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Representations of Domains via CF-approximation Spaces

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Introduction: Representation of Domains

The representation of domains, we mean:

- some general way
- one can characterize a domain.
- use a suitable family of a mathematical structure.
- equipped with set-inclusion order.

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Examples

Continuous domain can be represented

- Abstract bases $(B, \prec)$: using round ideals.([2,3])
- c-infs $(C, Con, \vdash)$: using states;([7])
- Formal topology: using (psudo)-formal points in suitable topological spaces.([9])
- ($\mathbb{F}$-augmented generalized) closure spaces; ([12])
- (Attribute continuous) formal contexts $(P_o, P_a, \models, \mathcal{F}_\tau)$. ([8])

Observations

Clearly,

- Representation via abstract bases is most natural due to its simplicity;
- The study scope of abstract bases is little narrow;
- Abstract bases $(B, \prec)$ are special generalized approximation spaces (U, R)(GA-space, for short).

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Our Intend

And we intend to

- generalize an abstract base to a CF-approximation space $(U, R, \mathcal{F})$, a GA-space (U, R), with some consistent family $\mathcal{F}$ of some finite sets.
- hope that continuous domains can be represented via CF-approximation spaces.

Basic notions

Definition

Let $(U, \prec)$ be a set equipped with a binary relation $\prec$. We say $\prec$ fully transitive if it is transitive and satisfies the strong interpolation property:

$$\forall |F| < \infty, F \prec z \Rightarrow \exists y \prec z \text{ such that } F \prec y,$$

where $F \prec y$ means for all $t \in F$, $t \prec y$. We call $(B, \prec)$ an abstract basis if $\prec$ is fully transitive.

Definition

Let $(B, \prec)$ be an abstract basis. A non-empty subset I of B is a round ideal if

(1) $\forall y \in I, x \prec y \Rightarrow x \in I$;

(2) $\forall x, y \in I, \exists z \in I$ such that $x \prec z$ and $y \prec z$.

All the round ideals of B in set-inclusion order is called the round ideal completion of B, denoted by $RI(B)$.

Theorem

For all abstract basis $(B, \prec)$, $RI(B)$ is a continuous domain. Conversely, if P is a continuous domain with a base B, then $(B, \ll)$ with $\ll$ be the restriction of the way-below relation to B, is an abstract basis and $RI(B, \ll) \cong (P, \leqslant)$.

- A *generalized approximation space* (*GA-space*, for short) is a pair (U, R), where U is a set, R is a binary relation on U.
- Define $R_s, R_p : U \to \mathcal{P}(U)$ such that for all $x \in U$, $R_s(x) = \{y \in U \mid xRy\}, R_p(x) = \{y \in U \mid yRx\}$.

Definition

Let (U, R) be a GA-space. For $A \subseteq U$, define

$$\underline{R}(A) = \{x \in U \mid \ R_s(x) \subseteq A\},$$
$$\overline{R}(A) = \{x \in U \mid \ R_s(x) \cap A \neq \emptyset\}.$$

The operators $\underline{R}, \overline{R} : \mathcal{P}(U) \to \mathcal{P}(U)$ are respectively called the lower and upper approximation operators in (U, R), which are key notions in GA-spaces.

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Lemma

Let (U, R) be a GA-space. Then the lower and upper approximation operators $\underline{R}$ and $\overline{R}$ have the following properties.

(1) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$, *where* A^c *is the complement of* $A \subseteq U$.

(2) $\underline{R}(U) = U$, $\overline{R}(\emptyset) = \emptyset$.

(3) *Let* $\{A_i \mid i \in I\} \subseteq \mathcal{P}(U)$. *Then* $\underline{R}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \underline{R}(A_i)$, $\overline{R}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \overline{R}(A_i)$.

(4) *If* $A \subseteq B \subseteq U$, *then* $\underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.

(5) *For all* $x \in U$, $\overline{R}(\{x\}) = R_p(x)$.

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Proposition

Let (U, R) be a GA-space. Then $\mathcal{T}_R = \{A \subseteq U \mid A \subseteq \underline{R}(A)\}$ is an Alexandrov topology.

Topology $\mathcal{T}_R$ is called a *topology induced by relation* R. And a set A in $(U, \mathcal{T}_R)$ is closed iff $\overline{R}(A) \subseteq A$.

Abstract base to CF-approximation space

- Change an abstract base $(B, \prec)$ to the triple $(B, \prec, \{\{b\} \mid b \in B\})$.
- $\{\downarrow^{\prec} b \mid b \in B\}$ is a base of $RI(B)$, where $\downarrow^{\prec} b = \{c \in B \mid c \prec b\}$.
- Change an abstract base $(B, \prec)$ to a GA-space (U, R) with R being transitive.

Abstract base to CF-approximation space

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- $\{\downarrow^{\prec} b \mid b \in B\}$ is a base of $RI(B)$, where $\downarrow^{\prec} b = \{c \in B \mid c \prec b\}$.
- Change an abstract base $(B, \prec)$ to a GA-space (U, R) with R being transitive.
- Change the family $\{\{b\} \mid b \in B\}$ to a suitable family $\mathcal{F}$ of some finite subsets of U.
- The family $\mathcal{F}$ can also induce a base of a continuous domain.

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CF-approximation spaces

Definition

Let (U, R) be a GA-space, R a transitive relation and $\mathcal{F} \subseteq \mathcal{P}_{fin}(U) \cup \{\emptyset\}$. If for all $F \in \mathcal{F}$, whenever $K \subseteq_{fin} \overline{R}(F)$, there always exists $G \in \mathcal{F}$ such that

$$K \subseteq \overline{R}(G), G \subseteq \overline{R}(F),$$

then $(U, R, \mathcal{F})$ is called a generalized approximation space with consistent family of finite subsets, or a CF-approximation space, for short.

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CF-closed sets

Definition

Let $(U, R, \mathcal{F})$ be a CF-approximation space, $E \subseteq U$. If for all $K \subseteq_{fin} E$, there always exists $F \in \mathcal{F}$ such that $K \subseteq \overline{R}(F) \subseteq E$ and $F \subseteq E$, then E is called a CF-closed set of $(U, R, \mathcal{F})$. The collection of all CF-closed sets of $(U, R, \mathcal{F})$ is denoted by $\mathfrak{C}(U, R, \mathcal{F})$.

Remark

(1) *If $\emptyset \in \mathfrak{C}(U, R, \mathcal{F})$, then $\emptyset \in \mathcal{F}$ by $\overline{R}(\emptyset) = \emptyset$.*

(2) *For CF-approximation space $(U, R, \mathcal{F})$, if $\mathcal{F} = \{\{x\} \mid x \in U\}$, then (U, R) is an abstract base, and all the CF-closed sets of $(U, R, \mathcal{F})$ are precisely all the round ideals of (U, R).*

Properties of CF-closed sets

Proposition

Let $(U, R, \mathcal{F})$ be a CF-approximation space. If $E \in \mathfrak{C}(U, R, \mathcal{F})$, then E is a closed set in $\mathcal{T}_R$.

Proposition

Let $(U, R, \mathcal{F})$ be a CF-approximation space, then
(1) for any $F \in \mathcal{F}$, $\overline{R}(F) \in \mathfrak{C}(U, R, \mathcal{F})$;
(2) if $E \in \mathfrak{C}(U, R, \mathcal{F})$, $A \subseteq E$, then $\overline{R}(A) \subseteq E$;
(3) if $\{E_i\}_{i\in I} \subseteq \mathfrak{C}(U, R, \mathcal{F})$ is a directed family, then $\bigcup_{i\in I} E_i \in \mathfrak{C}(U, R, \mathcal{F})$.

The above proposition above shows that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a dcpo.

Characterizations of CF-closed sets

Proposition

Let $(U, R, \mathcal{F})$ be a CF-approximation space. Then the following statements are equivalent:

(1) $E \in \mathfrak{C}(U, R, \mathcal{F})$;

(2) *The family* $\mathcal{A} = \{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}$ *is directed and* $E = \bigcup \mathcal{A}$;

(3) *There exists a family* $\{F_i\}_{i \in I} \subseteq \mathcal{F}$ *such that* $\{\overline{R}(F_i)\}_{i \in I}$ *is directed, and* $E = \bigcup_{i \in I} \overline{R}(F_i)$;

(4) *There always exists* $F \in \mathcal{F}$ *such that* $K \subseteq \overline{R}(F) \subseteq E$ *whenever* $K \subseteq_{fin} E$.

Way-below relation $\ll$ in $(\mathfrak{C}(U,R,\mathcal{F}),\subseteq)$

Theorem

Let $(U,R,\mathcal{F})$ be a CF-approximation space, $E_1,E_2\in\mathfrak{C}(U,R,\mathcal{F})$. Then $E_1\ll E_2$ if and only if there exists $F\in\mathcal{F}$ such that $E_1\subseteq\overline{R}(F)$ and $F\subseteq E_2$.

Corollary

Let $(U,R,\mathcal{F})$ be a CF-approximation space, $E\in\mathfrak{C}(U,R,\mathcal{F})$, $F\in\mathcal{F}$. The following statements hold:

(1) If $F\subseteq E$, then $\overline{R}(F)\ll E$;

(2) $\overline{R}(F)\ll\overline{R}(F)$ if and only if there exists $G\in\mathcal{F}$, such that $G\subseteq\overline{R}(G)=\overline{R}(F)$.

Representation Theorem

Theorem

Let $(U, R, \mathcal{F})$ be a CF-approximation space. Then $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain.

Theorem

Let L be a cont. domain, $\mathcal{F}_L = \{F \subseteq_{fin} L \mid F \text{ has a top}\}$ and $R_L = \ll$. Then $\mathfrak{C}(L, R_L, \mathcal{F}_L) = \{\Downarrow x \mid x \in L\}$.

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Theorem

(Representation Theorem) *A poset L is a continuous domain iff there is a CF-approximation space $(U, R, \mathcal{F})$ such that $L \cong (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.*

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Representations of some special domains

We have some special cases of representation theorem.

Theorem

Let $(U, R, \mathcal{F})$ be a CF-approximation space. If $(\{\overline{R}(F) \mid F \in \mathcal{F}\}, \subseteq)$ is a cusl (resp., sup-semilattice with bottom element), then $\mathfrak{C}(U, R, \mathcal{F})$ is a bc-domain(resp., continuous lattice). Conversely, above types of domains can be respectively represented in this way.

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Topological F-approximation spaces

Definition

Let R be a preorder, $\mathcal{F} \subseteq \mathcal{P}_{fin}(U) \cup \{\emptyset\}$. *Then* $(U, R, \mathcal{F})$ *is called a topological F-approximation space.*

Remark

A topological F-approximation space must be a CF-approximation space.

Representations of Algebraic domains

Theorem

Let $(U, R, \mathcal{F})$ be a topological F-approximation space. Then $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an algebraic domain. Conversely, any algebraic domain can be represented by some topological F-approximation space.

let $(L, \leqslant)$ be an algebraic domain. Set a topological F-approximation space $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$, where

$$\mathcal{F}_{K(L)} = \{F \subseteq_{fin} K(L) \mid F \text{ has top element}\},$$

$R_{K(L)} = \leqslant$ is a partial order. Then we have that

$$\mathfrak{C}(K(L), R_{K(L)}, \mathcal{F}_{K(L)}) = \{\downarrow x \cap K(L) \mid x \in L\}.$$

Since L is an algebraic domain, we know that

$$(\{\downarrow x \cap K(L) \mid x \in L\}, \subseteq) \cong (L, \leqslant).$$

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CF-approximation relations

Definition

Let $(U_1, R_1, \mathcal{F}_1)$, $(U_2, R_2, \mathcal{F}_2)$ *be CF-approximation spaces, and* $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ *a binary relation. If*

1. for all $F \in \mathcal{F}$, *there is* $G \in \mathcal{F}_2$ *such that* $F\ \Theta\ G$;

2. $(\forall F, F' \in \mathcal{F}_1,\ G \in \mathcal{F}_2)\ (F \subseteq \overline{R_1}(F'),\ F\ \Theta\ G) \Rightarrow (F'\ \Theta\ G)$;

3. $(\forall F \in \mathcal{F}_1,\ G, G' \in \mathcal{F}_2)(F\ \Theta\ G,\ G' \subseteq \overline{R_2}(G)) \Rightarrow (F\ \Theta\ G')$;

4. for all $F \in \mathcal{F}_1$, $G \in \mathcal{F}_2$, *if* $F\ \Theta\ G$, *then there are* $F' \in \mathcal{F}_1$, $G' \in \mathcal{F}_2$ *s. t.* $F' \subseteq \overline{R_1}(F)$, $G \subseteq \overline{R_2}(G')$ *and* $F'\ \Theta\ G'$; *and*

5. for all $F \in \mathcal{F}_1$, $G_1, G_2 \in \mathcal{F}_2$, *if* $F\ \Theta\ G_1$ *and* $F\ \Theta\ G_2$, *then there is* $G_3 \in \mathcal{F}_2$ *s. t.* $G_1 \cup G_2 \subseteq \overline{R_2}(G_3)$ *and* $F\ \Theta\ G_3$,

then Θ *is called a CF-approximation relation from* $(U_1, R_1, \mathcal{F}_1)$ *to* $(U_2, R_2, \mathcal{F}_2)$.

Identities and compsitions

- Given a CF-approximation space $(U, R, \mathcal{F})$, define a binary relation $\mathrm{Id}_{(U,R,\mathcal{F})} \subseteq \mathcal{F} \times \mathcal{F}$ such that for all $F, G \in \mathcal{F}$, $(F, G) \in \mathrm{Id}_{(U,R,\mathcal{F})} \Leftrightarrow G \subseteq \overline{R}(F)$.
- Let $(U_1, R_1, \mathcal{F}_1)$, $(U_2, R_2, \mathcal{F}_2)$, $(U_3, R_3, \mathcal{F}_3)$ be CF-approximation spaces, $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$, $\Upsilon \subseteq \mathcal{F}_2 \times \mathcal{F}_3$ be CF-approximation relations. Define $\Upsilon \circ \Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_3$, the composition of Υ and Θ by that for any $F_1 \in \mathcal{F}_1, F_3 \in \mathcal{F}_3$, $(F_1, F_3) \in \Upsilon \circ \Theta$ iff there exists $F_2 \in \mathcal{F}_2$ satisfying $(F_1, F_2) \in \Theta$ and $(F_2, F_3) \in \Upsilon$.

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Categories CF-GA and CDOM

- **CF-GA** objects: CF-approximation spaces;
  morphisms: CF-approximation relations.
  The identities are defined above.
  Compositions are compositions of binary relations
- **CDOM** objects: continuous domains;
  morphisms: Scott continuous maps.
  Identity map and compositions of maps

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Induced Scott continuous maps

Let Θ be a CF-approximation relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. For all $F \in \mathcal{F}_1$, set $\widetilde{\Theta}(F) = \bigcup\{\overline{R_2}(G) \mid F \,\Theta\, G \text{ and } G \in \mathcal{F}_2\}$. Define a map $f_\Theta : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \longrightarrow \mathcal{P}(U_2)$ such that for all $E \in \mathfrak{C}(U_1, R_1, \mathcal{F}_1)$,
$f_\Theta(E) = \bigcup\{\widetilde{\Theta}(F) \mid F \subseteq E \text{ and } F \in \mathcal{F}_1\}$.

Theorem

Let Θ be a CF-approximation relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. Then f_Θ is a Scott continuous map from $\mathfrak{C}(U_1, R_1, \mathcal{F}_1)$ to $\mathfrak{C}(U_2, R_2, \mathcal{F}_2)$.

Induced CF-approximation relations

Theorem

Let $f : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \longrightarrow \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$ be a Scott continuous map. Define $\Theta_f \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ such that

$$\forall F \in \mathcal{F}_1, G \in \mathcal{F}_2, F \ \Theta_f \ G \Leftrightarrow G \subseteq f(\overline{R_1}(F)).$$

Then Θ_f is a CF-approximation relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$.

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Equivalence of CF-GA and CDOM

Theorem

Let $f : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \longrightarrow \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$ be a Scott continuous map , Θ a CF-approximation relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. Then $\Theta_{f_\Theta} = \Theta$ and $f_{\Theta_f} = f$.

Theorem

*The categories **CF-GA** and **CDOM** are equivalent.*

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Thank You!