbf{POSET}_\delta$

Definition

Let P, Q be two posets. A mapping $f : P \to Q$ is called σ^Z-continuous if for any $A \in \Gamma^Z(Q)$, $f^{-1}(A) \in \Gamma^Z(P)$.

Let $\mathbf{POSET}_\delta$ denote the category whose objects are all posets and morphisms are σ^Z-continuous mappings.

δ_ZPALG

Definition

Let P be a poset and $x, y \in P$.

(1) x is called Z-beneath y, denoted by $x \prec_Z y$, if for any $A \in \Gamma^Z(P)$ with $y \in A^\delta$, $x \in A$.

(2) P is called δ_Z-prealgebraic if for each $x \in P$,

$$x \in \{y \in k_Z(P) : y \leq x\}^\delta,$$

where $k_Z(P) = \{a \in P : a \ is \ Z{-}compact, \ i.e., \ a \prec_Z a\}$.

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Call a δ_Z-prealgebraic complete lattice δ_Z-prealgebraic lattice for short. Clearly, $\Gamma^Z(P)$ is a δ_Z-prealgebraic lattice for any poset P.

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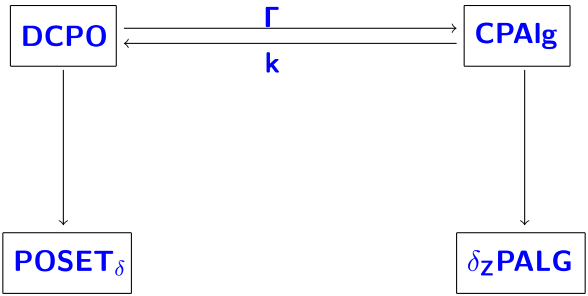
Denote by δ_Z**PALG** the category which has all δ_Z-prealgebraic lattices as objects and maps that have an upper adjoint and preserve the relation $\prec_Z$ as morphisms.

Motivation



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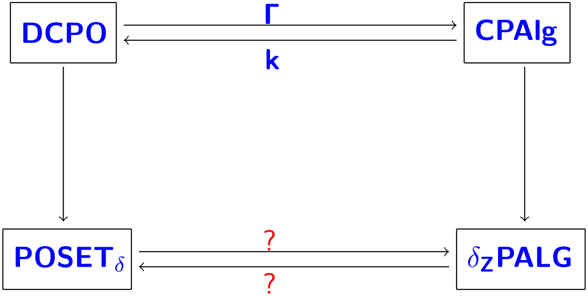

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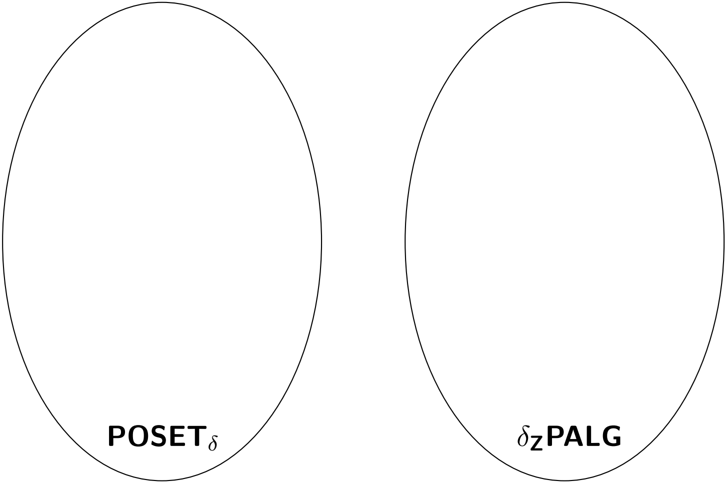

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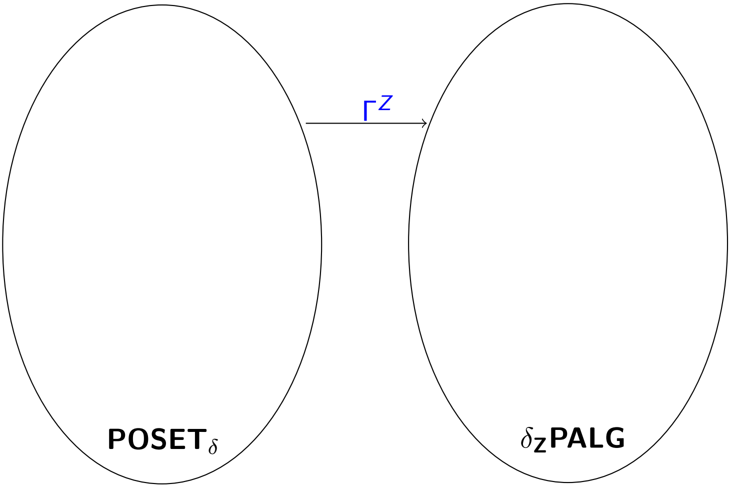

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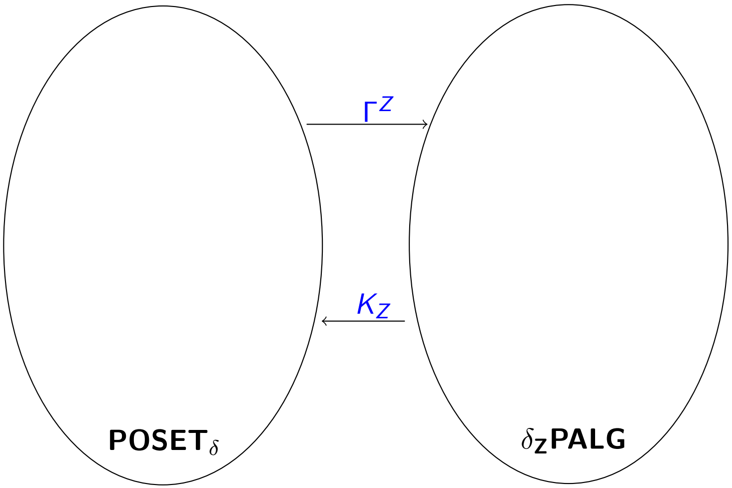

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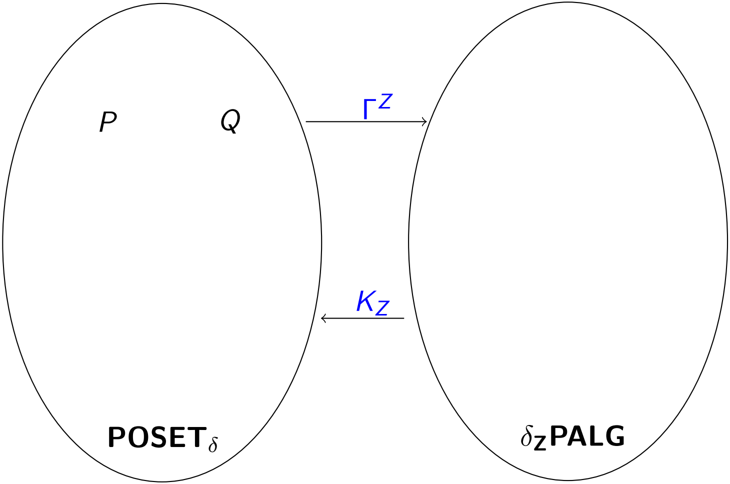

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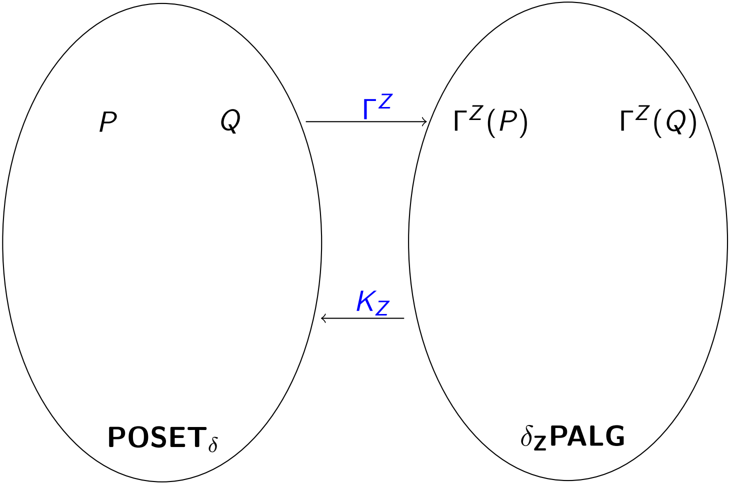

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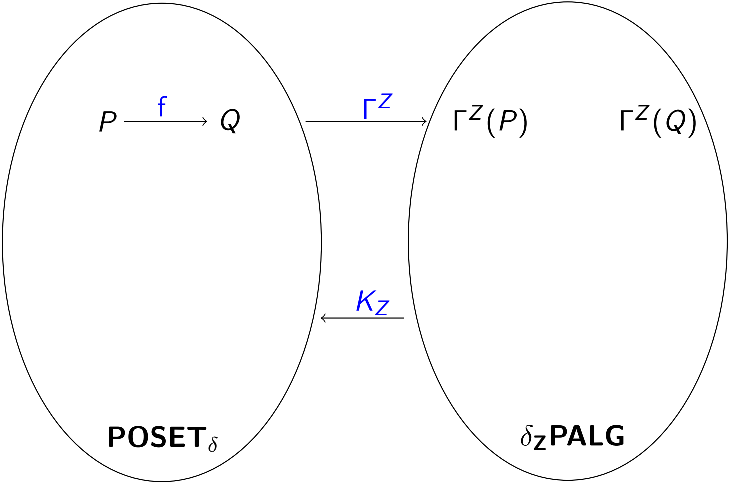

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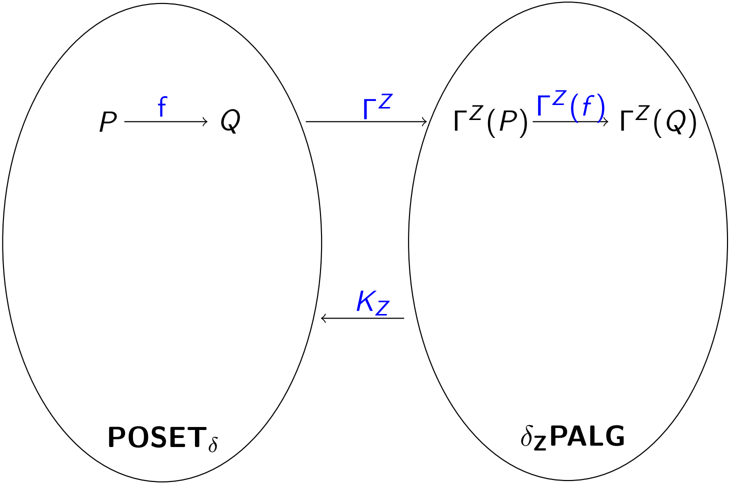

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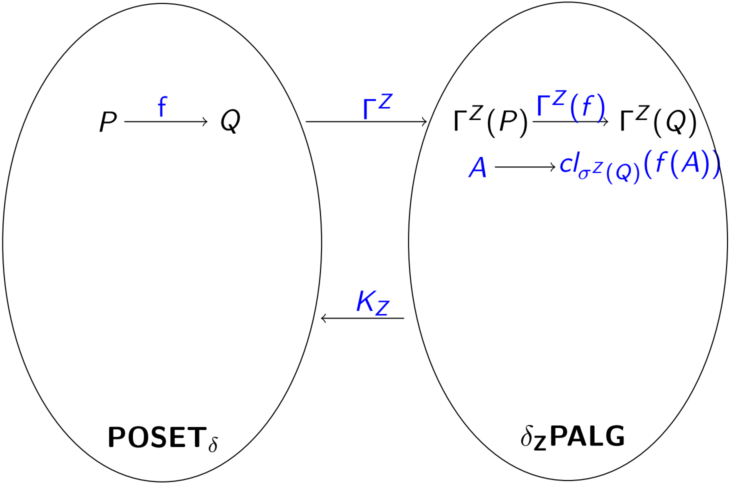

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_{\mathbf{Z}}\mathbf{PALG}$

Lemma

Let (g, d) be a Galois connection between two posets P and Q, where $g : P \to Q$, $d : Q \to P$. If for any $A \in \Gamma^Z(P)$, $g(A^\delta) \subseteq g(A)^\delta$, then d preserves the relation $\prec_Z$.

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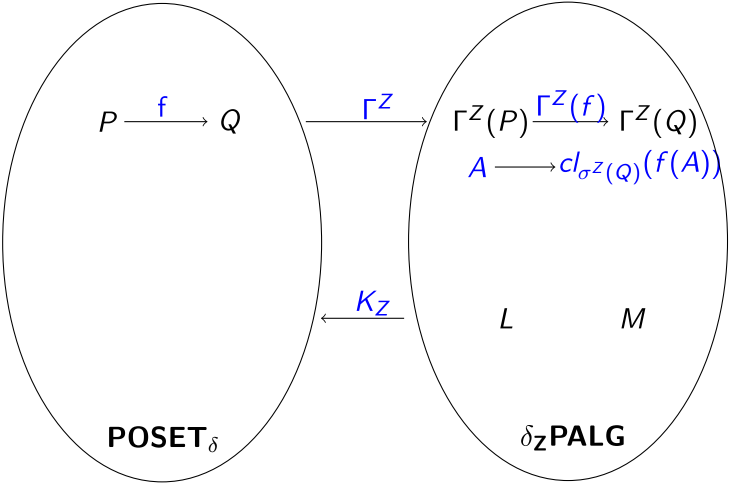
$\Gamma^Z(f)$ has an upper adjoint

$$h : \Gamma^Z(Q) \to \Gamma^Z(P) : C \longmapsto f^{-1}(C),$$

and $h(C^\delta) \subseteq h(C)^\delta$ for each $C \in \Gamma^Z(Q)$ holds.

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_{\mathbf{Z}}\mathbf{PALG}$



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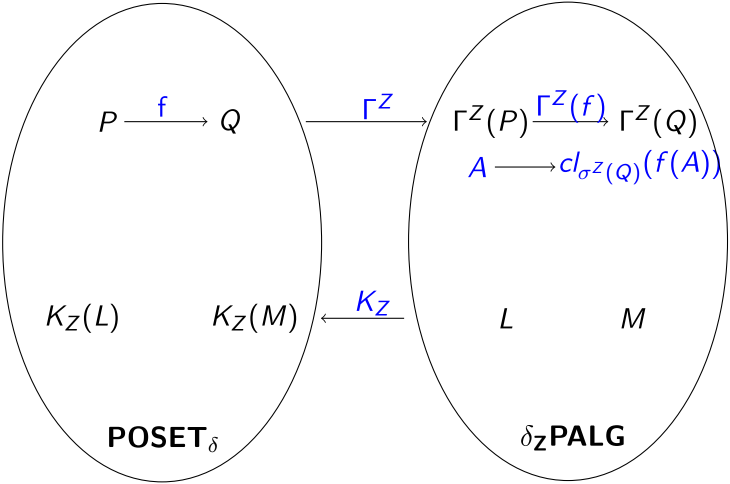

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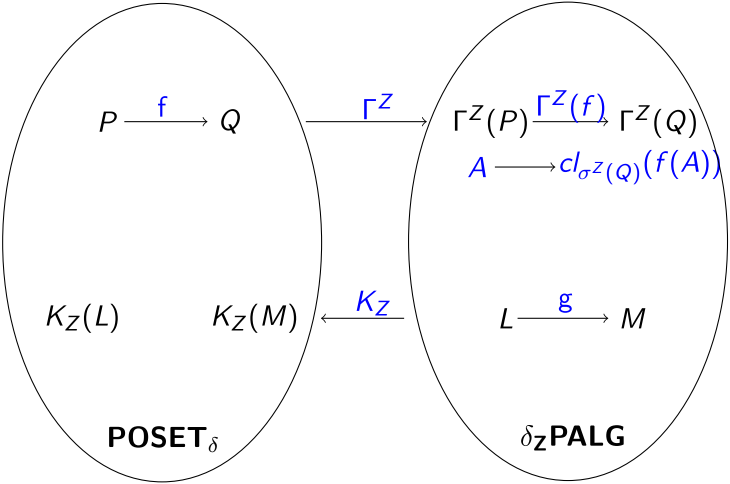

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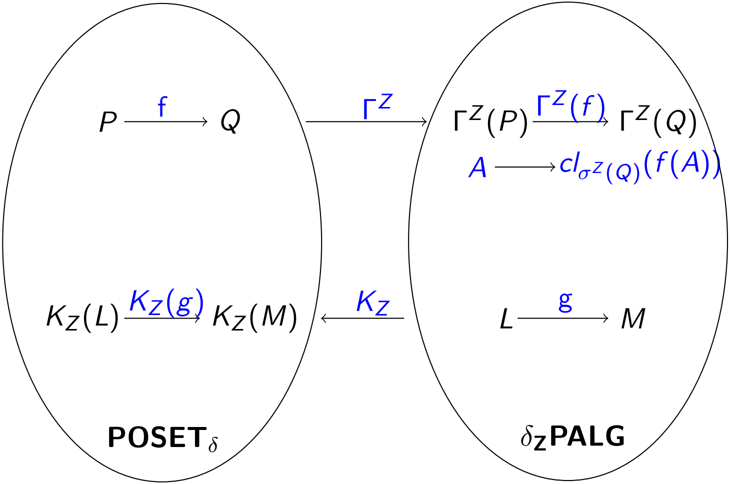

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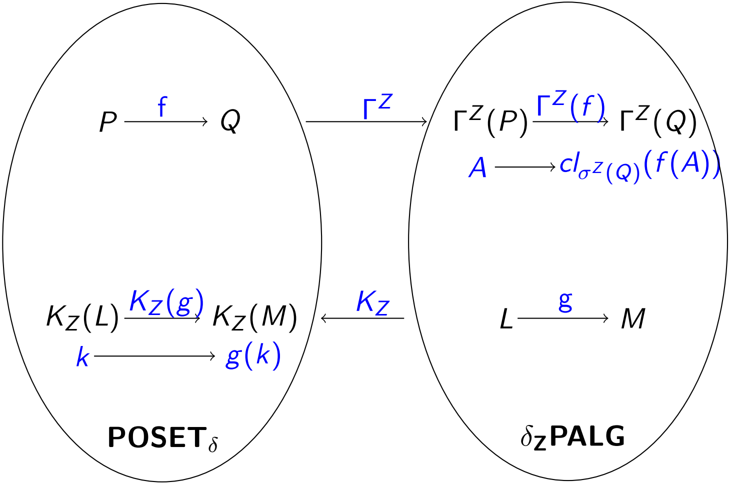

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_\mathbf{Z}\mathbf{PALG}$

$K_Z(g)$ is σ^Z-continuous

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_{\mathbf{Z}}\mathbf{PALG}$

$$K_Z(g) \text{ is } \sigma^Z\text{-continuous}$$

$$\Uparrow$$

$$\forall D \in Z(k_Z(L)),\ K_Z(f)(D^\delta \mid_{k_Z(L)}) \subseteq (K_Z(f)(D))^\delta \mid_{k_Z(M)}$$

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_{\mathbf{Z}}\mathbf{PALG}$

Lemma

If L is a *zcpo*, then $k_Z(L)$ is also a *zcpo*.

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_{\mathbf{Z}}\mathbf{PALG}$

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$$K_Z(f)(\downarrow_{k_Z(L)} \sup_{k_Z(L)} D) \subseteq \downarrow_{k_Z(M)} \sup_{k_Z(M)} K_Z(f)(D)$$

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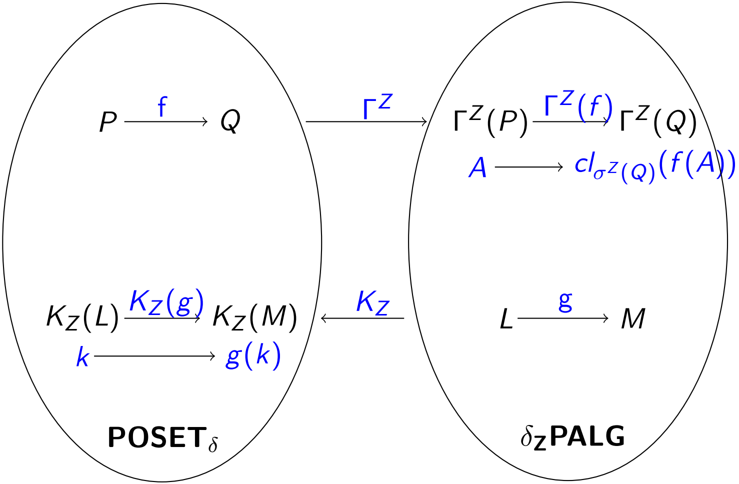
$$K_Z(f)(\downarrow_{k_Z(L)} \sup_{k_Z(L)} D) \subseteq \downarrow_{k_Z(M)} \sup_{k_Z(M)} K_Z(f)(D)$$

$$\Uparrow$$

$$K_Z(f)(\downarrow \sup D \cap k_Z(L)) \subseteq \downarrow \sup K_Z(f)(D) \cap k_Z(M)$$

Construct a pair of adjoints between $\mathbf{POSET}_\delta$ and $\delta_{\mathbf{Z}}\mathbf{PALG}$



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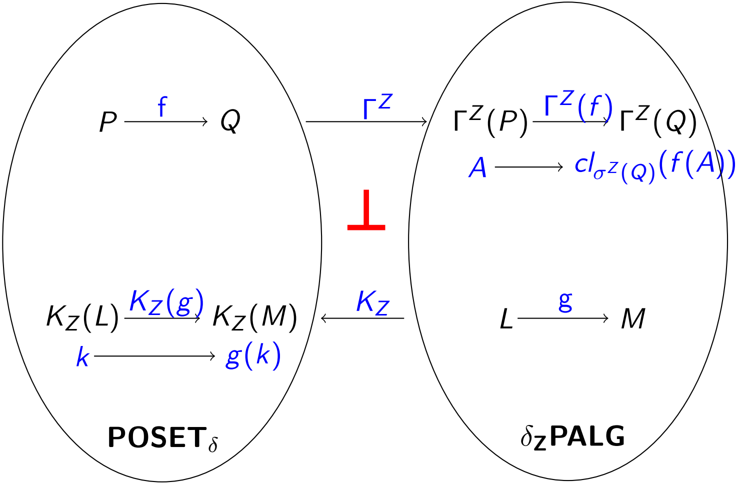
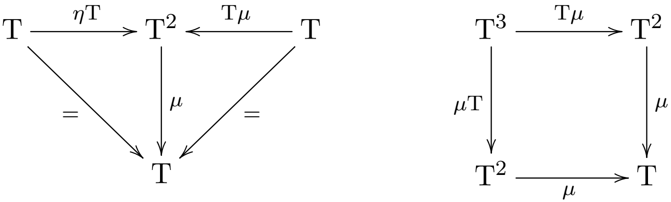


Monad

A monad on a category $\mathcal{A}$ is a triple $\mathbf{T} = (\mathrm{T}, \eta, \mu)$ consists of a functor $\mathrm{T} : \mathcal{A} \to \mathcal{A}$, together with two natural transformations $\eta : id \to \mathrm{T}$ and $\mu : \mathrm{T}^2 \to \mathrm{T}$ for which the following diagrams commute:



The relation between monad and adjoints

Proposition

Let $U : \mathcal{B} \to \mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ be functors such that $F \dashv U$ with $\eta : id \to UF$ and $\epsilon : FU \to id$ the unit and counit, respectively. Then $(UF, \eta, U\epsilon F)$ is a monad on $\mathcal{A}$.

A monad on $\mathbf{POSET}_\delta$

Theorem

The endofunctor $\delta = K_Z \circ \Gamma^Z$ together with two natural transformation $\eta : id \to \delta$ and $\mu = \Gamma^Z \epsilon K_Z : \delta^2 \to \delta$ is a monad on the category $\mathbf{POSET}_\delta$. More precisely, for each $P \in \mathbf{POSET}_\delta$, $\eta_P : P \to \delta(P)$ and $\mu_P : \delta^2(P) \to \delta(P)$ are defined as:

$$\forall p \in P, \eta(p) = \downarrow p,$$

$$\forall \mathcal{A} \in \delta^2(P), \mu(\mathcal{A}) = \sup \mathcal{A},$$

respectively.

Characterize the Eilenberg-Moore algebra of the monad δ

We call the poset P δcpo if for each $A \in \delta(P), \sup A$ exists.

Theorem

There exists a structure map $\xi : \delta(P) \to P$ in $\mathbf{POSET}_\delta$ such that (P, ξ) is an *Eilenberg-Moore* algebra of the monad (δ, η, μ) if and only if P is a δcpo.

The Eilenberg-Moore algebra

Definition

Let $\mathbf{T} = (\mathrm{T}, \eta, \mu)$ be a monad. A **T-algebra** of the monad **T** over the category $\mathcal{C}$ is an object A (the *underlying object*) together with an arrow $\alpha : \mathrm{TA} \to \mathrm{A}$ (the *structure mapping*) satisfying the following conditions:

(**Associativity**) $\alpha \circ \mu_A = \alpha \circ \mathrm{T}\alpha$, (**Unit law**) $\alpha \circ \eta_A = id_A$.

Thank you !