A domain-theoretic generalisation of the Henstock-Kurzweil integral for compact metric spaces

Abbas Edalat
Department of Computing
Imperial College London
ISDT 2022

Two shortcomings of Lebesgue integration theory

- **Motivation:** A map $f : [0,1] \to \mathbb{R}$ is Lebesgue integrable iff $|f|$ is Lebesgue integrable.
- $f(x) = (\sin 1/x^3)/x$ is not Lebesgue integrable but the improper integral $\int_\delta^1 (\sin 1/x^3)/x \, dx$ exists as $\delta \to 0^+$.
- The HK-integral generalises the Lebesgue integral and the improper Riemann integral.
- It satisfies all the basic properties of an integral as well as a monotone convergence theorem. Furthermore:
- If $F : [a,b] \to \mathbb{R}$ and $F'(x)$ exists for all $x \in [a,b]$, then:

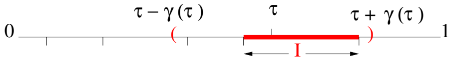
$$HK \int_a^b F'(x)\, dx = F(b) - F(a) \quad \text{Fundamental Thm. of Calculus}$$

Gauge and tagged partition

- Any map $\gamma : [0,1] \to \mathbb{R}^+$ is called a **gauge.** It generalises the norm of a partition of $[0,1]$ to depend on $x \in [0,1]$.

- A **tagged partition** P of $[0,1]$ is a finite collection $(t_i, I_i)_{i=1}^K$ where $(I_i)_{i=1}^K$ is a partition of $[0,1]$ by closed intervals I_i for $1 \le i \le K$ and $t_i \in I_i$ for each $1 \le i \le K$.

- The tagged partition $\dot{P} = (t_i, I_i)_{i=1}^K$ is said to be γ**-fine** (denoted $\dot{P} \prec \gamma$) if $I_i \subset (t_i - \gamma(t_i), t_i + \gamma(t_i))$ for $1 \le i \le K$.



Henstock-Kurzweil Integral

- Given a function $f : [0,1] \to \mathbb{R}_\perp$, the **Riemann sum** of the tagged partition $\dot{P} = (t_i, I_i)_{i=1}^{K}$ for f is given by,

$$S(f, P) = \sum_{i=1}^{K} f(t_k)(I_k^+ - I_k^-)$$

- A function $f : [0,1] \to \mathbb{R}_\perp$ has **HK-integral** $a \in \mathbb{R}$ if for each $\epsilon > 0$, there exists a gauge γ such that $|a - S(f, P)| < \epsilon$ for all γ-fine tagged partitions of $[0, 1]$.

Partitionable families of crescents of a space X

- Need a family of subsets to define partitions of X.
- For $\mathcal{B}$ any basis of open subsets of X closed under finite intersections, define the **crescents** generated by $\mathcal{B}$ as:

$$\mathbf{C}_{\mathcal{B}} := \{A_1 \cap \overline{A_2} : A_1, A_2 \in \mathcal{B}\}$$

- We always work with $\mathbf{C}_{\mathcal{B}}$ that is **partitionable**, i.e.:
(i) $\mathbf{C}_{\mathcal{B}}$ is closed under finite intersections.
(ii) For any $C, C_0 \in \mathbf{C}_{\mathcal{B}}$, there exists a finite collection of pairwise disjoint subsets $S_i \in \mathbf{C}_{\mathcal{B}}$, $(1 \leq i \leq m)$, satisfying:

$$C \setminus C_0 = S_1 \cup S_2 \cup \cdots \cup S_m.$$

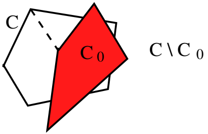
- For basis $\mathcal{B}$ closed under finite unions and intersections and closed under taking exterior, $\mathbf{C}_{\mathcal{B}}$ is partitionable.

Examples of partitionable collection of crescents

- **Example** $X = [0,1]$ and $\mathcal{B} =$ open intervals.
  $\mathbf{C}_{\mathcal{B}}$ is the family of all intervals and is partitionable.

- **Example** $\prod_{1\leq i\leq n}[a_i, b_i] \subset \mathbb{R}^n$, $\mathcal{B} =$ open hyper-rectangles.
  $\mathbf{C}_{\mathcal{B}}$ the family of all hyper-rectangles is partitionable.

- **Example** X a compact Riemannian manifold (eg $S^2 \subset \mathbb{R}^3$)
  $\mathcal{B} =$ open (geodisically) convex polyhedra.
  $\mathbf{C}_{\mathcal{B}}$ is the family of all convex polyhedra and is partitionable.



- **Example** $X = \{0,1\}^\omega$ the Cantor space. Let $\mathcal{B} =$ clopens.
  $\mathbf{C}_{\mathcal{B}} = \mathcal{B}$ is partitionable.

Tagged partition for compact metric X

- $P = \{R_i : 1 \leq i \leq n\}$ is a **partition** of $C \in \mathbf{C}_{\mathcal{B}}$ if $R_i \in \mathbf{C}_{\mathcal{B}}$ are pairwise disjoint for $1 \leq i \leq n$, and $C = \bigcup_{1 \leq i \leq n} R_i$.

$$\|P\| := \max\{diam(R_i) : 1 \leq i \leq n\}$$

- P' **refines** P, written $P \sqsubseteq P'$ if each crescent in P is the union of some crescents in P'.

- If $P = \{R_i : 1 \leq i \leq n\}$ is a partition of C and $t_i \in \overline{R_i}$, for $1 \leq i \leq n$, then the set of pairs

$$\dot{P} = \{(R_i, t_i) : 1 \leq i \leq n\}$$

is called a **tagged partition** of C.

Gauge and subordinate tagged partitions

- **Definition**

  A **gauge** on $\overline{C}$ for $C \in \mathbf{C}_{\mathcal{B}}$ is a map $\gamma : \overline{C} \to \mathbb{R}^+$.

- The tagged partition $\dot{P}$ is γ-**fine** on X, written $\dot{P} \prec \gamma$, if

  $$\overline{R_i} \subset O_{\gamma(t_i)}(t_i) \qquad \text{for } 1 \leq i \leq n,$$

  where $O_r(x)$ is the open ball of radius r centred at x.

- **Proposition** If γ is a gauge on $C \in \mathbf{C}_{\mathcal{B}}$, then there exists a γ-fine tagged partition of C.

  (Proof uses compactness of $\overline{C}$.)

Upper space of X and its probabilistic power domain

- μ: a normalised measure on compact metric space X.
- Upper space $\mathbf{U}X$: non-empty compact sets ordered with reverse inclusion, a bounded complete domain.

$$X \simeq \mathrm{Max}(\mathbf{U}X)$$

- Normalised probabilistic power domain $\mathbf{P}^1\mathbf{U}X$: normalised valuations on $\mathbf{U}X$.
- $\mathbf{M}^1X$: Normalised Borel measures with weak topology

$$\mathbf{M}^1X \simeq \mathrm{Max}(\mathbf{P}^1\mathbf{U}X) \quad \text{(AE95, Lawson97, AE97)}$$

- Let $\mathbf{P}^1_sD \subset \mathbf{P}^1D$: poset of simple valuations of domain D.

Partitions and simple valuations

- Partition $P = \{R_i : 1 \leq i \leq n\}$ of X induces $\mu_P \in \mathbf{P}^1_s\mathbf{U}X$:

$$\mu_P = \sum_{i=1}^{n} \mu(R_i)\delta_{\overline{R_i}} \quad \text{with} \quad \delta_C : \text{Point valuation on } C \in \mathbf{U}X$$

- Let $(\mathcal{P}, \sqsubseteq)$ be the **poset of partitions** of X.

- **Proposition**

  (i) $P \sqsubseteq P' \Rightarrow \mu_P \sqsubseteq \mu_{P'}$ in $\mathbf{P}^1\mathbf{U}X$.

  (ii) If P_i for $i \geq 0$ is an increasing sequence of partitions with $\lim_{i\to\infty} \|P_i\| = 0$, then $\sup_i \mu_{P_i} = \mu$.

  (ii) $\mu = \sup\{\mu_P : P \in \mathcal{P}\}$.

Integration of continuous versus unbounded functions

- $\mathbf{I}\mathbb{R}$: domain of compact intervals of $\mathbb{R}$ with reverse inclusion.
- Since $X \simeq \mathrm{Max}(\mathbf{U}X)$ is a dense subset of $\mathbf{U}X$, any function $f : X \to \mathbb{R} \subset \mathbf{I}\mathbb{R}$ has a continuous **envelope** $f^* : \mathbf{U}X \to \mathbf{I}\mathbb{R}$:

$$f^*(y) := \sup\{\inf f[O \cap X] : O \subset \mathbf{U}X \text{ is Scott open}, y \in O\}.$$

- For continuous f, get monotone map $\int f\, d(\cdot) : \mathbf{P}_s\mathbf{U}X \to \mathbf{I}\mathbb{R}$:

$$\int f\, d\left(\sum_{i=1}^{n} r_i \delta(y_i)\right) = \sum_{i=1}^{n} r_i f^*(y_i) = \sum_{i=1}^{n} r_i f[y_i]$$

- By continuity $\int f\, d\mu = \sup\{\int f\, d\mu_P : P \in \mathcal{P}\}$.
- However if f is unbounded $\int f\, d(\cdot)$ is not monotone.

Partition induced simple valuations way-below μ

- Get μ by partition induced simple valuations way-below it.
- For a compact $C \subset X$ and $\alpha > 0$, the α**-expansion** of C is

$$C_\alpha = \{x \in X : \exists y \in C.\, d(x, y) \leq \alpha\}.$$

- Given a partition $P = \{R_i : 1 \leq i \leq n\}$ of X and $\alpha > 0$, the α**-relaxation** of μ_P is the simple valuation defined as

$$\mu_{P,\alpha} := \sum_{1 \leq i \leq n} \mu(R_i)\delta_{(\overline{R_i})_\alpha},$$

- $\mu_{P,\alpha} \sqsubseteq \mu_P$ for any $\alpha > 0$ and $\mu = \sup\{\mu_{P,\alpha} : P \in \mathcal{P}, \alpha > 0\}$.
- However, $\mu_{P,\alpha} \ll \mu_P$ does **not** hold.

Partition induced simple valuations way-below μ

- For $1 > \beta > 0, \alpha > 0$, define

$$\mu_{P,\alpha,\beta} = \beta\delta_X + (1-\beta)\mu_{P,\alpha}.$$

- **Theorem 1** The collection

$$S_\mu := \{\mu_{P,\alpha,\beta} : P \in \mathcal{P}, \alpha > 0, 1 > \beta > 0\}$$

  is a directed set of normalised simple valuations way-below μ with supremum μ. And if $f : X \to \mathbb{R}$ is continuous

$$\int f\, d\mu = \sup\left\{\int f\, d\nu : \nu \in S_\mu\right\} \in \mathbb{R} \simeq \mathrm{Max}(\mathbf{I}\mathbb{R})$$

- The nagging problem for unbounded functions remains.

Directed sets of tagged partitions

- **Strategy:** Develop a directed set of gauge/tagged partition pairs to mirror that of simple valuations way-below μ.
- For partitions P_C and P'_C of $C \in \mathbf{C}_{\mathcal{B}}$, their lub in $\mathcal{P}_C$ is

$$P_C \vee P'_C = \{R \cap R' : R \in P_C, R' \in P'_C\}.$$

- For tagged-partitions $\dot{P}_C$ and $\dot{P}'_C$ of $C \in \mathbf{C}_{\mathcal{B}}$ with sets of tags T and T', respectively, define:

$$\dot{P}_C \sqsubseteq \dot{P}'_C \quad \text{if} \quad P_C \sqsubseteq P'_C \;\&\; T \subset T'$$

- Let $\dot{\mathcal{P}}_C$ be the poset of tagged partitions of C.

Directed sets of tagged partitions and gauge pairs

- **Definition** If $\gamma_C : \overline{C} \to \mathbb{R}^+$ is a gauge on C and if $\dot{P}_C$ is γ_C-fine, then we say $(\dot{P}_C, \gamma_C)$ is a **PG** pair.
- For PG pairs $(\dot{P}_C, \gamma_C)$, $(\dot{P}'_C, \gamma'_C)$ define

$$(\dot{P}_C, \gamma_C) \sqsubseteq (\dot{P}'_C, \gamma'_C) \text{ if } \dot{P}_C \sqsubseteq \dot{P}'_C \ \& \ \gamma_C \geq \gamma'_C$$

- Let $\dot{\mathcal{P}}\mathcal{G}_C$ be the poset of PG pairs for C.
- **Proposition** $\dot{\mathcal{P}}_C$ and $\dot{\mathcal{P}}\mathcal{G}_C$ are directed sets for $C \in \mathbf{C}_{\mathcal{B}}$.
- Proof uses some properties derived from partitionable $\mathbf{C}_{\mathcal{B}}$.

Simple valuations induced by PG pairs

- For each PG pair $(\dot{P}, \gamma)$, with $\dot{P} = \{(R_i, t_i) : 1 \leq i \leq n\}$, the normalised measure μ induces a simple valuation

$$\mu_{\dot{P},\gamma} = \sum_{1 \leq i \leq n} \mu(R_i)\delta_{t_i} \in \mathbf{M}^1 X \simeq \mathrm{Max}(\mathbf{P}^1 \mathbf{U} X)$$

- We have: $\mu_{P,\alpha,\beta} \sqsubseteq \mu_{\dot{P},\gamma}$
- **Theorem 2** (Using Portmanteau theorem) The net

$$\{\mu_{\dot{P},\gamma} : (\dot{P}, \gamma) \in \dot{\mathcal{P}}\mathcal{G}\} \subset \mathbf{M}^1 X$$

converges in weak topology to μ, i.e., $\lim_{(\dot{P},\gamma) \in \dot{\mathcal{P}}\mathcal{G}} \mu_{\dot{P},\gamma} = \mu$.

And if $f : X \to \mathbb{R}$ is continuous then

$$\lim_{(\dot{P},\gamma) \in \dot{\mathcal{P}}\mathcal{G}} \int f \, d\mu_{\dot{P},\gamma} = \int f \, d\mu.$$

Integral of continuous functions

- For integration of continuous maps, get equivalence of

$$S_\mu := \{\mu_{P,\alpha,\beta} : P \in \mathcal{P}, \alpha > 0, 1 > \beta > 0\}$$

and the simple measures induced by PG pairs

$$\mu_{\dot{P},\gamma} = \sum_{1 \le i \le n} \mu(R_i)\delta_{t_i} \quad \text{with} \quad \dot{P} = \{(R_i, t_i) : 1 \le i \le n\}$$

- **Theorem** For continuous $f : X \to \mathbb{R}$, these are equivalent:

  (i) $\int f\, d\mu = r$.

  (ii) For each $\epsilon > 0$, there exists a partition P and $1 > \alpha, \beta > 0$ such that $r \in \int f\, d\mu_{P,\alpha,\beta}$ and $\mathrm{diam}(\int f\, d\mu_{P,\alpha,\beta}) < \epsilon$.

  (iii) For each $\epsilon > 0$, there exists a gauge γ on X such that $|\int f\, d\mu_{\dot{P},\gamma} - r| < \epsilon$ for any γ-tagged partition $\dot{P}$.

Generalised HK-integral

- **Definition** A map $f : \overline{C} \to \mathbb{R}$ has a D_μ**-integral** with $\int_C f\, d\mu = r \in \mathbb{R}$ if there is a sub-net $\dot{\mathcal{P}}\mathcal{G}_C(f) \subset \dot{\mathcal{P}}\mathcal{G}_C$ with

$$\lim_{(\dot{P},\gamma)\in\dot{\mathcal{P}}\mathcal{G}_C(f)} \int f\, d\mu_{\dot{P},\gamma} = r.$$

- **Proposition**

$$\int_C f\, d\mu = r \quad \Longleftrightarrow$$

for all $\epsilon > 0$ there exists a gauge γ on $\overline{C}$ such that for any tagged partition $\dot{P} \prec \gamma$:

$$\left| \int_C f\, \mu_{\dot{P},\gamma} - r \right| < \epsilon.$$

D_μ-integral by lower and upper integrals

- Lee-Jhao 96 get lower/upper HK-integrals via “contraction”.
- We can do this directly here:

$$S^l(f,\gamma,\mu) = \inf\left\{\int f\, d\mu_{\dot{P},\gamma} : \dot{P} \prec \gamma\right\}$$

$$S^u(f,\gamma,\mu) = \sup\left\{\int f\, d\mu_{\dot{P},\gamma} : \dot{P} \prec \gamma\right\}$$

$$L\int_a^b f\, d\mu := \sup_\gamma S^\ell(f,\gamma,\mu) \qquad U\int_a^b f\, d\mu := \inf_\gamma S^u(f,\gamma,\mu)$$

- **Proposition**

$f : X \to \mathbb{R}_\perp$ is D_μ-integrable iff $L\int f\, d\mu = U\int f\, d\mu$.

Properties of HK-integral extend to D_μ-integral (I)

- **Null maps** If $f = 0$ a.e. with respect to μ then $\int f\,d\mu = 0$.
- **Linearity** If f_1 and f_2 are D_μ-integrable with $c_1, c_2 \in \mathbb{R}$, then $\int (c_1 f_1 + c_2 f_2)\,d\mu = c_1 \int f_1\,d\mu + c_2 \int f_2\,d\mu$.
- **Positivity** If $f \geq 0$ is D_μ-integrable then $\int f\,d\mu \geq 0$.
- **Cauchy condition.**

$$\int_C f\,d\mu \text{ exists} \iff$$

for each $\epsilon > 0$ there exists a gauge γ on $\overline{C}$ such that for any two PG pairs $(\dot{P}_1, \gamma), (\dot{P}_2, \gamma) \in \dot{\mathcal{P}}\mathcal{G}_C$ we have

$$\left| \int f\,d\mu_{\dot{P}_1,\gamma} - \int f\,d\mu_{\dot{P}_2,\gamma} \right| < \epsilon$$

Properties of HK-integral extend to D_μ-integral (II)

- **Additivity**

  Let $P = \{R_i : 1 \leq i \leq n\}$ be a partition of $R \in \mathbf{C}_{\mathcal{B}}$. Then $f : R \to \mathbb{R}$ is D_μ-integrable on R iff f is D_μ-integrable on R_i for $1 \leq i \leq n$ and, in which case,

$$\int_R f\, d\mu = \sum_{i=1}^{n} \int_{R_i} f\, d\mu.$$

- **Measurable sets**

  The characteristic function χ_E of a measurable set E is D_μ-integrable with $\int \chi_E\, d\mu = \mu(E)$.

Saks-Henstock lemma

- **Lemma** Suppose $f : X \to \mathbb{R}$ is D_μ-integrable and, for $\epsilon > 0$, let γ be a gauge such that $\dot{P} \prec \gamma$ implies:

$$\left| \int f \, d\mu_{\dot{P},\gamma} - \int f \, d\mu \right| \leq \epsilon.$$

Then for any γ-fine tagged sub-partition

$$\dot{Q} = \{(R_i, t_i) : 1 \leq i \leq n\}$$

of X with $R := \bigcup_{1 \leq i \leq n} R_i$, we have:

$$\left| \int f \, d\mu_{\dot{Q},\gamma} - \int_R f \, d\mu \right| \leq \epsilon.$$

Lebesgue integrable implies D_μ-integrable

- **Monotone Convergence Theorem**
  Let $(f_n)_{n\geq 0}$ be a monotone sequence of D_μ-integrable functions on X and put $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in X$.

  Then f is D_μ-integrable iff the sequence $(\int f_n \, d\mu)_{n\geq 0}$ is bounded in $\mathbb{R}$.

- **Theorem** If $f : X \to \mathbb{R}$ is Lebesgue integrable wrt μ, then it is D_μ-integrable and the two integrals coincide.

Invariance of D_μ-integral wrt change of basis in $[0,1]$

- Let μ be a normalised measure on $[0,1]$.
- Let $S_1, S_2 \subset [0,1]$ be two dense subsets and consider the two basis $\mathcal{B}_1$ and $\mathcal{B}_2$ generated by open intervals with endpoints in S_1 and S_2 respectively.
- We have two collections of crescents $\mathbf{C}_{\mathcal{B}_1}$ and $\mathbf{C}_{\mathcal{B}_2}$.
- **Proposition** $f : [0,1] \to \mathbb{R}$ is $D_\mu$integrable wrt $\mathbf{C}_{\mathcal{B}_1}$ iff it is D_μ-integrable wrt $\mathbf{C}_{\mathcal{B}_2}$, and if so the two integrals coincide.
- This extends to maps $f : [0,1]^n \to \mathbb{R}$ with $\mathcal{B}$ either as open hyper-rectangles or as open convex polyhedra.
- It also extends to any Riemannian surface with $\mathcal{B}$ as open (geodisically) convex polyhedra.

D_μ-integrable but non-Lebesgue integrable maps

- Let $X = \{0,1\}^\omega$ the Cantor space with μ as the product uniform measure on $\{0,1\}$.
- Let $g : \{0,1\}^\omega \to [0,1]$ be the continuous map $x \mapsto g(x) = \sum_{i=0}^{\infty} x_i/2^{i+1}$.
- **Proposition** A map $f : [0,1] \to \mathbb{R}$ is HK-integrable iff $f \circ g : \{0,1\}^\omega \to \mathbb{R}$ is D_μ-integrable.

A D_μ-integrable but non-Lebesgue integrable map

- The map $f(x) = (\sin 1/x^3)/x$ is not Lebesgue integrable but is HK-integrable:
- By the change of variable $u = 1/x^3$:

$$\lim_{a \to 0} \int_a^1 f(x)\,dx = \frac{\pi}{6} - \frac{1}{3}\int_0^1 \frac{\sin u}{u}\,du$$

- Thus, the map $f \circ g : \{0,1\}^\omega \to \mathbb{R}$ is D_μ-integrable but not Lebesgue integrable.

Some further work

- Finding simple partitionable collection of crescents in concrete compact metric spaces.
- D_u-integrable, but non-Lebesgue integrable maps on Riemannian manifolds.
- Invariance of D_μ-integral under change of basis.
- Extension to σ-compact spaces.