Lattice-Free and Point-Free: Vickers Duality for Subbases of Stably Locally Compact Spaces

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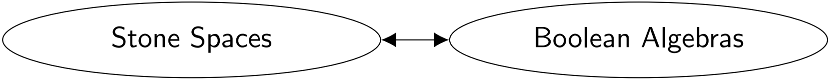
Classic Stone Duality



- Stone space = compact Hausdorff 0-dimensional space X.
- Order clopens $\mathcal{CO}(X)$ by inclusion, i.e.

$$O \leq N \qquad \Leftrightarrow \qquad O \subseteq N.$$

- Note $(\mathcal{CO}(X), \subseteq)$ is a Boolean algebra:

$$\emptyset \subseteq O \subseteq X. \qquad \text{(bounded)}$$

$$O \wedge N = O \cap N. \qquad \text{($\wedge$-semilattice)}$$

$$O \vee N = O \cup N. \qquad \text{($\vee$-semilattice)}$$

$$O^c = X \setminus O. \qquad \text{(complemented)}$$

$$M \wedge (N \vee O) = (M \wedge N) \vee (M \wedge O). \qquad \text{(distributive)}$$

Theorem (Stone 1936)

Every Boolean algebra arises this way. Moreover,

$$\text{Boolean Homomorphisms} \qquad \leftrightarrow \qquad \text{Continuous Maps}.$$

Classic Wallman Duality

- Can extend this to compact T_1 spaces X as follows.
- Take any $\cap$-$\cup$-basis (= closed under $\cap$ and $\cup$) $B \subseteq \mathcal{O}(X)$.
- Again B is a distributive lattice with maximum $1 = X$. Also

$$p \nleq q \quad \Rightarrow \quad \exists r \geq q \ (r \neq 1 = p \vee r). \qquad \text{(Subfit)}$$

Theorem (Wallman 1938)

Every subfit unital distributive lattice arises in this way.

- Can even take a more general $\cup$-subbasis $S \subseteq \mathcal{O}(X)$.
- Then S is a subfit unital $\vee$-semilattice.

Theorem (B.-Kubiś 2020)

Every subfit unital $\vee$-semilattice arises in this way. Moreover,

$$\text{Relational Homomorphisms} \quad \leftrightarrow \quad \text{Continuous Maps.}$$

Another Wallman Duality

- What about general subbases S of compact T_1 spaces?
- Then the inclusion ordering might collapse to equality.
  - E.g. take a compact subset of $\mathbb{R}$ of measure 1 and take S to be the subbasis of all regular open subsets of measure $1/2$.
- Instead we can consider subbasic (non-)covers of X, i.e.

$$\mathcal{C} = \{F \subseteq S : F \text{ is finite and } X \neq \bigcup F\}.$$

- Then $\mathcal{C}$ is an abstract simplicial complex, i.e.

$$F \subseteq G \in \mathcal{C} \qquad \Rightarrow \qquad F \in \mathcal{C}.$$

- Define $p \leq q$ when $\{q\} \cup F \in \mathcal{C} \Rightarrow \{p\} \cup F \in \mathcal{C}$, for all $F \subseteq S$.
- $\leq$ is a preorder. Call $\mathcal{C}$ proper if $p \leq q \leq p \;\Rightarrow\; p = q$.

Theorem (Wallman 1938)

Every proper abstract simplicial complex arises in this way.

Related Dualities

- Other dualities focus on locally compact sober (LCS) spaces.

Theorem (Hofmann-Lawson 1978)

Every continuous frame arises as $\mathcal{O}(X)$ for LCS space X. Also,

$$\text{Frame Homomorphisms} \quad \leftrightarrow \quad \text{Continuous Maps.}$$

- Generalisations to (semi)lattice bases of LCS spaces:
  - Compact Hausdorff: Shirota (1952), De Vries (1962).
  - Stably Compact: Jung-Sünderhauf (1996), van Gool (2012).
  - General Locally Compact Sober: B. (2021), Kawai (2021).
- Key feature of above: replacing $\subseteq$ with $\Subset$, i.e.

$$O \Subset N \quad \Leftrightarrow \quad \exists \text{ compact } K \ (O \subseteq K \subseteq N).$$

- So abstractly we have a set B with an idempotent relation $\prec$ (i.e. $p \prec q \ \Leftrightarrow \ \exists r \ (p \prec r \prec q)$) satisfying various properties.

The Compact Cover Relation

- What about arbitrary subbases S of stably locally compact X? (= LCS and $K \cap L$ compact for compact saturated $K, L \subseteq X$)
- Wallman suggests we should again consider covers.
- However, covers of the entirety of X may no longer suffice.
  - E.g. take X to be any space with a dense point x (e.g. $\{0\}$ is dense in $[0,1]$ or $[0,1] \cup [2,3]$ etc. with the upper topology).
  - Then $X = \bigcup F$ precisely when $x \in s$, for some $s \in F$, which gives no information about the topology on $X \setminus \{x\}$.
- Instead we must consider (compact) covers of smaller subsets, e.g. formed from finite subbasic intersections.
- Specifically, define $\vdash$ on $\mathsf{F}S = \{F \subseteq S : |F| < \infty\}$ by

$$F \vdash G \qquad \Leftrightarrow \qquad \bigcap F \Subset \bigcup G.$$

- What properties does $\vdash$ possess? Can we use $\vdash$ to recover X? Can these unify previous dualities for stably compact X?

The Compact Cover Relation

- Take a subbasis S of a LCS space X and define $\vdash$ on FS by

$$F \vdash G \qquad \Leftrightarrow \qquad \bigcap F \Subset \bigcup G.$$

- We immediately see that $\vdash$ is monotone, i.e.

$$F \vdash G \qquad \Rightarrow \qquad \{s\} \cup F \vdash G \quad \text{and} \quad F \vdash G \cup \{s\}.$$

- Conversely, $\vdash$ also satisfies Gentzen's cut rule, i.e.

$$F \vdash G \qquad \Leftarrow \qquad \{s\} \cup F \vdash G \quad \text{and} \quad F \vdash G \cup \{s\}.$$

Proof.

- Say $\{s\} \cup F \vdash G$ and $F \vdash G \cup \{s\}$, i.e. we have compact K, L with

$$s \cap \bigcap F \subseteq K \subseteq \bigcup G \qquad \text{and} \qquad \bigcap F \subseteq L \subseteq \bigcup G \cup s.$$

- Then $K \cup (L \setminus s)$ is compact and $\bigcap F \subseteq K \cup (L \setminus s) \subseteq \bigcup G$. □

Entailments

- So the compact cover relation $\vdash$ is an entailment, i.e.

$$F \vdash G \qquad \Leftrightarrow \qquad \{s\} \cup F \vdash G \quad \text{and} \quad F \vdash G \cup \{s\}.$$

- Entailments are always transitive on singletons, i.e.

$$p \vdash q \vdash s \qquad \Rightarrow \qquad p \vdash s.$$

Proof.

- If $p \vdash q \vdash s$ then $p \vdash \{q, s\}$ and $\{p, q\} \vdash s$, by monotonicity.
- Then Gentzen's cut rule yields $p \vdash s$. □

- The obvious extension to larger subsets can fail, e.g.

$$p \Subset q \cup r \quad \text{and} \quad q \cap r \Subset s \qquad \not\Rightarrow \qquad p \Subset s$$

- But other extensions do hold, e.g.

$$p \Subset q, \quad p \Subset r \quad \text{and} \quad q \cap r \Subset s \qquad \Rightarrow \qquad p \Subset s.$$

$$p \Subset q \cup r, \quad q \Subset s \quad \text{and} \quad r \Subset s \qquad \Rightarrow \qquad p \Subset s.$$

- This leads us to consider a certain 'diagonal' relation on $\mathsf{FF}S$.

The Diagonal Relation

- For any set S, define the overlap relation on $\mathsf{F}S$ by

$$F \between G \qquad \Leftrightarrow \qquad F \cap G \neq \emptyset.$$

- The selections any $\mathcal{Q} \subseteq \mathsf{F}S$ are given by

$$\mathcal{Q}_{\between} = \{G \in \mathsf{F}S : \forall F \in \mathcal{Q}\ (F \between G)\}.$$

- If $S \subseteq \mathsf{P}X = \{Y : Y \subseteq X\}$ then, for any $\mathcal{Q} \in \mathsf{FF}S$,

$$\bigcap_{F \in \mathcal{Q}} \bigcup F = \bigcup_{F \in \mathcal{Q}_{\between}} \bigcap F.$$

- The diagonal relation on $\mathsf{FF}S$ is defined by

$$\mathcal{Q} \bowtie \mathcal{R} \qquad \Leftrightarrow \qquad \mathcal{Q}_{\between} \between \mathcal{R}_{\between},$$

  i.e. every selection of $\mathcal{Q}$ overlaps every selection of $\mathcal{R}$.

Diagonal Transitivity

- If $S \subseteq \mathsf{P}X$ then

$$\mathcal{Q} \bowtie \mathcal{R} \qquad \Rightarrow \qquad \bigcap_{H \in \mathcal{Q}} \bigcup H \subseteq \bigcup_{H \in \mathcal{R}} \bigcap H.$$

- Thus if S is a subbasis of an LCS space X and $\mathcal{Q} \bowtie \mathcal{R}$ then

$$\forall H \in \mathcal{Q}\,(\bigcap F \Subset \bigcup H) \text{ and } \forall H \in \mathcal{R}\,(\bigcap H \Subset \bigcup G) \quad \Rightarrow \quad \bigcap F \Subset \bigcup G.$$

- This kind of 'diagonal transitivity' applies to all entailments.

Theorem (Jung-Kegelmann-Moshier 1999) (Vickers 2004)

If $\vdash$ is an entailment on $\mathsf{F}S$, for any $F, G \in \mathsf{F}S$ and $\mathcal{Q}, \mathcal{R} \in \mathsf{FF}S$,

$$F \vdash \mathcal{Q} \bowtie \mathcal{R} \vdash G \qquad \Rightarrow \qquad F \vdash G.$$

- Note singleton transitivity is the case $\mathcal{Q} = \mathcal{R} = \{\{s\}\}$.

Diagonal Interpolation vs Divisibility

- The compact cover relation $\vdash$ on $\mathsf{F}S$, for a subbasis S of LCS X, is also diagonally interpolative, i.e. conversely

$$F \vdash G \qquad \Rightarrow \qquad \exists \mathcal{Q}, \mathcal{R} \in \mathsf{FF}S \ (F \vdash \mathcal{Q} \bowtie \mathcal{R} \vdash G).$$

- Proved by refining G to obtain $\mathcal{R}$ and taking $\mathcal{Q} = \mathcal{R}_{\between}$.
- Actually this shows $F \vdash \mathcal{Q} \bowtie \mathcal{R} \vdash_{1\exists} G$ where

$$\mathcal{R} \vdash_{1\exists} G \qquad \Leftrightarrow \qquad \forall H \in \mathcal{R} \ \exists s \in G \ (H \vdash s).$$

- Thus the compact cover relation $\vdash$ is divisible.

Diagonal Idempotents $\to$ Stably Locally Compact Spaces

- Given $\vdash$ on $\mathsf{F}S$, we call $\mathcal{Q} \subseteq \mathsf{F}S$ a quasi-ideal if

$$F \in \mathcal{Q} \qquad \Leftrightarrow \qquad \exists \mathcal{G} \in \mathsf{F}\mathcal{Q}\ (F \vdash \mathcal{G}_{\between}).$$

- If $\vdash$ is monotone and diagonally idempotent then the quasi-ideals form a stably continuous frame ($\approx$ Vickers 2004).
- Hofmann-Lawson duality then yields an SLC space X where the elements of S can be identified with certain open subsets.
- However, these will not always form a subbasis of X:
  - E.g. let S be the collection of open subintervals of $(0,1)$ formed from consecutive dyadic rationals, i.e.

$$S = \{((k-1)/2^n, k/2^n) : 1 \leq k \leq 2^n\}.$$

  - Define $\vdash$ on $\mathsf{F}S$ by

$$F \vdash G \qquad \Leftrightarrow \qquad \bigcap F \Subset \mathrm{int}(\mathrm{cl}(\bigcup G)). \qquad (*)$$

  - Then $\vdash$ is monotone, diagonally idempotent and the spectrum of the quasi-ideals can be identified with $(0,1)$.
  - However, S is certainly not a subbasis for $(0,1)$.
- Crucially, the relation $\vdash$ given in $(*)$ is not divisible.

Tight Subsets

- To unify previous dualities, we would also need to obtain the space X directly from filter-like subsets of S.
- Accordingly, given $\vdash$ on $\mathsf{F}S$, we call $T \subseteq S$ tight if

$$\exists F \in \mathsf{F}S \; (T \supseteq F \vdash G) \qquad \Leftrightarrow \qquad T \cap G \neq \emptyset$$

- As long as $\vdash$ is monotone, this splits into

$$T \supseteq F \vdash G \qquad \Rightarrow \qquad T \cap G \neq \emptyset. \qquad \text{(Prime)}$$
$$\exists F \in \mathsf{F}T \; (F \vdash s) \qquad \Leftarrow \qquad s \in T. \qquad \text{(Round)}$$

- Motivation: take a subbasis S of LCS X. Given $x \in X$, let

$$S_x = \{t \in S : x \in t\}.$$

- Then S_x tight w.r.t. the compact cover relation $\vdash$ on $\mathsf{F}S$.
- If X is stable, every tight subset is of the form S_x for some x.

The Tight Spectrum

- Given $\vdash$ on $\mathrm{F}S$, let $\mathrm{T}S = \{T \subseteq S : T \text{ is tight}\}$. For $p \in S$, let

$$\mathrm{T}_p = \{T \in \mathrm{T}S : p \in T\}$$

- The tight spectrum is the space $\mathrm{T}S$ with the topology generated by $(\mathrm{T}_p)_{p \in S}$ (i.e. $(\mathrm{T}_p)_{p \in S}$ is a subbasis of $\mathrm{T}S$).

Theorem (B.-Kubiś 2021)

If $\vdash$ is a monotone, diagonally transitive and divisible relation on $\mathrm{F}S$ then the tight spectrum is a stably locally compact space and

$$F \vdash G \qquad \Rightarrow \qquad \bigcap_{p \in F} \mathrm{T}_p \Subset \bigcup_{p \in G} \mathrm{T}_p.$$

- $\Leftarrow$ will also hold when $\vdash$ satisfies 'strong transitivity', i.e.

$$F \Vdash \mathcal{Q} \bowtie \mathcal{R} \vdash G \qquad \Rightarrow \qquad F \vdash G,$$

where $F \Vdash G$ means $H \vdash G$ whenever $H \vdash f$, for all $f \in F$.

A Unifying Duality

- In this way, we obtain a duality between subbases of stably locally compact spaces and abstract ‘cover relations’.
- Can also show that continuous maps between the spaces correspond to certain relations between the subbases.
- Can also use this to obtain various previous dualities for bases of SLC spaces (by De Vries, Jung-Sünderhauf, etc.)
- General procedure: Given a distributive lattice S and some ‘compatible’ idempotent $\prec$, define a cover relation by

  $$F \vdash G \qquad \Leftrightarrow \qquad \exists H \in \mathsf{F}S \; (\bigwedge F \leq \bigvee H \text{ and } H \prec G),$$

  where $H \prec G$ means $\forall h \in H \; \exists g \in G \; (h \prec g)$.
- Then tight subsets are precisely the round prime filters.
- $\Rightarrow$ round prime filters form a stably locally compact space.
- Can also be used to obtain new dualities...

A New Duality

- Grätzer (1978) extended Stone duality to distributive $\vee$-semilattices and compactly based sober spaces.
- Recently, Celani-González (2020) extended Grätzer's duality to even non-distributive $\vee$-semilattices.
- Hansoul-Poussart (2009) and Bezhanishvili-Jansana (2011) obtained an analog of Grätzer's duality for distributive $\vee$-semilattices and but with spectral spaces instead (i.e. stably compact compactly based spaces).
- Can we likewise extend this to non-distributive $\vee$-semilattices?
- Yes – can define a cover relation $\vdash$ on any $\vee$-semilattice S by

  $$F \vdash G \quad \Leftrightarrow \quad \forall p, q \in S \left(\forall f \in F \,(p \leq f \vee q) \;\Rightarrow\; p \leq \bigvee G \vee q\right)$$

  (intuitively () part is saying $p \setminus q \subseteq \bigcap F$ implies $p \setminus q \subseteq \bigcup G$).
- Future Work: Generalise to (non compactly based) stably locally compact spaces using an idempotent relation $\prec$ on S.