The dimension Dind of finite topological T_0-spaces

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1. Introduction

The results presented here are obtained by a joint work with D. N. Georgiou, A. C. Megaritis and F. Sereti.

A.V. Arhangelskii introduced the dimension Dind and some properties of this dimension have been studied by several authors, say W. Kulpa (1971/72) and V. Chatyrko and B. Pasynkov (2002) for normal Hausdorff spaces.

In this talk, we will consider on the relations between Dind and the fundamental dimensions ind, Ind and dim for finite T_0-spaces.

Several open questions for further investigations on Dind in the classes of Alexandroff and finite spaces are also asked.

2. Preliminaries

First, we remind the fundamental concepts of dimension of topological spaces.

Definition 1 The *covering dimension* of a topological space X, denoted by $\dim X$, is defined as follows:

1. $\dim X = -1$ iff $X = \emptyset$.

2. For $k \in \{0, 1, \ldots\}$, $\dim X \leqslant k$ if for every finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathcal{V}$ of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $\mathrm{ord}(\mathcal{V}) \leqslant k$, where $\mathrm{ord}(\mathcal{V}) \leq k$ if $|\{V \in \mathcal{V} : x \in V\}| \leq k+1$ for every $x \in X$.

Definition 2 The *small inductive dimension* of a topological space X, denoted by $\mathrm{ind}\, X$, is defined as follows:

1. $\operatorname{ind} X = -1$ iff $X = \emptyset$.

2. For $k \in \{0, 1, \ldots\}$, $\operatorname{ind} X \leqslant k$ if for every point $x \in X$ and for every nbd U of x, there exists a nbd V of x such that $V \subset U$ and $\operatorname{ind} \operatorname{Bd} V \leqslant k - 1$.

Definition 3 The *large inductive dimension* of a topological space X, denoted by $\operatorname{Ind} X$, is defined as follows:

1. $\operatorname{Ind} X = -1$ iff $X = \emptyset$.

2. For $k \in \{0, 1, \ldots\}$, $\operatorname{Ind} X \leqslant k$ if for every closed set F of X and for every open set U of X with $F \subset U$, there exists an open set V of X such that $F \subset V \subset U$ and $\operatorname{Ind} \operatorname{Bd} V \leqslant k - 1$.

A. V. Arhangelskii introduced the dimension Dind.

Definition 4 The dimension Dind is defined as follows:

1. $\operatorname{Dind} X = -1$ iff $X = \emptyset$.

2. $\operatorname{Dind} X \leqslant k$, where $k \in \{0, 1, \dots\}$, if for any finite open cover $\mathcal{U}$ of X there exists a finite family $\mathcal{V}$ of pairwise disjoint open subsets of X such that $\mathcal{V} \prec \mathcal{U}$ and $\operatorname{Dind}\left(X \setminus \bigcup\{V : V \in \mathcal{V}\}\right) \leqslant k - 1$.

3. Fundamental properties on Dind for finite T_0-spaces

We begin with some fundamental properties on Dind for finite T_0-spaces.

Proposition 1 *Let X be a finite T_0-space and A a closed subset of X. Then we have*

$$\mathrm{Dind}\, A \leqslant \mathrm{Dind}\, X.$$

Proposition 2 *Let X be a finite T_0-space and $\mathbf{U}_x$ the minimal nbd of x for each $x \in X$. Then* $\mathrm{Dind}\, X \leqslant k$ *iff there exists a family $\{V_i\}_{i=1}^m$ of open sets of X such that:*

(1) $\{V_i\}_{i=1}^m \prec \{\mathbf{U}_x : x \in X\}$,

(2) $V_i \cap V_j = \emptyset$ *if* $i \neq j$ *and*

(3) $\mathrm{Dind}\,(X \setminus \bigcup_{i=1}^m V_i) \leqslant k - 1$. $\square$

Proposition 3 *If* X *is the sum* $\oplus_{s\in S}X_s$ *of finite* T_0*-spaces* X_s, $s \in S$, *then we have that*

$$\operatorname{Dind} X = \sup\{\operatorname{Dind} X_s : s \in S\}.$$

We show the following theorem by use of the above Propositions 1-3.

Theorem 1 *For every* $k \in \{0, 1, \ldots\}$, *there exists a finite* T_0*-space* X *such that* $\operatorname{Dind} X = k$.

Proof. We shall prove the theorem by induction on k. Let X_0 be a one point space. Then $\operatorname{Dind} X_0 = 0$. (Moreover, for every discrete finite space X we have $\operatorname{Dind} X = 0$.)

Let $k \geq 1$, we assume that there exists a finite T_0-space (Y_0, τ_{Y_0}) such that $\operatorname{Dind}(Y_0) = k - 1$.

We consider the space $X = \{w\} \cup Y_0 \cup Y_1$, where (Y_1, τ_{Y_1}) is a space which is homeomorphic to (Y_0, τ_{Y_0}), $Y_0 \cap Y_1 = \emptyset$ and $w \notin Y_0 \cup Y_1$. We consider the topology τ on X, which has as a basis the family

$$\beta = \{\emptyset\} \cup \{\{w\} \cup U : U \in \tau_{Y_0}\} \cup \{\{w\} \cup U : U \in \tau_{Y_1}\}.$$

We have that $\mathbf{U}_w = \{w\}$, $\mathbf{U}_y = \{w\} \cup \mathbf{U}_y^i$ for $y \in Y_i$, where $\mathbf{U}_y^i$ is the minimal nbd of $y \in Y_i$ in Y_i for $i = 0, 1$.

We shall show that $\operatorname{Dind} X = k$.

We observe that a family $\mathcal{V}$ of pairwise disjoint open subsets of X which refines the family of minimal open nbds are exactly one of the following three cases:

1. $\mathcal{V} = \{\{w\}\}$,

2. $\mathcal{V} = \{\{w\} \cup \mathbf{U}_y^0\}$ for $y \in Y_0$, and

3. $\mathcal{V} = \{\{w\} \cup \mathbf{U}^1_y\}$ for $y \in Y_1$.

For the first case, by Proposition 3, we have

$$\begin{aligned}\text{Dind}\,(X \setminus \{w\}) &= \text{Dind}\,(Y_0 \oplus Y_1) \\ &= \max\{\text{Dind}\,(Y_0), \text{Dind}\,(Y_1)\} \\ &= k-1.\end{aligned}$$

For the second case, by Proposition 1, we have that for every $\mathbf{U}^0_y \in \tau_{Y_0}$,

$k-1 = \text{Dind}\,(Y_1) = \text{Dind}\,(X \setminus (\{w\} \cup Y_0)) \leqslant \text{Dind}\,(X \setminus (\{w\} \cup \mathbf{U}^0_x)) \leqslant \text{Dind}\,(X \setminus \{w\}) = k-1.$

Similarly, for every $\mathbf{U}^1_x \in \tau_{Y_1}$, we have that $\text{Dind}\,(X \setminus (\{w\} \cup \mathbf{U}^1_x)) = k-1$. Thus, by Proposition 2, we have that $\text{Dind}\,X = k$. $\square$

We recall that a pairwise disjoint cover $\mathcal{U}$ of X is called a *partition* of X.

The notion of the partition gives the following characterization of zero dimensional spaces with respect to Dind.

Proposition 4 Dind $X = 0$ *iff there exists a partition of* X *consisting of minimal open nbds.*

$\square$

4. Relations between Dind and other dimensions for finite T_0-spaces

We study relations between the dimension Dind and the dimensions Ind, ind and dim.

First, we notice that for any Alexandroff space X, we have

1. R. Berghammer and M. Winter showed that $\operatorname{Ind} X \leq \operatorname{ind} X$ holds ([2019]).

2. D. W. Bass showed that $\operatorname{Ind} X \leq \dim X$ holds ([1969]).

Lemma 1 *Let X be a finite T_0-space. Then there exists a cover $\{\mathrm{U}_{x_1}, \ldots, \mathrm{U}_{x_s}\}$ of X consisting of minimal open nbds such that for each $k \in \{1, \ldots, s\}$, there exists a closed subset F_k of X such that $x_k \in F_k$ and $\mathrm{U}_{F_k} = \mathrm{U}_{x_k}$, where U_{F_k} is the smallest open set containing F_k.*

Theorem 2 *For a finite T_0-space X, we have* $\operatorname{Dind} X \leqslant \operatorname{Ind} X$.

Proof. We shall prove the theorem by induction on $\operatorname{Ind} X = n$.

Let $n = 0$. Then by Lemma 1, there exists a cover $\mathcal{C} = \{\mathbf{U}_{x_1}, \ldots, \mathbf{U}_{x_s}\}$ of X consisting of minimal open nbds such that for each $k \in \{1, \ldots, s\}$, there exists a closed subset F_k of X such that $x_k \in F_k$ and $\mathbf{U}_{F_k} = \mathbf{U}_{x_k}$. Since $\operatorname{Ind} X = 0$, we have $\operatorname{Bd} \mathbf{U}_{x_k} = \emptyset$ for each $k \in \{1, \ldots, s\}$.

Then we may assume that $\mathbf{U}_{x_j} \cap \mathbf{U}_{x_k} = \emptyset$ for each $j \neq k \in \{1, \ldots, s\}$.

Indeed, let $\mathbf{U}_{x_j} \cap \mathbf{U}_{x_k} \neq \emptyset$. Then $x_j \in \operatorname{Cl} \mathbf{U}_{x_k}$ and $x_k \in \operatorname{Cl} \mathbf{U}_{x_j}$. Now, we suppose that $x_j \notin \mathbf{U}_{x_k}$, then $x_j \in \operatorname{Bd} \mathbf{U}_{x_k}$, a contradiction.

Thus $x_j \in \mathbf{U}_{x_k}$, and hence $\mathbf{U}_{x_j} \subset \mathbf{U}_{x_k}$. Similarly, we have that $\mathbf{U}_{x_k} \subset \mathbf{U}_{x_j}$. Therefore, $\mathbf{U}_{x_j} = \mathbf{U}_{x_k}$.

Let $\mathcal{C}' = \mathcal{C} - \{\mathbf{U}_{x_k}\}$. We continue the similar process, we can have a partition $\mathcal{C}^*$ of X by minimal open nbds of X. Hence, we have that $\operatorname{Dind} X = 0$.

Let $n \geq 1$, and we assume that the theorem is true for all finite T_0-spaces with $\operatorname{Ind} Y \leqslant n - 1$.

Let $\operatorname{Ind} X = n$. We shall prove that $\operatorname{Dind} X \leqslant n$. It suffices to construct a family $\mathcal{V}$ of open sets of X which satifies the conditions (1)-(3) of Proposition 2.

By Lemma 1, there exists a cover

$$\mathcal{C} = \{\mathbf{U}_{x_1}, \ldots, \mathbf{U}_{x_s}\}$$

of X consisting of minimal open nbds such that for each $k \in \{1, \ldots, s\}$, there exists a closed subset F_k of X such that $x_k \in F_k$ and $\mathbf{U}_{F_k} = \mathbf{U}_{x_k}$. We consider the family $\mathcal{V} = \{V_1, \ldots, V_s\}$ of pairwise disjoint open subsets of X, where

$$V_1 = \mathbf{U}_{x_1} \setminus (\mathrm{Cl}\,\mathbf{U}_{x_2} \cup \ldots \cup \mathrm{Cl}\,\mathbf{U}_{x_s})),$$

$$V_2 = \mathbf{U}_{x_2} \setminus (\mathrm{Cl}\,\mathbf{U}_{x_3}) \cup \ldots \cup \mathrm{Cl}\,\mathbf{U}_{x_s}),$$

$\vdots$

$$V_s = \mathbf{U}_{x_s}.$$

Then $\mathcal{V} \prec \mathcal{C}$. We prove that

$$X \setminus (V_1 \cup \ldots \cup V_s) \subseteq \bigcup_{k=1}^{s} \mathrm{Bd}\,\mathbf{U}_{x_k}.$$

Let $x \in X \setminus (V_1 \cup \ldots \cup V_s)$. Since $\mathcal{C}$ is a cover of X, there is $k \in \{1, \ldots, s-1\}$ such that $x \in \mathbf{U}_{x_k}$. Let

$$k_0 = \max\{k : x \in \mathbf{U}_{x_k}\}.$$

Then, we have that $x \in \mathrm{Cl}\,\mathbf{U}_{x_{k_0+1}} \cup \ldots \cup \mathrm{Cl}\,\mathbf{U}_{x_s}$ (otherwise, $x \in V_{k_0}$). Hence, $x \in \mathrm{Cl}\,\mathbf{U}_{x_\ell} \setminus \mathbf{U}_{x_\ell} = \mathrm{Bd}\,\mathbf{U}_{x_\ell}$ for some $\ell > k_0$.

By subspace theorem for Ind, we have

$$\mathrm{Ind}\,(X \setminus (V_1 \cup \ldots \cup V_s)) \leqslant \mathrm{Ind}\,(\bigcup_{k=1}^{s} \mathrm{Bd}\,\mathbf{U}_{x_k}).$$

Since $\mathrm{Ind}\,X = n$,

$$\mathrm{Ind}\,\mathrm{Bd}\,\mathbf{U}_{x_k} = \mathrm{Ind}\,\mathrm{Bd}\,\mathbf{U}_{F_k} \leqslant n-1,$$

$k = 1, \ldots, s$, and therefore, we have

$$\mathrm{Ind}\,(\bigcup_{k=1}^{s} \mathrm{Bd}\,\mathbf{U}_{x_k}) \leqslant n-1.$$

Hence, by inductive assumption, we conclude that

$$\mathrm{Dind}\,(X \setminus (V_1 \cup \ldots \cup V_s)) \leqslant n-1.$$

Thus, by Proposition 2, $\operatorname{Dind} X \leqslant n$. $\square$

The following is a direct consequence of the results presented above and Theorem 2.

Corollary 1 *Let X be a finite T_0-space. Then we have that*

$$\operatorname{Dind} X \leqslant \operatorname{ind} X \text{ and } \operatorname{Dind}(X) \leqslant \dim X.$$

We shall find a finite T_0-space X for which the converse of Theorem 2 does not hold.

Example 1 For each natural number $n \geqslant 1$ there is a finite T_0-space X_n such that

$$\operatorname{Dind} X_n = 1 \text{ and } \operatorname{Ind} X_n = \operatorname{ind} X_n = n.$$

Proof. Let $n \geqslant 1$ be a natural number, and

$$X_n = \{x_0, x_1, \ldots, x_{2n-1}, x_{2n}\}.$$

We induce a topology on X_n by defining the minimal nbd $\mathbf{U}_i$ of a point $x_i \in X_n$ for each $i = 0, 1, 2, \ldots, 2n-1, 2n$. Let

$$\mathbf{U}_0 = \{x_0\}, \mathbf{U}_1 = \{x_0, x_1\} \text{ and } \mathbf{U}_2 = \{x_0, x_2\}.$$

Let i be a natural number with $1 \leqslant i \leqslant n-1$, and we suppose that $\mathbf{U}_{2i-1}$ and $\mathbf{U}_{2i}$ are defined. Then we define

$$\mathbf{U}_{2i+1} = \mathbf{U}_{2i-1} \cup \{x_{2i+1}\}$$

and

$$\mathbf{U}_{2i+2} = \mathbf{U}_{2i-1} \cup \{x_{2i+2}\}.$$

It is obvious that X_n is a finite T_0-space.

Since $X_n \setminus \mathbf{U}_{2n-1} = \{x_2, x_4, \ldots, x_{2n}\}$ is a discrete subspace of X_n, we have Dind $(X_n \setminus U_{2n-1}) = 0$. Thus we have Dind $(X_n) \leqslant 1$.

On the other hand, since every nbd of a point in X_n contains x_0, every disjoint family of the minimal nbds is a singleton. Obviously, $\mathbf{U}_k \neq X_n$ for each k with $0 \leqslant k \leqslant 2n$. Hence, we have that Dind $X_n \geqslant 1$ and hence Dind $X_n = 1$.

Next, we show that Ind $X_n = n$.

(1) We show that Ind $X_n \geqslant n$ by induction on n.

Let $n = 1$. Then $F = \{x_2\}$ be a closed set of $X_1 = \{x_0, x_1, x_2\}$, and $\mathbf{U}_2 = \{x_0, x_2\}$ is the minimal open set containing F. It follows that Bd $\mathbf{U}_2 = \{x_1\}$ and Ind (Bd $\mathbf{U}_2$) $= 0$. Hence Ind $X_1 \geqslant 1$.

Let $n \geqslant 2$ and we suppose that $\operatorname{Ind} X_{n-1} \geqslant n-1$. We consider on the closed set $F = \{x_2\}$ of X_n. Then $\mathbf{U}_2 = \{x_0, x_2\}$ is the minimal open set containing F. It follows that $\operatorname{Bd} \mathbf{U}_2 = X_n \setminus \{x_0, x_2\} = \{x_1, x_3, x_4, \ldots, x_{2n-1}, x_{2n}\}$ is homeomorphic to X_{n-1}. Hence, by the inductive assumption, we have that $\operatorname{Ind}(\operatorname{Bd}\mathbf{U}_2) \geqslant n-1$. Hence $\operatorname{ind} X_n \geq \operatorname{Ind} X_n \geqslant n$.

(2) Next, we show that $\operatorname{ind} X_n \leqslant n$ by induction on n.

To show that $\operatorname{ind} X_1 \leqslant 1$, let U be a non-empty open set of X_1. Since $x_0 \in U$, it follows that $\operatorname{Bd} U \subseteq \{x_1, x_2\}$. Since $\{x_1, x_2\}$ is discrete, we have $\operatorname{ind} \operatorname{Bd} U \leq 0$. Thus, $\operatorname{ind} X_1 \leqslant 1$.

Let $n \geq 2$ and we suppose that $\operatorname{ind} X_{n-1} \leqslant n-1$. Let U be a non-empty open set of X_n. Since $x_0 \in U$, it follows that $\operatorname{Bd} U \subseteq \{x_1, x_2, \ldots, x_{2n-1}, x_{2n}\}$. We put

$$Y = \{x_1, x_2, \ldots, x_{2n-1}, x_{2n}\}.$$

Since $\{x_2\}$ is an isolated point of Y, Y is a disjoint sum of $\{x_2\}$ and $\{x_1, x_3, x_4, \ldots, x_{2n-1}, x_{2n}\}$. It is easy to show that $\{x_1, x_3, x_4, \ldots, x_{2n-1}, x_{2n}\}$ is homeomorphic to X_{n-1}. Thus, we have that

$$\text{ind}\,\{x_1, x_3, x_4, \ldots, x_{2n-1}, x_{2n}\} \leqslant n-1.$$

Hence $\text{ind}\,Y \leqslant n-1$, and thus $\text{ind}\,(\text{Bd}\,U) \leqslant n-1$. Therefore, $\text{Ind}\,X_n \leq \text{ind}\,X_n \leqslant n$. □

Furthermore, we can extend Example 1 as follows.

Example 2 For each pair of natural numbers $m \leqslant n$, there is a finite T_0-space X_{mn} such that $\text{Dind}\,X_{mn} = m$ and $\text{Ind}\,X_{mn} = \text{ind}\,X_{mn} = n$.

In order to construct the space of Example 2, we need a further consideration on the proof of Theorem 1.

Lemma 2 *Let Y_0 and Y_1 be finite T_0-spaces with $Y_0 \cap Y_1 = \emptyset$ and $X = \{w\} \cup Y_0 \cup Y_1$ be the space described in the proof of Theorem 1. Then we have the following:*

(1) $\max\{\operatorname{Dind} Y_0, \operatorname{Dind} Y_1\} \leq \operatorname{Dind} X$
$\leq \max\{\operatorname{Dind} Y_0, \operatorname{Dind} Y_1\} + 1$, *and*

if $\operatorname{Dind} Y_0 = \operatorname{Dind} Y_1$, *then*

$$\operatorname{Dind} X = \operatorname{Dind} Y_0 + 1.$$

(2) $\operatorname{ind} X = \max\{\operatorname{ind} Y_0, \operatorname{ind} Y_1\} + 1$.

(3) $\operatorname{Ind} X = \max\{\operatorname{Ind} Y_0, \operatorname{Ind} Y_1\} + 1$.

Proof. (1) Since Y_0 and Y_1 are closed in X, it is clear that $\max\{\operatorname{Dind} Y_0, \operatorname{Dind} Y_1\} \leqslant \operatorname{Dind} X$.

Further, we can show that

$$\text{Dind}\, X \leq \max\{\text{Dind}\, Y_0, \text{Dind}\, Y_1\} + 1$$

by a similar argument in the proof of Theorem 1.

Now, we suppose that $\text{Dind}\, Y_0 = \text{Dind}\, Y_1 = m$ and we will show that $\text{Dind}\,(X \setminus \bigcup \mathcal{V}) \geqslant m$ for every family $\mathcal{V}$ of open sets of X which satisfies the conditions (1)-(3) in Proposition 2.

Since every non-empty open set of X contains the point w, we can assume that $\mathcal{V} = \{\mathbf{U}_x\}$, where $\mathbf{U}_x$ is the minimal nbd of some point $x \in X$.

Case 1. $x = w$. We notice that $\mathcal{V} = \{\mathbf{U}_w\} = \{w\}$. Since $X \setminus \{w\}$ is a topological sum $Y_0 \oplus Y_1$, it follows from Proposition 3 that $\text{Dind}\,(X \setminus \{w\}) = m$.

Case 2. $x \in Y_0$. We notice that $X \setminus \mathbf{U}_x \supseteq Y_1$. Hence, $\mathrm{Dind}\,(X \setminus \mathbf{U}_x) \geqslant \mathrm{Dind}\, Y_1 = m$.

Case 3. $x \in Y_1$. As exactly the same as Case 2, we can show that $\mathrm{Dind}\,(X \setminus \mathbf{U}_x) \geqslant m$.

It follows that $\mathrm{Dind}\, X \geqslant m + 1$, and hence $\mathrm{Dind}\, X = m + 1$.

(2) Let $m = \max\{\mathrm{ind}\, Y_0, \mathrm{ind}\, Y_1\}$. Notice that $\{w\}$ is the minimal nbd of w in X and $\mathrm{Bd}_X\{w\} = Y_0 \cup Y_1$ is a topological sum of Y_0 and Y_1. Then $\mathrm{ind}\, \mathrm{Bd}_X\{w\} = m$. Hence $\mathrm{ind}_w X = m + 1$, and hence $\mathrm{ind}\, X \geq m + 1$.

Next, we take a point y in Y_0. Let $\mathbf{U}_y^0$ be the minimal nbd of y in Y_0. Then $\mathbf{U}_y = \{w\} \cup \mathbf{U}_y^0$ is the minimal nbd of y in X. Since $\mathrm{Bd}_X(\mathbf{U}_y) \subseteq Y_0 \cup Y_1$, it follows that

$$\mathrm{ind}\,(\mathrm{Bd}_X(\mathbf{U}_y)) \leqslant \mathrm{ind}(Y_0 \cup Y_1) = m.$$

Hence $\text{ind}_y X \leqslant m+1$. Similarly, we have that

$$\text{ind}_y X \leqslant m+1$$

for each $y \in Y_1$.

It follows that $\text{ind}\, X = m+1$.

(3) Let $\max\{\text{Ind}\, Y_0, \text{Ind}\, Y_1\} = m$, F be a closed set of X and U an open set of X such that $F \subseteq U$. Since every non-empty open set of X contains w, we have that $w \notin \text{Bd}_X U$. Hence, $\text{Bd}_X U \subseteq Y_0 \cup Y_1$, and $Y_0 \cup Y_1$ is a topological sum of Y_0 and Y_1. Hence, $\text{Ind}\, \text{Bd}_X(U) \leq m$, and hence $\text{Ind}\, X \leqslant m+1$.

We will show that $\text{Ind}\, X \geqslant m+1$.

Let $\text{Ind}\, Y_1 = m$, $F = Y_0$ and $U = \{w\} \cup Y_0$. Then F is a closed set of X and U is an open set of X with $F \subseteq U$. We notice that U is the minimal nbd of F.

It is easy to show that $\text{Bd}_X(U) = Y_1$, and hence $\text{Ind}(\text{Bd}_X(U)) = \text{Ind}\, Y_1 = m$. Hence $\text{Ind}\, X \geqslant m+1$, and thus, $\text{Ind}\, X = m+1$. □

Remark 1 We notice that Lemma 2 does not hold for dim, i.e.,

$$\dim X \neq \max\{\dim Y_0, \dim Y_1\} + 1.$$

In fact, let $Y = \{1, 2\}$ be the two points set with the discrete topology and $X = \{w\} \cup Y_0 \cup Y_1$. Then $\dim Y = 0$, but since every minimal nbd contains the point w, we have that

$$\dim X = 3 > \max\{\dim Y_0, \dim Y_1\} + 1 = 1. \square$$

The following is a direct consequence of Theorem 1 and Lemma 2.

Corollary 2 *For each natural number n, there is a finite T_0-space X_n such that*

$$\text{Dind}\, X_n = \text{ind}\, X_n = \text{Ind}\, X_n = n.$$

Proof. Let n be a natural number. By Theorem 1, there is a finite T_0-space Y_n such that $\operatorname{Dind} Y_n = n$.

By use of Lemma 2, we have a finite T_0-space Z_n such that $\operatorname{Dind} X_n \leq \operatorname{ind} X_n = \operatorname{Ind} X_n = n$. Then, $X_n = Y_n \oplus Z_n$ is desired. $\square$

Proof of Example 2. Let m, n be natural numbers with $m \leqslant n$. By Example 1, there is a finite T_0-space X_n such that

$$\operatorname{Dind} X_n = 1 \text{ and } \operatorname{Ind} X_n = \operatorname{ind} X_n = n.$$

On the other hand, by Corollary 2, we have a finite T_0-space X_m such that

$$\operatorname{Dind} X_m = \operatorname{Ind} X_m = \operatorname{ind} X_m = m.$$

Then

$$Y_{mn} = X_m \oplus X_n$$

is desired. $\square$

As we mentioned above, we have that Dind $X \leqslant \dim X$. Similar to Example 2, we have the following.

Example 3 For each pair of natural numbers $m \leqslant n$, there is a finite T_0-space Z_{mn} such that Dind $Z_{mn} = m$ and $\dim Z_{mn} = n$.

To show Example 3, we need one more construction of spaces.

Let Y be a finite T_0-space and Y_0 and Y_1 copies of Y with $Y_0 \cap Y_1 = \emptyset$. Let w be a point with $w \notin Y_0 \cap Y_1$. We put $X = \{w\} \cup Y_0 \cup Y_1$ and we induce a topology τ on X defining the minimal nbd $\mathbf{U}_x$ for $x \in X$ as follows:

Let $\mathbf{U}_w = \{w\} \cup Y_1$. For each $y \in Y$, we denote copies of y in Y_0 and Y_1 by y^0 and y^1, respectively. For each $y \in Y$, let

$$\mathbf{U}_{y^0} = \mathbf{U}_y^0 \cup \mathbf{U}_y^1 \text{ and } \mathbf{U}_{y^1} = \mathbf{U}_y^1,$$

where $\mathbf{U}_y^i$ is the minimal nbd of y^i in Y_i for $i = 0, 1$. Then X is a finite T_0-space, and we say that X is *the finite T_0-space constructed by a base space Y*.

Then, we have the following.

Proposition 5 *Let Y be a finite T_0-space and X the finite T_0-space constructed by a base space Y. Then we have that*

$$\mathrm{d}(X) = \mathrm{d}(Y) + 1,$$

where d *is any of the dimensions* Dind , dim, ind *and* Ind*.*

Proof (1) First, we show that

$$\mathrm{Dind}\, X = \mathrm{Dind}\, Y + 1.$$

Let $\mathcal{V} = \{\mathbf{U}_w\} = \{\{w\} \cup Y_1\}$. Then $X \setminus \bigcup \mathcal{V} = Y_0$. Hence,

$$\mathrm{Dind}\,(X \setminus \bigcup \mathcal{V}) = \mathrm{Dind}\, Y_0 = \mathrm{Dind}\, Y,$$

and hence, by Proposition 2, we have

$$\mathrm{Dind}\, X \leqslant \mathrm{Dind}\, Y + 1.$$

Next, we show that $\mathrm{Dind}\, X \geqslant \mathrm{Dind}\, Y + 1$ by induction on $\mathrm{Dind}\, Y$.

If $\mathrm{Dind}\, Y = -1$, then $X = \{w\}$ and hence $\mathrm{Dind}\, X = 0$.

Let $n \geqslant 0$ and we assume the proposition holds for every finite T_0-space X constructed by a base space Y with $\mathrm{Dind}\, Y \leqslant n-1$.

Let X be the space constructed by a base space Y with $\mathrm{Dind}\, Y = n$.

Let $\mathcal{V}$ be a disjoint family of X consisting of the minimal nbds in X.

(i) If $\mathbf{U}_w \in \mathcal{V}$, then it follows that $\mathcal{V} = \{\mathbf{U}_w\}$. Hence, we have that

$$\mathrm{Dind}\,(X \setminus \bigcup \mathcal{V}) = \mathrm{Dind}\, Y_0 = \mathrm{Dind}\, Y = n.$$

(ii) If $\mathbf{U}_w \notin \mathcal{V}$, then for $i = 0, 1$, we put

$$\mathcal{V}_i = \{V \in \mathcal{V} : V \text{ is the minimal nbd of a point in } Y_i\}.$$

Then $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$. For each $V \in \mathcal{V}_1$ there is $y \in Y$ such that $V = \mathbf{U}_y^1$. We put $W(V) = V \cup \mathbf{U}_y^0$. Then $W(V)$ is the minimal nbd of y^0 in X and $W(V) \cap V' = \emptyset$ for each $V' \in \mathcal{V} \setminus \{V\}$. Hence

$$\mathcal{V}' = \mathcal{V}_0 \cup \{W(V) : V \in \mathcal{V}_1\}$$

is a disjoint family by the minimal nbds of points in Y_0, and $\bigcup \mathcal{V} \subset \bigcup \mathcal{V}'$. Thus, we may assume that $\mathcal{V} = \mathcal{V}_0$.

Then $X \setminus \bigcup \mathcal{V} = \{w\} \cup Y_0' \cup Y_1'$, where Y_0' and Y_1' are copies of a subspace $Y' \subset Y$ in Y_0 and Y_1, respectively. Hence $X \setminus \bigcup \mathcal{V}$ can be considered as the finite T_0-space constructed by a base space Y'. Since Y' is a complement of a union of a disjoint family of open sets of Y and $\operatorname{Dind} Y = n$, it follows that $\operatorname{Dind} Y' \geqslant n - 1$.

If $\operatorname{Dind} Y' = n - 1$, then by the inductive assumption, we have

$$\operatorname{Dind}(X \setminus \bigcup \mathcal{V}) \geqslant \operatorname{Dind} Y' + 1 \geqslant (n-1) + 1 = n.$$

If $\operatorname{Dind} Y' = n$, then

$$\operatorname{Dind}(X \setminus \bigcup \mathcal{V}) \geqslant \operatorname{Dind} Y_0' = \operatorname{Dind} Y' = n.$$

Hence, we have that $\operatorname{Dind} X \geqslant n+1$, and hence $\operatorname{Dind} X = \operatorname{Dind} Y + 1$.

(2) Second, we show $\dim X = \dim Y + 1$.

Let $\dim Y = n$.
First, we show that $\dim X \leqslant n + 1$.

Let $\mathcal{V}$ be a finite open cover of X. There is $V_w \in \mathcal{V}$ such that $w \in V_w$. Since, $\{w\} \cup Y_1 \subset V_w$, we may assume that for each $V \in \mathcal{V}$ with $V \neq V_w$, V is the minimal nbd of a point of Y_0. Since $\{V \cap Y_0 : V \in \mathcal{V}\}$ is a finite open cover of Y_0 and Y_0 is homeomorphic to Y, there is an open refinement $\mathcal{W}$ of $\{V \cap Y_0 : V \in \mathcal{V}\}$ such that $\operatorname{ord}(\mathcal{W}) \leqslant n$.

For each $W \in \mathcal{W}$ we put

$$W^X = W \cup W^1,$$

where W^1 is a copy of W in Y_1, and

$$\mathcal{W}^X = \{\mathbf{U}_w\} \cup \{W^X : W \in \mathcal{W}\}.$$

It is easy to see that $\mathcal{W}^X$ is a refinement of $\mathcal{V}$, and $\operatorname{ord}(\mathcal{W}^X) = \operatorname{ord}(\mathcal{W}) + 1 \leqslant n + 1$. Hence $\dim X \leqslant n + 1$.

Next, we show that $\dim X \geqslant n + 1$. Since $\dim Y = n$, there is a finite open cover $\mathcal{V}$ of Y such that $\operatorname{ord}(\mathcal{W}) \geqslant n$ for each open refinement $\mathcal{W}$ of $\mathcal{V}$. We put

$$\mathcal{V}^X = \{\mathbf{U}_w\} \cup \{V^0 \cup V^1 : V \in \mathcal{V}\},$$

where V^i is a copy of V in Y_i for $i = 0, 1$. Let $\mathcal{W}^X$ be an open refinement of $\mathcal{V}^X$. There is $W_w \in \mathcal{W}^X$ such that $w \in W_w$. Then $W_w = \mathbf{U}_w$ and hence $W_w \cap Y_0 = \emptyset$.

It follows that $\{W \cap Y_0 : W \in \mathcal{W}^X \setminus \{W_w\}\}$ is an open cover of Y_0 and refines $\{V^0 : V \in \mathcal{V}\}$. Hence

$$\text{ord}\{W \cap Y_0 : W \in \mathcal{W}^X \setminus \{W_w\}\} \geqslant n.$$

Since $W \cap W_w = W \cap \mathbf{U}_w \neq \emptyset$ for each non-empty element $W \in \mathcal{W}^X$, it follows that $\text{ord}(\mathcal{W}^X) \geqslant n + 1$. Hence $\dim X \geqslant n + 1$.

(3) Third, we show $\text{ind}\, X = \text{ind}\, Y + 1$.

We show the equality by induction on $\text{ind}\, Y$. If $\text{ind}\, Y = -1$, then $X = \{w\}$ and hence $\text{ind}\, X = 0$.

Let $n \geqslant 0$ and we assume that $\text{ind}\, X = \text{ind}\, Y + 1$ holds for every finite T_0-space X constructed by a base space Y with $\text{ind}\, Y \leqslant n - 1$.

Let X be the space constructed by a base space Y with $\text{ind}\, Y = n$.

We show that

$$\operatorname{ind}(\mathrm{Bd}_X(\mathbf{U}_x)) \leqslant n$$

for every $x \in X$, where $\mathbf{U}_x$ is the minimal nbd of $x \in X$.

(i) First, we show that $\operatorname{ind}(\mathrm{Bd}_X(\mathbf{U}_w)) = n$. Note that $\mathrm{Bd}_X(\mathbf{U}_w) = Y_0$, therefore

$$\operatorname{ind}(\mathrm{Bd}_X(\mathbf{U}_w)) = \operatorname{ind} Y_0 = n.$$

(ii) Let $x \in X$ with $x \neq w$. Then $x = y^0$ or $x = y^1$ for some $y \in Y$. Then

$$\mathrm{Bd}_X(\mathbf{U}_{y^0}) = \{w\} \cup \mathrm{Bd}_{Y_0}(\mathbf{U}_y^0) \cup \mathrm{Bd}_{Y_1}(\mathbf{U}_y^1)$$

and

$$\mathrm{Bd}_X(\mathbf{U}_{y^1}) = \{w\} \cup \mathrm{Cl}_{Y_0}(\mathbf{U}_y^0) \cup \mathrm{Bd}_{Y_1}(\mathbf{U}_y^1)$$

$$= \Big(\{w\} \cup \mathrm{Bd}_{Y_0}(\mathbf{U}_y^0) \cup \mathrm{Bd}_{Y_1}(\mathbf{U}_y^1)\Big) \cup \mathrm{Cl}_{Y_0}(\mathbf{U}_y^0).$$

Since for the copies $\mathsf{Bd}_{Y_0}(\mathbf{U}_y^0)$ and $\mathsf{Bd}_{Y_1}(\mathbf{U}_y^1)$ of the same subspace of Y, and

$$\mathsf{ind}(\mathsf{Bd}_{Y_0}(\mathbf{U}_y^0)) = \mathsf{ind}(\mathsf{Bd}_{Y_1}(\mathbf{U}_y^1)) \leqslant n-1.$$

Since $\mathsf{Bd}_X(\mathbf{U}_{y^0})$ is the finite T_0-space constructed by a base space homeomorphic to $\mathsf{Bd}_{Y_0}(\mathbf{U}_y^0)$, by the inductive assumption, it follows that

$$\mathsf{ind}(\mathsf{Bd}_X(\mathbf{U}_{y^0})) \leqslant n.$$

Moreover, since $\mathsf{ind}(\mathsf{Cl}_{Y_0}(\mathbf{U}_y^0)) \leqslant \mathsf{ind}(Y_0) = n$, by the sum theorem for ind, we have

$$\mathsf{ind}(\mathsf{Bd}_X(\mathbf{U}_{y^1})) \leqslant n.$$

This implies the equality $\mathsf{ind}(X) = \mathsf{ind}(Y) + 1$.

(4) Finally, we show $\mathsf{Ind}\, X = \mathsf{Ind}\, Y + 1$.

We show the equality by induction on $\mathsf{Ind}\, Y$.

If $\mathsf{Ind}\, Y = -1$, then $X = \{w\}$ and hence $\mathsf{Ind}\, X = 0$.

Let $n \geqslant 0$ and we assume that $\operatorname{Ind} X = \operatorname{Ind} Y + 1$ holds for every finite T_0-space X constructed by a base space Y with $\operatorname{Ind} Y \leqslant n - 1$.

Let X be the space constructed by a base space Y with $\operatorname{Ind} Y = n$.

We show that $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant n$ for every closed set F of X, where $\mathbf{U}_F$ is the minimal open set containing F.

Let F be an arbitrary closed subset of X.

(i) First, we consider on the closed set $F = \{w\}$. Then we have $\operatorname{Bd}_X(\mathbf{U}_{\{w\}}) = \operatorname{Bd}_X(\mathbf{U}_w) = Y_0$ and therefore, $\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_{\{w\}})) = n$.

(ii) We suppose that $w \in F$. Then $\operatorname{Bd}_X(\mathbf{U}_F) \subseteq Y_0$. Therefore,

$$\operatorname{Ind}(\operatorname{Bd}_X(\mathbf{U}_F)) \leqslant \operatorname{Ind}(Y_0) = n.$$

(iii) We assume that $w \notin F$. Observe that $F \subseteq Y_0$. The copy of F in Y_1 we will denote also by F. Then for the minimal open set $\mathbf{U}_F$ in X containing F we have $\mathbf{U}_F = \mathbf{U}_F^0 \cup \mathbf{U}_F^1$ and

$$\mathrm{Bd}_X(\mathbf{U}_F) = \{w\} \cup \mathrm{Bd}_{Y_0}(\mathbf{U}_F^0) \cup \mathrm{Bd}_{Y_1}(\mathbf{U}_F^1),$$

where $\mathbf{U}_F^i$ is the minimal open set in Y_i containing F for $i = 0, 1$. Since for the copies $\mathrm{Bd}_{Y_0}(\mathbf{U}_F^0)$ and $\mathrm{Bd}_{Y_1}(\mathbf{U}_F^1)$ of the same subspace of Y we have

$$\mathrm{Ind}(\mathrm{Bd}_{Y_0}(\mathbf{U}_F^0)) = \mathrm{Ind}(\mathrm{Bd}_{Y_1}(\mathbf{U}_F^1)) \leqslant n - 1,$$

By the inductive assumption, it follows that

$$\mathrm{Ind}(\mathrm{Bd}_X(\mathbf{U}_F)) \leqslant n.$$

This implies the equality $\mathrm{Ind}\, X = \mathrm{Ind}\, Y + 1$. $\square$

Example 4 For each natural number n, there is a finite T_0-space X_n such that

$$\mathrm{Dind}\, X_n = \dim X_n = \mathrm{ind}\, X_n = \mathrm{Ind}\, X_n = n.$$

Proof We construct the finite T_0-space X_n by induction on n. Let $X_0 = \{0\}$. It is obvious that

$$\mathrm{Dind}\, X_0 = \dim X_0 = \mathrm{ind}\, X_0 = \mathrm{Ind}\, X_0 = 0.$$

Let $X_1 = \{0, 1, 2\}$ and

$$\tau = \{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, X_1\}.$$

It is obvious that (X_1, τ) is a finite T_0-space. Furthermore, it is easy to see that

$$\mathrm{Dind}\, X_1 = \dim X_1 = \mathrm{ind}\, X_1 = \mathrm{Ind}\, X_1 = 1.$$

Let $n \geqslant 1$ and we assume that X_n is constructed. Let X_{n+1} be the finite T_0-space constructed by a base space X_n.

Then, it follows from Proposition 6 that

$$\operatorname{Dind} X_{n+1} = \dim X_{n+1} = \operatorname{ind} X_{n+1}$$
$$= \operatorname{Ind} X_{n+1} = n+1. \ \square$$

Example 5 We consider the finite T_0-space $X = \{x_1, \ldots, x_n\}$, where $n \geqslant 4$, which has as a basis the family $\beta = \{\emptyset, \{x_k\}\} \cup \{\{x_k, x_i\} : i \neq k\}$, for some fixed k. Then

$$\operatorname{Dind} X = \operatorname{Ind} X = \operatorname{ind} X = 1, \text{ and}$$

$$\dim X = n-2.$$

Proof of Example 3 Let m, n be natural numbers with $m \leqslant n$. By Example 5, there is a finite T_0-space X_n such that

$$\operatorname{Dind}(X_n) = 1 \text{ and } \dim X_n = n.$$

On the other hand, by Example 4, we have a finite T_0-space X_m such that

$$\operatorname{Dind} X_m = \dim X_m = m.$$

Then $Z_{mn} = X_n \oplus X_m$ is a finite T_0-space such that $\operatorname{Dind} Z_{mn} = m$ and $\dim Z_{mn} = n$. $\square$

5. Open questions

Question 1 Is there a characterization of Dind of a finite T_0-space by using matrices?

Question 2 Is there an algorithm for the computation of Dind of a finite T_0-space?

Question 3 Let d be one the dimensions ind, Ind and dim. What conditions must an Alexandroff T_0-space X satisfy so that Dind $X = \mathrm{d}(X)$?

Question 4 What conditions must an Alexandroff T_0-space X satisfy so that

$$\mathrm{ind}(X) = \dim(X) = \mathrm{Ind}(X) = \mathrm{Dind}\,(X)?$$

Question 5 Let $k \in \{0, 1, \ldots\}$. Is there a universal space in the class of all Alexandroff T_0-spaces X of weight $\leqslant \tau$, where τ is an infinite cardinal, such that Dind $X \leqslant k$?

References

[B1] Bass, D. W., Covering dimension and quasi-order spaces, J. London Math. Soc. (2) 1 (1969), 505–508.

[B2] Bass, D. W., Dimension and finite spaces, J. London Math. Soc. 44 (1969), 159–162.

[BW] Berghammer, R. and Winter, M., Order- and graph-theoretic investigation of dimensions of finite topological spaces and Alexandroff spaces, Monatsh. Math. 90 (2019), no. 1, 33–78.

[CP] Chatyrko, V. A. and Pasynkov, B. A., On sum and product theorems for dimension Dind, Houston J. Math. 28 (2002), no. 1, 119–131.

[CHH] Chatyrko, V. A., Han, Sang-Eon and Hattori, Y., The small inductive dimension of Alexandroff spaces, Filomat 30 (2016), no. 11, 3007–3014.

[GHMS] Georgiou, D. N., Hattori Y., Megaritis, A. C. and Sereti, F.The dimension Dind of finite topological T_0-spaces, Math. Slovaca 72 (2022) 813–829.

[GM] Georgiou, D. N., and Megaritis, A. C., Covering dimension and finite spaces, Appl. Math. Comput. 218 (2011), no. 7, 3122–3130.

[GMM] Georgiou, D. N., Megaritis, A. C. and Moshokoa, S. P., Small inductive dimension and Alexandroff topological spaces, Topology Appl. 168 (2014), 103–119.