Partial frames, their free frames and their congruence frames

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We begin by introducing three objects that will repeatedly arise in this talk:

- a partial frame L
- its free frame $\mathcal{H}_{\mathcal{S}}L$
- its congruence frame $\mathcal{C}_{\mathcal{S}}L$.

Partial frames are:
• meet-semilattices, where
• not all subsets need have joins.

A selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration, which we call the "designated" ones; an $\mathcal{S}$-frame then must have joins of (at least) all such subsets and binary meet must distribute over these.

$\mathcal{S}$-frame maps preserve finite meets and designated joins; and, in particular, the top and bottom elements.

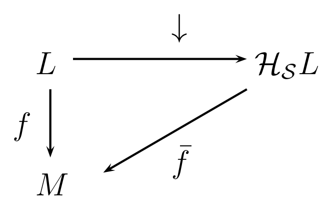
Throughout this talk, L refers to an arbitrary $\mathcal{S}$-frame.

We are indebted to earlier work by other authors in this field, for example:
• J. Paseka, Covers in Generalized Frames, in: General Algebra and Ordered Sets (Horni Lipova 1994), Palacky Univ. Olomouc, Olomouc pp. 84-99.
• E.R. Zenk, Categories of Partial Frames, Algebra Univers. 54 (2005) 213-235.
• D. Zhao, Nuclei on Z-Frames, Soochow J. Math. 22 (1) (1996) 59-74.
• D. Zhao, On Projective Z-frames, Canad. Math. Bull. 40(1) (1997) 39-46.

Examples Here are some examples of different selection functions and their corresponding $\mathcal{S}$-frames.
1. If all joins are specified, we are have the notion of a frame.
2. If countable joins are specified, we have the notion of a σ-frame.
3. If joins of subsets with cardinality less than some (regular) cardinal κ are specified, we have the notion of a κ-frame.
4. If only finite joins are specified, we have the notion of a bounded distributive lattice.

The free frame over a partial frame

A subset J of L is an $\mathcal{S}$-ideal of L if J is a non-empty downset closed under designated joins. The collection of all $\mathcal{S}$-ideals of L is a full frame, denoted $\mathcal{H}_{\mathcal{S}}L$.
For any $x \in L$, the principal downset, $\downarrow x = \{t \in L : t \leq x\}$, is an $\mathcal{S}$-ideal.
The map $\downarrow : L \to \mathcal{H}_{\mathcal{S}}L$ is the embedding of L into the free frame over L, meaning that every $\mathcal{S}$-frame map f to a frame M factors via this map.



The congruence frame of a partial frame

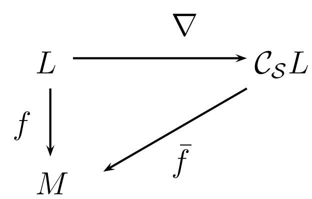
We call $\theta \subseteq L \times L$ an $\mathcal{S}$-congruence on L if it satisfies the following:
(C1) θ is an equivalence relation on L.
(C2) $(a, b), (c, d) \in \theta$ implies that $(a \wedge c, b \wedge d) \in \theta$.
(C3) If $\{(a_\alpha, b_\alpha) : \alpha \in \mathcal{A}\} \subseteq \theta$ and $\{a_\alpha : \alpha \in \mathcal{A}\}$ and $\{b_\alpha : \alpha \in \mathcal{A}\}$ are designated subsets of L, then $(\bigvee_{\alpha \in \mathcal{A}} a_\alpha, \bigvee_{\alpha \in \mathcal{A}} b_\alpha) \in \theta$.

Using this, the standard quotient map $L \to L/\theta$ becomes an $\mathcal{S}$-frame map.

The collection of all $\mathcal{S}$-congruences on L is denoted by $\mathcal{C}_\mathcal{S}L$; it is a full frame.

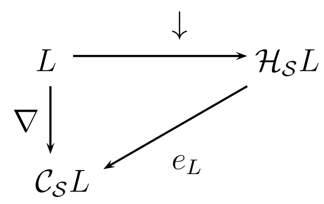
We define
$\nabla_a = \{(x, y) : x \vee a = y \vee a\}$ and
$\Delta_a = \{(x, y) : x \wedge a = y \wedge a\}$.
These closed and open $\mathcal{S}$-congruences together generate $\mathcal{C}_\mathcal{S}L$.
$\nabla_1 = L \times L$ is the top element and
$\nabla_0 = \{(x, x) : x \in L\}$ is the bottom element of $\mathcal{C}_\mathcal{S}L$.

The map $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ is an embedding of L into its congruence frame. It has the universal property that if $f : L \to M$ is an $\mathcal{S}$-frame map into a frame M with complemented image, then there exists a frame map $\bar{f} : \mathcal{C}_{\mathcal{S}}L \to M$ such that $f = \bar{f} \circ \nabla$.



All three objects in one diagram

Definition For any $\mathcal{S}$-frame L, define $e_L : \mathcal{H}_\mathcal{S} L \to \mathcal{C}_\mathcal{S} L$ to be the unique frame map such that $e_L(\downarrow a) = \nabla_a$ for all $a \in L$; that is, making the following diagram commute:



Further, the frame map e_L is one-one.

Booleanness conditions for partial frames

There are two relatively obvious options for the notion of Booleanness here:

First: Every element has a complement. We use this as our definition of a Boolean $\mathcal{S}$-frame.

Second: an analogue using the Booleanization, or least dense quotient, of a full frame.
For this, we need the following $\mathcal{S}$-congruence:
For $x \in L$, set $P_x = \{t \in L : t \wedge x = 0\}$.
Let $\pi_L = \{(x, y) : P_x = P_y\}$.
We call π_L the Madden congruence of L.
The quotient map induced by the Madden congruence,
$p_L : L \to L/\pi_L$ is dense, onto and the universal such.
We refer to this as the Madden quotient of L, and call L d-reduced if its Madden quotient is an isomorphism.

An $\mathcal{S}$-frame map $h : L \to M$ is called skeletal if $(h \times h)[\pi_L] \subseteq \pi_M$.

Theorem In the following list of conditions, each one implies the succeeding one, but not conversely.

(a) $\mathcal{H}_{\mathcal{S}}L$ is a Boolean frame.
(b) L is a Boolean frame.
(c) L is a Boolean $\mathcal{S}$-frame.
(d) L is a d-reduced $\mathcal{S}$-frame.

Theorem characterizing (a) The following are equivalent:

(i) The frame $\mathcal{H}_{\mathcal{S}}L$ is Boolean.
(ii) The $\mathcal{S}$-frame embedding $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ is an isomorphism.
(iii) Every $\theta \in \mathcal{C}_{\mathcal{S}}L$ has the form $\theta = \nabla_a$, for some $a \in L$.

Theorem characterizing (c) The following are equivalent:

(i) L is Boolean; that is, every element of L is complemented.
(ii)The embedding $e : \mathcal{H}_{\mathcal{S}}L \to \mathcal{C}_{\mathcal{S}}L$ is an isomorphism.
(iii) Every $\mathcal{S}$-congruence θ of L is an arbitrary join of $\mathcal{S}$-congruences of the form ∇_a, for some $a \in L$.

Closed and open maps

Definition Let $h : L \to M$ be an $\mathcal{S}$-frame map.
A function $r : M \to L$ is a *right adjoint* of h if

$$h(x) \leq m \iff x \leq r(m) \text{ for all } x \in L, m \in M.$$

A function $\ell : M \to L$ is a *left adjoint* of h if

$$\ell(m) \leq x \iff m \leq h(x) \text{ for all } x \in L, m \in M.$$

Example This is an example of an (onto) $\mathcal{S}$-frame map which has neither a right nor a left adjoint.

Let L be the σ-frame consisting of all countable and co-countable subsets of $\mathbb{R}$, and $\mathbf{2}$ denote the 2-element chain. Define $h : L \to \mathbf{2}$ by $h(C) = 0$ if C is countable and $h(D) = 1$ if D is co-countable. Then h is a σ-frame map. However it has no right adjoint since there is no largest $A \in L$ with $h(A) = 0$. Similarly it has no left adjoint.

Definition Let $h : L \to M$ be an $\mathcal{S}$-frame map.
We call h *closed* if, for all $m \in M$, there exists $x \in L$ with

$$(h \times h)^{-1}(\nabla_m) = \nabla_x$$

We call h *open* if, for all $m \in M$, there exists $x \in L$ with

$$(h \times h)^{-1}(\Delta_m) = \Delta_x$$

Theorem Let $h : L \to M$ be an $\mathcal{S}$-frame map.

(a) The map h is closed iff h has a right adjoint, r, and for all $x \in L, m \in M$,

$$r(h(x) \vee m) = x \vee r(m).$$

(b) The map h is open iff h has a left adjoint, ℓ, and for all $x \in L, m \in M$,

$$\ell(h(x) \wedge m) = x \wedge \ell(m).$$

Note Any frame map with a Boolean domain is closed. The corresponding statement for $\mathcal{S}$-frames is false, as our earlier example shows. However, if the domain has a Boolean free frame, the result does hold.
We also note that, as for full frames, every open map is skeletal, but not conversely.

The embedding of a partial frame into its free frame or its congruence frame

Theorem Let L be an $\mathcal{S}$-frame and $\downarrow : L \to \mathcal{H}_{\mathcal{S}}L$ the embedding into its free frame.

(a) The map $\downarrow$ has a right adjoint iff $\downarrow$ is closed iff $\downarrow$ is an isomorphism.

(b) The map $\downarrow$ has a left adjoint iff L is a complete lattice.

(c) The map $\downarrow$ is open iff L is a frame.

(d) The map $\downarrow$ is skeletal for any L.

Theorem Let L be an $\mathcal{S}$-frame and $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ the embedding into its congruence frame.

(a) The map ∇ is closed iff ∇ is an isomorphism.

(b) The map ∇ is open iff L is a Boolean frame.

(c) The map ∇ is skeletal iff L is a d-reduced $\mathcal{S}$-frame.

THE END