Free directed-completions and double-dense-completions of S-posets

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Let S be a monoid.

- A $(right)$ *S-act* is a set A equipped with a map

$$\lambda : A \times S \longrightarrow A$$

such that

$$\lambda(a, 1) = a \text{ and } \lambda(a, st) = \lambda(\lambda(a, s), t)$$

for all $a \in A$ and $s, t \in S$.

- Over the past three decades, an extensive theory of the properties of S-acts has been developed.

- A comprehensive survey was published in 2000 by Kilp et al.: M. Kilp, U. Knauer, A. Mikhalev, Monoids, Acts and Categories, Walter de Gruyter, Berlin, 2000.

Let S be a pomonoid.

- A $(right)$ S-$poset$ is a poset A together with a monotone map $\lambda : A \times S \longrightarrow A$, called the $action\ of\ S\ on\ A$, such that

$$a1 = a \quad \text{and} \quad a(st) = (as)t,$$

for any $a \in A$ and $s, t \in S$, where $A \times S$ is considered as a poset with componentwise order and we denote $\lambda(a, s)$ by as.

S-poset, as a particular kind of ordered algebraic structure, we will consider two completions about it,

1. free directed-completion,
2. double-dense-completion.

Base on this, we consider the intermediate structure and provide a construction for the Δ_1-object of an S-poset.

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Free directed-completions of S-posets

Definition 1

An S-poset A is

(i) *directed complete* if it is directed complete as a poset. Further, it is called *continuous* if for every directed subset $D \subseteq A$ and $s \in S$, $(\bigvee D)s = \bigvee(Ds)$;

(ii) *complete* if it is complete as a poset.

Free directed-completions of S-posets

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Definition 2

An *S-poset map* (or *homomorphism*) is an action preserving monotone map between S-posets, and is an *S-poset embedding* if it is also an order-embedding.
Furthermore, it is an *S-poset isomorphism* if it is also surjective.

Free directed-completions of S-posets

Definition 3

An *S-poset extension* of an S-poset A is a pair (α, C), where C is an S-poset and $\alpha : A \longrightarrow C$ is an S-poset embedding.

Free directed-completions of S-posets

Definition 3

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Definition 4

A *directed-completion* of an S-poset A is an S-poset extension (α, C) such that

(i) C is a directed complete S-poset;

(ii) $\alpha(A)$ is directed-join-dense in C.

Such a completion is *continuous* if C is continuous, and is *compact* if $\alpha(a)$ is a compact element in C for each $a \in A$.

Theorem 1

Let (α, C) be a directed-completion of an S-poset A. Then the following statements are equivalent:

(i) (α, C) is compact and continuous;

(ii) for any continuous and directed complete S-poset E, if $f : A \longrightarrow E$ is an S-poset homomorphism, then there exists a unique S-poset homomorphism $\bar{f} : C \longrightarrow E$ preserving directed joins such that $\bar{f} \circ \alpha = f$;

(iii) there exists an S-poset isomorphism $\eta : C \longrightarrow Id(A)$ with $\eta(\alpha(a)) = \downarrow a$ for all $a \in A$, where the action $\lambda : Id(A) \times S \longrightarrow Id(A)$ is given by

$$\lambda(I, s) = \downarrow \{as \mid a \in I\}$$

for all $I \in Id(A)$ and $s \in S$.

This completion is called as the *free directed-completion* of A.

- Denote it by $\mathbb{F}_{\sqcup}(A)$, and the embedding of A into $\mathbb{F}_{\sqcup}(A)$ by α_A.

Given two S-posets A and B, and an S-poset homomorphism $f : A \longrightarrow B$, there is a unique S-poset homomorphism

$$\mathbb{F}_{\sqcup}(f) : \mathbb{F}_{\sqcup}(A) \longrightarrow \mathbb{F}_{\sqcup}(B)$$

preserving directed joins such the diagram in Figure 1 commutes.

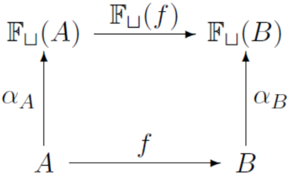

Fig. 1. The commutative diagram for the free directed-completions

In fact, the formula for $\mathbb{F}_{\sqcup}(f)$ has the form, for any $y \in \mathbb{F}_{\sqcup}(A)$,

$$\mathbb{F}_{\sqcup}(f)(y) = \bigvee^{\uparrow}\{\alpha_B \circ f(x) \mid x \in A, \alpha_A(x) \leqslant y\}.$$

Dually, we denote the free filtered-completion of an S-poset A by $\mathbb{F}_{\sqcap}(A)$, and the embedding of A into $\mathbb{F}_{\sqcap}(A)$ by β_A.

• We write $\mathbb{F}_{\sqcap}(f)$ for the preserving filtered meets S-poset homomorphism extension of an S-poset homomorphism $f : A \longrightarrow B$, which is given by

$$\mathbb{F}_{\sqcap}(f)(y) = \bigwedge^{\downarrow}\{\beta_B \circ f(x) \mid x \in A, y \leqslant \beta_A(x)\}.$$

for any $y \in \mathbb{F}_{\sqcap}(A)$.

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Intermediate structures

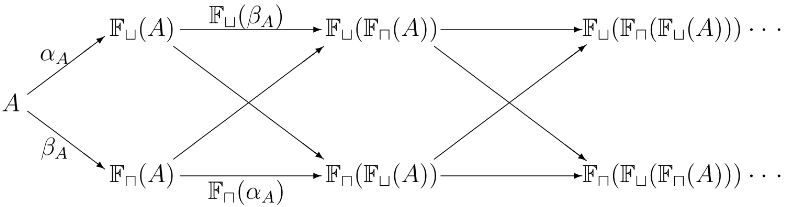

Fig. 2. The hierarchy of completions of S-poset A

Intermediate structures

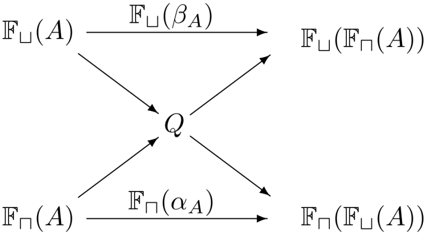

Fig. 3. An interpolant Q in the hierarchy of completions of S-poset A

Intermediate structures

Let

$$Int(A) = \mathbb{F}_{\sqcup}(A) \cup \mathbb{F}_{\sqcap}(A).$$

Define an order relation $\sqsubseteq$ on $Int(A)$ by, for all $y_1, y_2 \in \mathbb{F}_{\sqcup}(A)$ and $x_1, x_2 \in \mathbb{F}_{\sqcap}(A)$,

- $y_1 \sqsubseteq y_2 \Leftrightarrow y_1 \leqslant_{\sqcup} y_2$;
- $x_1 \sqsubseteq x_2 \Leftrightarrow x_1 \leqslant_{\sqcap} x_2$;
- $y_1 \sqsubseteq x_1 \Leftrightarrow \forall\, p, q \in A,$ if $\alpha_A(p) \leqslant y_1$ and $x_1 \leqslant \beta_A(q)$ then $p \leqslant q$;
- $x_1 \sqsubseteq y_1 \Leftrightarrow \exists\, p \in A$ such that $x_1 \leqslant \beta_A(p)$ and $\alpha_A(p) \leqslant y_1$.

Define $= : \sqsubseteq \bigcap \sqsupseteq$,
and define an order relation $\leqslant$ on $Int(A)$ by ${}^{\sqsubseteq}\!/\!_{=}$. Thus,

- $\leqslant$ is a partial order.

Intermediate structures

Denote the action of S-poset $\mathbb{F}_{\sqcup}(A)$ by $\lambda_{\sqcup}$ and the action of S-poset $\mathbb{F}_{\sqcap}(A)$ by $\lambda_{\sqcap}$.

Define $\lambda_{Int} : Int(A) \times S \longrightarrow Int(A)$ by

$$\lambda_{Int}(m, s) = \begin{cases} \lambda_{\sqcup}(m, s), & m \in \mathbb{F}_{\sqcup}(A), \\ \lambda_{\sqcap}(m, s), & m \in \mathbb{F}_{\sqcap}(A). \end{cases}$$

for all $m \in Int(A)$ and $s \in S$,

Proposition 1

$Int(A)$ is an S-poset.

We refer to $Int(A)$ with order $\leqslant$ and action λ_{Int} as the *intermediate structure*.

Intermediate structures

Define $F : Int(A) \longrightarrow \mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(A))$, by for all $a \in Int(A)$,

$$F(a) = \begin{cases} \mathbb{F}_{\sqcup}(\beta_A)(a), & a \in \mathbb{F}_{\sqcup}(A), \\ \alpha_{\mathbb{F}_{\sqcap}(A)}(a), & a \in \mathbb{F}_{\sqcap}(A), \end{cases}$$

and define $G : Int(A) \longrightarrow \mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(A))$, by for all $a \in Int(A)$,

$$G(a) = \begin{cases} \beta_{\mathbb{F}_{\sqcup}(A)}(a), & a \in \mathbb{F}_{\sqcup}(A), \\ \mathbb{F}_{\sqcap}(\alpha_A)(a), & a \in \mathbb{F}_{\sqcap}(A). \end{cases}$$

Proposition 2

F and G are S-poset embeddings.

Intermediate structures

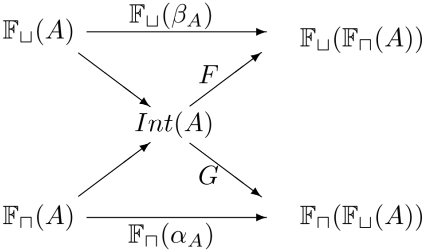

Fig. 4. The relationships between intermediate structure and completions

Corollary 1

S-poset $Int(A)$ is an interpolant in Figure 3.

Intermediate structures

What is the largest interpolant Q?

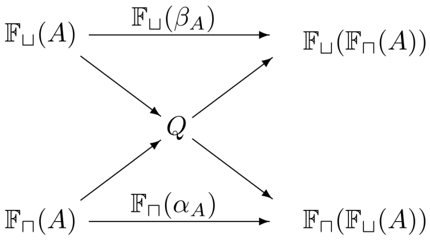

Fig. 3. An interpolant Q in the hierarchy of completions of S-poset A

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Double-dense-completions of S-posets

Definition 5

A *double-dense-completion* of an S-poset A is an S-poset extension (α, C) such that
(i) C is a complete S-poset;
(ii) $\alpha(A)$ is both join-dense and meet-dense in C.

Double-dense-completions of S-posets

Definition 5

A *double-dense-completion* of an S-poset A is an S-poset extension (α, C) such that
(i) C is a complete S-poset;
(ii) $\alpha(A)$ is both join-dense and meet-dense in C.

Definition 6

An S-poset A is called *LU-compatible* if for any $s, t \in S$ and $X = \lambda(Y^u, s)$ for some $Y \subseteq A$,

$$(\lambda(X, t))^l = (\lambda(X^{lu}, t))^l.$$

Double-dense-completions of S-posets

Example 1

(1) Every complete S-poset is LU-compatible.

(2) Let $S = \{0, x, 1\}$. Define $\cdot$ and $\leqslant$ on S as follows:

$\cdot$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

Table 1. The values of operation $\cdot$

Fig. 5. Hasse diagram of the poset S

Clearly, S is a pomonoid.

Double-dense-completions of S-posets

Let $A = \{a, b, c, d, e\}$. Define an order $\leqslant$ on A as shown in Figure 6 and define an action $\lambda : A \times S \longrightarrow A$ as follows:

λ	a	b	c	d	e
0	e	e	e	e	e
x	c	d	e	e	e
1	a	b	c	d	e

Table 2. The values of action λ

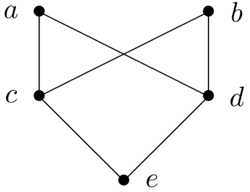

Fig. 6. Hasse diagram of the poset A

Then A is an LU-compatible S-poset which is not complete.

Double-dense-completions of S-posets

Let A be an S-poset and let $DM_1(A) = \{X \subseteq A \mid X^{ul} = X\}$.

Define $\alpha : A \longrightarrow DM_1(A)$ by $\alpha(a) = \downarrow a$ for all $a \in A$, and define $\lambda_1 : DM_1(A) \times S \longrightarrow DM_1(A)$ by

$$\lambda_1(X, s) = (\lambda_A(X^u, s))^l$$

for all $X \in DM_1(A)$ and $s \in S$.

Theorem 4

Let A be an S-poset. Then $(\alpha, DM_1(A))$ is a double-dense-completion of A if and only if A is LU-compatible.

Definition 7

An S-poset A is *UL-compatible* if for any $s, t \in S$ and $X \subseteq A$ with $X^{ul} = X$,

$$(\lambda(X, st))^{ul} = (\lambda((\lambda(X, s))^{ul}, t))^{ul}.$$

Let $DM_2(A) = \{X \subseteq A \mid X^{ul} = X\}$.
Define $\alpha : A \longrightarrow DM_2(A)$ by $\alpha(a) = \downarrow a$ for all $a \in A$, and define $\lambda_2 : DM_2(A) \times S \longrightarrow DM_2(A)$ by

$$\lambda_2(X, s) = (\lambda(X, s))^{ul}$$

for all $X \in DM_2(A)$ and $s \in S$.

Theorem 5

Let A be an S-poset. Then $(\alpha, DM_2(A))$ is a double-dense-completion of A if and only if A is UL-compatible.

Remark 1

Consider the S-poset A mentioned in Example 1(2).

- It is easy to verify that A is also UL-compatible. Meanwhile, $\lambda_1(A, x) = A \neq \{c, d, e\} = \lambda_2(A, x)$.

Hence $DM_1(A) \ncong DM_2(A)$, which means that double-dense-completions of an S-poset are not unique in general.

Let S be a pogroup.

Proposition 3

Each S-poset is LU-compatible and UL-compatible.

Let S be a pogroup.

Proposition 3

Each S-poset is LU-compatible and UL-compatible.

Theorem 4

Each S-poset has a unique double-dense-completion.

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Δ_1-objects

Let A be an S-poset. Define A^δ as

$$\{X \in DM_1(Int(A)) \ : \ X^u \cap \mathbb{F}_\sqcup(A) \text{ is filtered and } X \cap \mathbb{F}_\sqcap(A) \text{ is directed}\}.$$

Lemma 1

Let S be a pogroup. Then A^δ is a sub-S-poset of $DM_1(Int(A))$.

Δ_1-objects

Define $\bar{F} : A^\delta \longrightarrow \mathbb{F}_\sqcup(\mathbb{F}_\sqcap(A))$, by for all $X \in A^\delta$,

$$\bar{F}(X) = \bigvee\nolimits^{\uparrow} \alpha_{\mathbb{F}_\sqcap(A)}(X \cap \mathbb{F}_\sqcap(A)),$$

and define $\bar{G} : A^\delta \longrightarrow \mathbb{F}_\sqcap(\mathbb{F}_\sqcup(A))$, by for all $X \in A^\delta$,

$$\bar{G}(X) = \bigwedge\nolimits^{\downarrow} \beta_{\mathbb{F}_\sqcup(A)}(X^u \cap \mathbb{F}_\sqcup(A)).$$

Proposition 4

$\bar{F}$ and $\bar{G}$ are S-poset embeddings.

Δ_1-objects

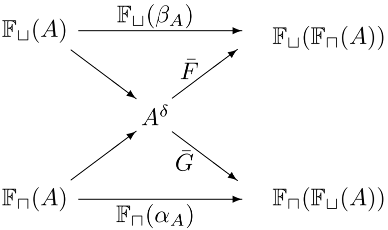

Fig. 7. The relationships between A^{δ} and completions

Proposition 5

A^{δ} is an interpolant in Figure 3.

Δ_1-objects

Suppose Q is an interpolant as in Figure 3.

Denote the S-poset embeddings from $\mathbb{F}_{\sqcup}(A)$ and $\mathbb{F}_{\sqcap}(A)$ to Q by g' and f', and the S-poset embeddings from Q to $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(A))$ and $\mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(A))$ by f and g, respectively. Then

- $f \circ f' = \alpha_{\mathbb{F}_{\sqcap}(A)}$,
- $f \circ g' = \mathbb{F}_{\sqcup}(\beta_A)$,
- $g \circ f' = \mathbb{F}_{\sqcap}(\alpha_A)$ and
- $g \circ g' = \beta_{\mathbb{F}_{\sqcup}(A)}$.

Define $\gamma : Q \longrightarrow A^{\delta}$ by for all $q \in Q$,

$$\gamma(q) = \{a \in \mathbb{F}_{\sqcap}(A) \mid \alpha_{\mathbb{F}_{\sqcap}(A)}(a) \leqslant f(q)\}^{ul}.$$

Theorem 5

Let S be a pogroup. Then A^{δ} is the Δ_1-object.

Δ_1-objects

Theorem 6

Let S be a pogroup and A be an S-lattice. Then the Δ_1-object is precisely the double-dense-completion of the intermediate structure.

Thank you!