# Quasiexact posets and the moderate meet-continuity

Zhaorong He

Cooperated with Zhongqiang Yang and Dongsheng Zhao

NTU, Singapore

July 4, 2022

# 1. Motivation

Scott proposed a model for information systems using the Scott topology and a binary relation $\prec$ in connection with the information models. For continuous lattices, the relation $\prec$ coincides with the way-below relation.

D. S. Scott, Outline of a mathematical of theory of computing, Proceedings of the Fourth Annual Princeton Conference on Information Sciences and System, Princeton University Press, 1970, 169-176. DOI: 10.1002/9781118400715.ch5

# 1. Motivation

The class of continuous complete lattices was introduced by Scott.

D. S. Scott, Continuous lattices, in: Toposes, algebraic geometry and logic, Springer Lecture Notes in Mathematics 274, Springer-Verlag, 1974, pp. 97-136. DOI: https://doi.org/10.1007/BFb0073967

However, for general complete lattices, the aforementioned two relations may be distinct.

# 1. Motivation

One of the notable feathers of continuous lattices is that they admit a unique compact Hausdorff topology for which the meet operation is continuous. This topology, referred to as the CL-topology, turns out to be 'order intrinsic' - it can be defined merely using the lattice structure.

Gierz and Lawson characterized those complete lattices for which the CL-topology is Hausdorff and called them generalized continuous lattices.

G. Gierz and J. Lawson, Generalized continuous and hyper-continuous lattices, The Rocky Mountain Journal of Mathematics, 11 (1981), 271-296. https://www.jstor.org/stable/44236598

# 1. Motivation

Gierz et al. introduced the quasicontinuous posets and showed that a complete lattice is generalized continuous if and only if it is quasicontinuous. The key result for establishing the major properties of quasicontinuous dcpos is the Rudin's Lemma.


G. Gierz, J. D. Lawson, and A. R. Stralka, Quasicontinuous posets, Houston Journal of Mathematics, 9 (2) (1983), 191-208. https://www.math.uh.edu/~hjm/vol09-2.html

M. E. Rudin, Directed sets which converge, In L.F. McAuley and M.M. Rao, editors, General Topology and Modern Analysis, University of California, Riverside, 1980, pages 305–307. Academic Press, 1981. ISBN: 012481820X

# 1. Motivation

Coecke and Martin introduced two orders: the Bayesian order on classical states and the spectral order on quantum states.

They revealed that the corresponding sets are dcpos with an intrinsic notion of approximation.

The operational significance of the orders involved conclusively establishes that physical information has a natural domain-like theoretic structure.

B. Coecke and K. Martin, *A partial order on classical and quantum states*, Technical Report PRG-RR-02-07, Oxford University, 2002. Electronic ISBN: 978-3-642-12821-9, http://web.comlab.ox.ac.uk/oucl/publications/tr/rr-02-07.html

# 1. Motivation

Mushburn called the approximation mentioned by Coecke and Martin the weak way-below relation, and defined two topologies on posets: the way-below topology and the weak way-below topology.

These topologies coincide with the Scott topology for continuous posets, but are very different for non-continuous posets.

J. Mushburn, A comparison of three topologies on ordered sets, Topology Proceedings, 31 (1) (2007), 197-217. https://ecommons.udayton.edu/mth_fac_pub/18

Mushburn also showed that while domain representable spaces must be Baire, this is not the case with respect to the new topologies.

Thus, Mushburn defined the weak domains and weak domain representable spaces and constructed an example to show that weak domain representable spaces need not be Baire.

J. Mushburn, Linearly ordered topological spaces and weak domain representability, Topology Proceedings, 35 (2010), 149-164. https://ecommons.udayton.edu/mth_fac_pub/20

# 1. Motivation

The class of meet continuous lattices was first introduced by Birkhoff.

G. Birkhoff, “Lattice Theory," Vol. 25. American Mathematical Society Colloquium Publications, 1967. ISBN: 9780821810255

Much investigations on meet continuity for lattices and semilattices, refer to

J. R. Isbell, Meet-continuous lattices, Symposia Mathematica, Vol. XVI (Convegno sui Gruppi Topologici e Gruppi di Lie, INDAM, Rome, 1974), pp. 41–54. Academic Press, London, 1975.

# 1. Motivation


- K. H. Hofmann and A. R. Stralka, The algebraic theory of compact Lawson semilattices: applications of Galois connections to compact semilattices, Dissertationes Mathematicae, 137 (1976), 1-54, https://www.infona.pl/resource/bwmeta1.element.zamlynska-0a64c2e4-4598-4b8a-9017-1624fd87ce6b
- G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove and D. Scott, Continuous Lattices and Domains, Encyclopedia of Mathematics and its Applications, vol. 93, Cambridge University Press, 2003. ISBN: 9780521803380

# 1. Motivation

Kou et al. extended the notion of meet continuity to general dcpos and proved that a dcpo is continuous iff it is meet continuous and quasicontinuous.

H. Kou, Y. Liu and M. Luo, On meet continuous dcpo, Domain and Process Semantic Structures in Computation. Kluwer, 2001. DOI: https://doi.org/10.1007/978-94-017-1291-0_5

# 1. Motivation

The investigation of the more general meet-continuous posets and quasicontinuous posets, refer to


X. Mao and L. Xu, Quasicontinuity of posets via Scott topology and sobrification. Order, 23 (2006), 359-369. DOI: https://doi.org/10.1007/s11083-007-9054-4

X. Mao and L. Xu, Meet continuity properties of posets. Theoretical Computer Science, 410 (2009), 4234-4240. DOI: https://doi.org/10.1016/j.tcs.2009.06.017

## Definition

For any poset $P$ and $G, H \in \mathcal{P}^*(P)$, we say that $G$ is weakly way below $H$ and write $G \ll_w H$, if for every directed subset $D \subseteq P$, $\bigvee D \in H$ implies $d \in \uparrow G$ for some $d \in D$.

Write

$$\mathrm{fin}_w(x) = \{F \subseteq P : F \text{ is finite}, F \ll_w x\}.$$

## Definition

A poset $P$ is said to be quasiexact if for each $x \in P$, the family $\mathrm{fin}_w(x)$ is directed and $\bigcap\{\uparrow F : F \in \mathrm{fin}_w(x)\} = \uparrow x$.

# 2. Quasiexact dcpos

## Rudin's Lemma

For any poset $P$, let $\mathcal{F}$ be a directed family of nonempty finite subsets of $P$. There exists a directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $D \cap F \neq \varnothing$ for every $F \in \mathcal{F}$.

## Lemma

*For any dcpo $P$, let $G \in \mathcal{P}^*(P)$ and $x \in P$ with $G \ll_w x$. If $\mathcal{F}$ is a directed family of nonempty finite subsets $F \subseteq \downarrow x$ with $\bigcap_{F\in\mathcal{F}} \uparrow F \subseteq \uparrow x$, then there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \uparrow G$.*

**Theorem**

*For any dcpo $P$, the following statements are equivalent:*

(a) $P$ *is quasiexact;*

(b) *for any* $x \in P$, *the collection* $fin_{(bl)w}(x)$ *is directed and* $\bigcap\{\uparrow F : F \in fin_{(bl)w}(x)\} = \uparrow x$;

(c) *for any* $x \in P$, *there exists a directed subset* $\mathcal{F} \subseteq fin_w(x)$ *such that* $\bigcap_{F \in \mathcal{F}} \uparrow F = \uparrow x$;

(d) *for any* $x \in P$, *there exists a directed subset* $\mathcal{F} \subseteq fin_{(bl)w}(x)$ *such that* $\bigcap_{F \in \mathcal{F}} \uparrow F = \uparrow x$.

# 2. Quasiexact dcpos

## Proposition

*The cartesian product $\prod_{i\in I} P_i$ of a family of quasiexact dcpos is a quasiexact dcpo, provided that at most finitely many does not have a bottom element $\perp_i$.*

## Proposition

(i) *Every exact dcpo is quasiexact. Hence, every weak domain is a quasiexact dcpo.*

(ii) *Every quasicontinuous domain is a quasiexact dcpo.*

# 2. Quasiexact dcpos

Quasicontinuous domains are not necessarily with the relation $\ll_w$ weakly increasing.

Quasiexact posets need not to be exact or be with the relation $\ll_w$ weakly increasing.

## Example

*Let* $P = \{a, b, c, d\} \cup \{x_n : n \in \mathbb{N}\}$ *with the order*

(i) $a < b < c < d$;

(ii) $x_m < x_n$ *whenever* $m < n$ *and* $m, n \in \mathbb{N}$;

(iii) $x_n < c$ *for any* $n \in \mathbb{N}$,

*where* $\{a, b, c, d\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing$.

# 2. Quasiexact dcpos

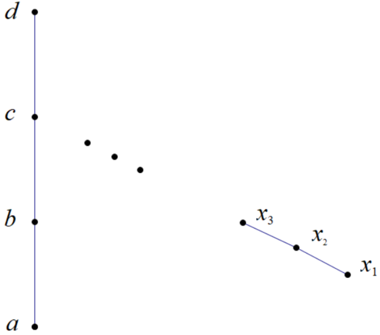


Figure: An exact quasicontinuous domain in which $\ll_w$ is not weakly increasing

# 2. Quasiexact dcpos

Quasicontinuous domains need not to be exact, even when the relation $\ll_w$ is weakly increasing.

## Example

*Let* $P = (\mathbb{N} \times \{1, 2\}) \cup \{\top\}$ *and define an order by the following rules:*

(i) $(m, i) < (n, i)$ *if* $m < n$ *for all* $m, n \in \mathbb{N}$ *and* $i = 1, 2$*;*

(ii) $(n, i) < \top$ *for all* $n \in \mathbb{N}$ *and* $i = 1, 2$*.*

# 2. Quasiexact dcpos

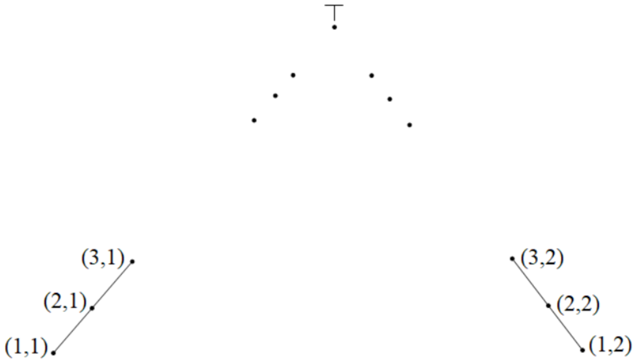


Figure: A non-exact quasicontinuous domain with $\ll_w$ weakly increasing

# 2. Quasiexact dcpos

Quasiexacness does not imply stronger properties of $\ll_w$.

Weak domains are not necessarily quasicontinuous domain.

Hence quasiexact dcpos with the relation $\ll_w$ weakly increasing are not necessarily quasicontinuous.

**Example (Johnstone space)**

*Let* $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ *with ordering defined by*

(i) $(a, m) < (a, n)$ *if* $m < n$ *for all* $a, m, n \in \mathbb{N}$;

(ii) $(a, m) < (b, \omega)$ *if* $m \leq b$ *for all* $a, b, m \in \mathbb{N}$.

# 3. The wf topology and the moderate meet-continuity

When the family $\{\uparrow_w x : x \in P\}$ generates a topology on a poset $P$, then this topology is called the weak way-below topology (wwb topology, for short), denoted by $\tau_{wwb}(P)$. The topological space $(P, \tau_{wwb}(P))$ is simply written as $(P, \tau_{wwb})$.

J. Mushburn, A comparison of three topologies on ordered sets, Topology Proceedings, 31 (1) (2007), 197-217.
https://ecommons.udayton.edu/mth_fac_pub/18

**Lemma**

*If $P$ is a quasiexact poset, then $\{\uparrow_w F : F \subseteq P, F \text{ is finite}\}$ generates a topology on $P$.*

# 3. The wf topology and the moderate meet-continuity

Whenever the family $\{\Uparrow_w F : F \subseteq P, F \text{ is finite}\}$ generates a topology on $P$, we call it the weak way-below finitely determined topology (briefly, wf topology) on $P$, denoted by $\tau_{wf}(P)$.

## Remark

(1) For any quasiexact poset $P$, we have $\Uparrow_w F \subseteq \mathrm{int}_{\tau_{wf}}(\uparrow F)$ for every $F \in \mathcal{P}^*(P)$.

(2) If a poset $P$ admits both the wwb topology and the wf topology, then $\tau_{wwb}(P) \subseteq \tau_{wf}(P)$.

(3) If $P$ is an exact poset, then $\sigma(P) \subseteq \tau_{wf}(P)$.

Mushburn constructed an example to show the wwb topology can be strictly finer than the Scott topology.

Every quasicontinuous dcpo admits the wf topology. However, the following example shows that quasicontinuous dcpos do not necessarily admit the wwb topology.

**Example**

*Let* $P = (\mathbb{N} \times \{1, 2\}) \cup \{\top\}$ *with the order*

(i) $(m, i) < (n, i)$ *if* $m < n$ *for all* $m, n \in \mathbb{N}$ *and* $i = 1, 2$;

(ii) $(n, i) < \top$ *for all* $n \in \mathbb{N}$ *and* $i = 1, 2$.

# 3. The wf topology and the moderate meet-continuity

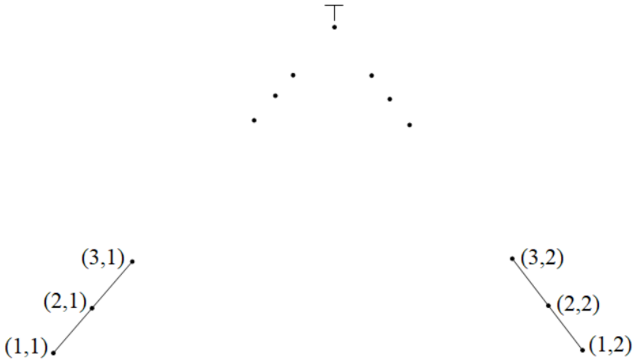


Figure: A quasicontinuous domain admitting the wf topology

# 3. The wf topology and the moderate meet-continuity

**Proposition**

*A poset $P$ is quasiexact if for any nonempty $H \subseteq P$ and any $x \in P$, $H \ll_w x$ implies there exists a finite $F \subseteq \uparrow H$ such that $F \ll_w x$.*

**Definition**

A poset $P$ admitting the wf topology is said to be moderately meet continuous if for any $x \in P$ and directed subset $D$, $x \leq \bigvee D$ implies that $x \in \mathrm{cl}_{\tau_{wf}}(\downarrow D \cap \downarrow x)$.

**Remark**

Let $P$ be a moderately meet continuous poset. Then $\mathrm{int}_{\tau_{wf}}(\uparrow F) \subseteq \bigcup\{\Uparrow_w x : x \in F\}$ for any $F \in \mathcal{P}^*(P)$.

# 3. The wf topology and the moderate meet-continuity

## Corollary

*Let $P$ be a moderate meet-continuous quasiexact poset. Then*

$$\Uparrow_w F = \bigcup\{\Uparrow_w x : x \in F\} = int_{\tau_{wf}}(\uparrow F)$$

*for any $F \in \mathcal{P}^*(P)$.*

## Theorem

*If $P$ is a moderately meet continuous quasiexact poset, then $\sigma(P) \subseteq \tau_{wwb}(P) = \tau_{wf}(P)$.*

# 3. The wf topology and the moderate meet-continuity

**Theorem**

*Every moderately meet continuous quasiexact poset is exact and meet continuous.*

**Corollary**

*A poset $P$ is a domain if $P$ is a moderate meet-continuous quasiexact dcpo with the relation $\ll_w$ weakly increasing.*

**Problem 1.** What is the property $p$ such that a poset $P$ is exact if and only if it is quasiexact and has property $p$?

**Problem 2.** Under what conditions, a quasiexact dcpo is quasicontinuous?

# Acknowledgement

**Thank you for your attentions!**