A Hofmann-Mislove theorem for c-well-filtered spaces

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- Background
- c-well-filtered spaces
- A Hofmann-Mislove theorem

- In 1981, Hofmann and Mislove proved that there exists a bijection between the nonempty Scott open filters on the open set lattice and the compact saturated subsets in a sober space X.

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  This result is known as the Hofmann-Mislove Theorem I.
- Moreover, they also showed that for a locally compact sober space X, there is a bijection between the family of nonempty Scott open filters of the compact saturated sets and the open set lattice.

  This result is known as the Hofmann-Mislove Theorem II.

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- In 2017, Yang and Shi proved that the compact saturated subsets of a countably sober space correspond bijectively to the Scott open countable filters of its open-set lattice.

Problem

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locally compact	locally Lindelöf
+	+
well-filtered	c-well-filtered

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Does the Hofmann-Mislove theorem II hold for countably well-filtered spaces?

c-well-filtered spaces

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Definition

A T_0 space X is c-well-filtered if for every countably filtered family $\{K_i\}_{i\in I}$ of saturated Lindelöf subsets of X and each open subset U with $\bigcap_{i\in I} K_i \subseteq U$, there is a $K_{i_0} \subseteq U$ for some $i_0 \in I$.

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Proposition

Suppose that X is a countable set. Then topological space (X, τ) is c-well-filtered.

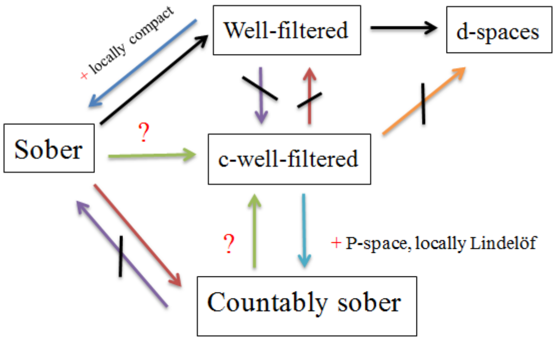

Figure: The relationship between several types of topological spaces

well-filtered $\not\Rightarrow$ c-well-filtered

Example

Consider the real number set $\mathbb{R}$ with the co-countable topology τ_{coc}, where $\tau_{coc} = \{U \subseteq \mathbb{R} : \mathbb{R} \backslash U \text{ is countable}\} \bigcup \{\emptyset\}$. It is known that the topological space $(\mathbb{R}, \tau_{coc})$ is a well-filtered space, but not c-well-filtered.

c-well-filtered $\not\Rightarrow$ well-filtered

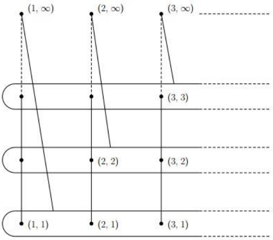

Figure: Johnstone's dcpo $\mathbb{J}$

It is easy to see that $\mathbb{J}$ is countable. So, $\Sigma\mathbb{J}$ is a c-well-filtered space. But $\Sigma\mathbb{J}$ is not a well-filtered space.

c-well-filtered $\not\Rightarrow$ d-spaces

Figure: dcpo $\mathbb{N}_\omega$

Let $L = \mathbb{N}_\omega$ with the Alexandroff topology $a(L)$. It is clear that $\{\omega\} \in a(L)$, but $\{\omega\}$ is not a Scott open subset. Therefore, the Alexandroff space $\Gamma L = (L, a(L))$ is a c-well-filtered space but not a d-space.

locally Lindelöf c-well-filtered $\Rightarrow$ countably sober

Let X be a topological space. The set of all saturated Lindelöf subsets of X is denoted by $\mathcal{LQ}(X)$.

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1. A topological space X is locally Lindelöf if for every open subset U of X and for every point $x \in U$, there exists $K \in \mathcal{LQ}(X)$ such that $x \in \mathrm{int}(K) \subseteq K \subseteq U$.

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Let X be a topological space. The set of all saturated Lindelöf subsets of X is denoted by $\mathcal{LQ}(X)$.

1. A topological space X is locally Lindelöf if for every open subset U of X and for every point $x \in U$, there exists $K \in \mathcal{LQ}(X)$ such that $x \in \text{int}(K) \subseteq K \subseteq U$.
2. A topological space X is called a P-space if and only if the intersection of any countable open sets in X is an open set.

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We denote that $\mathbb{Z}_+$ represents a countable set.

1. A subset C of a topological space X is countably irreducible if C is nonempty and if for any closed subsets $\{B_i : i \in \mathbb{Z}_+\}$, $C \subseteq \bigcup_{i\in\mathbb{Z}_+} B_i$ implies that $C \subseteq B_i$ for some $i \in \mathbb{Z}_+$.
2. A topological space X is countably sober if and only if for every countably irreducible closed subset A of X, there exists a unique element $x \in X$ such that $A = \downarrow x$.

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Proposition

Let X be a P-space. If X is a locally Lindelöf and c-well-filtered space, then X is countably sober.

Problems

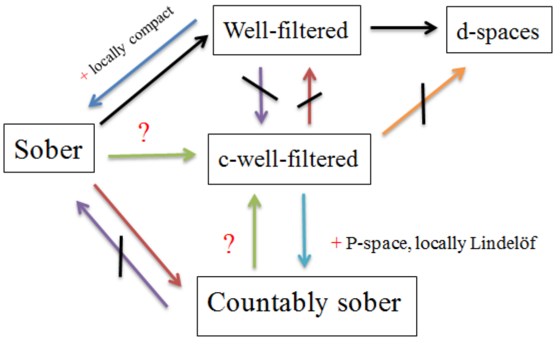
There are still two problems.

Question 1

Sober $\Rightarrow$ c-well-filtered?

Question 2

Countably sober $\Rightarrow$ c-well-filtered?



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Some properties of c-well-filtered spaces

Let L be a poset.

1. A nonempty subset $D \subseteq L$ is countably directed if for every $E \in \mathrm{Count}D$, there exists $d \in D$ such that $E \subseteq {\downarrow} d$.

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   All σ-Scott open subsets of L form a topology, called the σ-Scott topology and denoted as $\sigma_c(L)$.

Some properties of c-well-filtered spaces

Similar to well-filtered spaces, c-well-filtered spaces have some similar results too.

Proposition

A topological space X is c-well-filtered if and only if for every closed subset C of X and each countably filtered family $\{K_i\}_{i\in I}$ of saturated Lindelöf subsets, if $C \cap K_i \neq \emptyset$ for all $i \in I$, then $\bigcap_{i\in I} K_i \cap C \neq \emptyset$.

Some properties of c-well-filtered spaces

Proposition

Let (X, τ) be a c-well-filtered T_0-space. Then $\Omega(X)$ is a countably directed complete poset and $\tau \subseteq \sigma_c(\Omega(X))$, where $\Omega(X) = (X, \leq_\tau), \leq_\tau$ is the specialization order of (X, τ).

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Proposition

Let (X,τ) be a c-well-filtered space and A a saturated subset of X. Then (A,τ_A) is c-well-filtered in the inherited topology.

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Proposition

A retract of a c-well-filtered space is c-well-filtered.

A countably approximating poset L is defined as:

1. a countably directed complete poset– supremums of countably directed subsets of L exist.
2. the countably way-below relation on L is approximating.
   - $x, y \in L$, x is said to be countably way-below y (in symbol, $x \ll_c y$) if for every countably directed subset $D \subseteq L$ with $y \leq \sup D$, $x \leq d$ for some $d \in D$.
   - the countably way-below relation is said to be approximating if for all $a \in L$, $\{x \mid x \ll_c a\}$ is countably directed and $\sup\{x \mid x \ll_c a\} = a$.

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A nonempty subset F of L is called a countable filter if it is a countably filtered upper set.

Saturated Lindelöf subsets

Proposition

Let X be a c-well-filtered space. Then $K = \bigcap \mathcal{C}$ is a nonempty saturated Lindelöf set for each countable filter base $\mathcal{C}$ of nonempty saturated Lindelöf subsets of X. Hence, $(\mathcal{LQ}(X), \supseteq)$ is a countably directed complete poset.

Saturated Lindelöf subsets

Proposition

Let X be a P-space.

- Let $K_1, K_2 \in \mathcal{LQ}(X)$ and consider the following assertions:

  (a) There exists $U \in \mathcal{O}(X)$ such that $K_1 \supseteq U \supseteq K_2$, i.e. $\mathrm{int}(K_1) \supseteq K_2$;

  (b) $K_1 \ll_c K_2$ in $\mathcal{LQ}(X)$.

  If X is c-well-filtered, then $(a) \Rightarrow (b)$; if X is locally Lindelöf, then $(b) \Rightarrow (a)$.
- If X is a locally Lindelöf and c-well-filtered space, then $(\mathcal{LQ}(X), \supseteq)$ is a countably approximating poset.

Hofmann-Mislove theorem

Lemma

Let X be a c-well-filtered space and $U \in \mathcal{O}(X)$. Then the set

$$\phi^{'}(U) = \{K \in \mathcal{LQ}(X) : K \subseteq U\}$$

is a σ-Scott open countable filter in $(\mathcal{LQ}(X), \supseteq)$.

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The set of all σ-Scott open countable filters of $(\mathcal{LQ}(X), \supseteq)$ is denoted by $\mathrm{OCFilt}_\sigma((\mathcal{LQ}(X), \supseteq))$.

Hofmann-Mislove theorem

Theorem

Let X be a locally Lindelöf and c-well-filtered P-space, then the mapping

$$\phi^{'} : \mathcal{O}(X) \rightarrow OCFilt_{\sigma}((\mathcal{LQ}(X), \supseteq))$$

is an order isomorphism.

Thank you !