An Insight into Trigonometric Identities and Trigonometric Equations

Elementary Trigonometric Results

Recall definition of sine, cosine and tangent of an angle.
Let \( P(x, y) \) be a point on a circle with radius \( r = \sqrt{x^2 + y^2} \) in the Cartesian Plane
As usual, \( \angle XOP \) is measured in the direction from \( OX \) to \( OP \), taking anticlockwise as positive.
Let \( \theta = \angle XOP \)
Then \( \sin \theta = \frac{y}{r}, \ \cos \theta = \frac{x}{r}, \ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x} \)

Relations between trigonometric functions of angles of opposite signs

Observe the following diagram where points A and A’, B and B’ are reflection of each other in the x-axis (think of A’ to A as well as A to A’)

We can summarize the change of coordinates corresponding to change of sign of angle (namely a reflection) as this :
\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ -y \end{pmatrix} \text{ as } \theta \rightarrow -\theta, \text{ for all values of } \theta, \text{ positive or negative}
\]
We thus have the result
\[
\sin(-\theta) = -\sin \theta, \ \cos(-\theta) = \cos \theta \text{ and } \tan(-\theta) = -\tan \theta \quad \ldots \ldots \quad (1)
\]

Relations between trigonometric functions of angles by a difference of one right angle

The following diagram shows how coordinates are changed as a result of adding one right angle to a given one. The various cases are:
\( \angle XOP \) is changed to \( \angle XOQ \), \( \angle XOQ \) is changed to \( \angle XOR \), 

We can summarize the change of coordinates as a result of the addition of one right angle as this:

\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x \end{pmatrix} \text{ as } \theta \rightarrow \theta + \frac{\pi}{2}, \text{ where } \theta \text{ is in any quadrant}
\]

We thus have

If \( \cos \theta = \frac{x}{r} \) and \( \sin \theta = \frac{y}{r} \),

then \( \sin(\theta + \frac{\pi}{2}) = \frac{x}{r} = \cos \theta \), \( \cos(\theta + \frac{\pi}{2}) = \frac{-y}{r} = -\sin \theta \)

So,

\[
\sin(\theta + \frac{\pi}{2}) = \cos \theta, \quad \cos(\theta + \frac{\pi}{2}) = -\sin \theta \quad \text{and} \quad \tan(\theta + \frac{\pi}{2}) = -\tan \theta
\]

for all values of \( \theta \) ................................................................................................. (2)

(The reader should not just take \( \angle XOP \) as an example of \( \theta \), instead, think about \( \angle XOQ, \angle XOR \) and \( \angle XOS \) as well )

With results (1) and (2), which are valid for all values of \( \theta \), we deduce all results related to complimentary and supplementary angles.

For example,

\[
\cos(180^\circ - \theta) = \cos(90^\circ + 90^\circ - \theta) = -\sin(90^\circ - \theta) = -\cos(-\theta) = -\cos \theta
\]

which is true for all values of \( \theta \)

**Formulae for addition and subtraction of angles.**
The above figure shows two angles \( p \) and \( q \) of any size, given by \( \angle AOC \) and \( \angle AOB \) respectively, where A, B and C are points on a circle, centre O, radius \( r \). (It need not be \( p > q \) as depicted)

Let OA be lying on the OX axis and so the angles are measured using OX as initial direction.

We find \( C(r \cos p, r \sin p) \), \( B(r \cos q, r \sin q) \)

By using the cosine rule, we have, for all possible positions of B and C,

\[
2(\text{OB})(\text{OC}) \cos(p - q) = (\text{OB})^2 + (\text{OC})^2 - (\text{BC})^2
\]

So,

\[
2r^2 \cos(p - q) = r^2 + r^2 - [(r \cos p - r \cos q)^2 + (r \sin p - r \sin q)^2]
\]

\[
= 2r^2 - r^2[\cos^2 p - 2 \cos p \cos q + \cos^2 q] - r^2[\sin^2 p - 2 \sin p \sin q + \sin^2 q]
\]

\[
= 2r^2 - r^2[1 + 1 - 2 \cos p \cos q - 2 \sin p \sin q]
\]

\[
= 2r^2[\cos p \cos q + \sin p \sin q]
\]

So, \( \cos(p - q) = \cos p \cos q + \sin p \sin q \), for all values of \( p \) and \( q \) ................. (3)

From results (1) , (2) and (3) we can establish all formulae related to additions and subtractions of angles,

For example

\[
\cos(p + q) = \cos(p - q) = \cos p \cos q - \sin p \sin q = \cos p \cos q - \sin p \sin q
\]

**Trigonometric Equations**

Let us investigate the three simple equations

\[
\sin \theta = a, \text{ where } -1 \leq a \leq 1
\]

\[
\cos \theta = b, \text{ where } -1 \leq b \leq 1
\]

\[
\tan \theta = c, \text{ for all values of } c
\]
First note that all three equations have two solutions in any interval of length $2\pi$, for example, we can find two solutions in each of $(0.5\pi, 2.5\pi), (-0.9\pi, 1.1\pi)$ etc.

So, to begin with, it is usually convenient to obtain two in the range of $[-\pi, \pi]$

Next, we note that if $\alpha$ is a solution, say to $\sin \theta = a$, where $-1 \leq a \leq 1$, then

$$\sin \alpha = a, \quad \sin(\alpha + 2\pi) = a, \quad \sin(\alpha - 2\pi) = a, \quad \sin(\alpha + 2n\pi) = a,$$

where $n$ is any integer.

So, if $\alpha$, $\beta$ are two solutions to $\sin \theta = a$, where $-1 \leq a \leq 1$, in the interval $[-\pi, \pi]$, then the complete solution set is $\theta = \alpha + 2n\pi, \beta + 2n\pi$, though of course, it works out to be $\theta = n\pi + (-1)^n(\alpha)$, which is equivalent to $\theta = n\pi + (-1)^n(\beta)$

(More discussions are left as an exercise)

Exercise (without answers attached)

Recall these results:

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta \quad \text{and} \quad \tan(-\theta) = -\tan \theta,$$

for all values of $\theta$ ................................. (1)

$$\sin(\theta + \frac{\pi}{2}) = \cos \theta, \quad \cos(\theta + \frac{\pi}{2}) = -\sin \theta \quad \text{and} \quad \tan(\theta + \frac{\pi}{2}) = -\tan \theta$$

for all values of $\theta$ ................................. (2)

$$\cos(p - q) = \cos p \cos q + \sin p \sin q, \quad \text{for all values of} \quad p \text{ and } q \quad \text{......................... (3)}$$

1. Testify the rule :

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix} \text{ as } \theta \to \theta + 90^0, \text{ where } P(x, y) \text{ is the point on a circle of radius } r, \theta = \angle XOP, \text{ for the following cases}

2. By using Results (1) and (2) above, express the following in terms of $\sin \theta, \cos \theta$ and $\tan \theta$ only

   (i) $\sin(180^0 - \theta)$  (ii) $\cos(180^0 - \theta)$  (iii) $\tan(180^0 - \theta)$  
   (iv) $\sin(180^0 + \theta)$  (v) $\cos(180^0 + \theta)$  (vi) $\tan(180^0 + \theta)$

3. By using Results (1) and (2) above, express the following in terms of $\sin \theta, \cos \theta, \tan \theta$ and $\cot \theta$ only

   (i) $\sin(270^0 - \theta)$  (ii) $\cos(270^0 - \theta)$  (iii) $\tan(270^0 - \theta)$
   (iv) $\sin(270^0 + \theta)$  (v) $\cos(270^0 + \theta)$  (vi) $\tan(270^0 + \theta)$

4. By using Results (1), (2) and (3) above, obtain the formulae

$$\sin(p + q) = \sin p \cos q + \cos p \sin q \quad \text{and} \quad \sin(p - q) = \sin p \cos q - \cos p \sin q$$
5. \( \alpha, \beta \) are two different solutions of the equation
\[ \sin \theta = a, \text{ where } |a| \leq 1 \]
State a relation between \( \alpha \) and \( \beta \) if
(i) \( \alpha \) and \( \beta \) are in the same quadrant
(ii) \( \alpha \) and \( \beta \) are in different quadrant
(iii) there is no information about the quadrants of \( \alpha \) and \( \beta \).

6. \( \alpha, \beta \) are two different solutions of the equation
\[ \cos \theta = a, \text{ where } |a| \leq 1 \]
State a relation between \( \alpha \) and \( \beta \)

7. \( \alpha, \beta \) are two different solutions of the equation
\[ \tan \theta = a \]
What is the values of \( \alpha - \beta \)?

8. In the solution for the equation \( \cos(2x - 70^\circ) = 0.5 \), values of \( x \) required for
are in the interval \((-150^\circ, 570^\circ)\)
How many possible values of \( x \) are there?
Exercise (with answers attached)

Recall these results:

\[
sin(-\theta) = -\sin \theta, \cos(-\theta) = \cos \theta \quad \text{and} \quad \tan(-\theta) = -\tan \theta,
\]
for all values of \( \theta \) ........................................ (1)

\[
sin(\theta + \frac{\pi}{2}) = \cos \theta, \quad \cos(\theta + \frac{\pi}{2}) = -\sin \theta \quad \text{and} \quad \tan(\theta + \frac{\pi}{2}) = -\tan \theta
\]
for all values of \( \theta \) ........................................ (2)

\[
cos(p - q) = \cos p \cos q + \sin p \sin q, \quad \text{for all values of} \quad p \quad \text{and} \quad q \quad \text{..........} \quad (3)
\]

1. Testify the rule:

\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x \end{pmatrix} \quad \text{as} \quad \theta \rightarrow \theta + 90^\circ, \quad \text{where} \quad P(x, y) \quad \text{is the point on a circle of radius}
\]
\( r, \quad \theta = \angle XOP, \quad \text{for the following cases} \)

(i) \( \theta = 240^\circ \) \quad (ii) \( \theta = -240^\circ \) \quad (iii) \( \theta = 500^\circ \)

2. By using Results (1) and (2) above, express the following in terms of \( \sin \theta, \cos \theta \) and \( \tan \theta \) only

\[
\begin{array}{llll}
(i) & \sin(180^\circ - \theta) & (ii) & \cos(180^\circ - \theta) \\
(iv) & \sin(180^\circ + \theta) & (v) & \cos(180^\circ + \theta) \\
\end{array}
\]

Answers:

\[
\begin{array}{llll}
(i) & \sin \theta & (ii) & -\cos \theta \\
(iv) & -\sin \theta & (v) & -\cos \theta \\
\end{array}
\]

3. By using Results (1) and (2) above, express the following in terms of \( \sin \theta, \cos \theta, \tan \theta \) and \( \cot \theta \) only

\[
\begin{array}{llll}
(i) & \sin(270^\circ - \theta) & (ii) & \cos(270^\circ - \theta) \\
(iv) & \sin(270^\circ + \theta) & (v) & \cos(270^\circ + \theta) \\
\end{array}
\]

Answers:

\[
\begin{array}{llll}
(i) & -\cos \theta & (ii) & -\sin \theta \\
(iv) & -\cos \theta & (v) & \sin \theta \\
\end{array}
\]

4. By using Results (1), (2) and (3) above, obtain the formulae

\[
sin(p + q) = \sin p \cos q + \cos p \sin q \quad \text{and} \quad \sin(p - q) = \sin p \cos q - \cos p \sin q
\]

Solution

\[
\begin{align*}
sin(p + q) &= -\cos(\frac{\pi}{2} + p + q) = -[\cos(\frac{\pi}{2} + p) \cos q - \sin(\frac{\pi}{2} + p) \sin q] \\
&= \sin p \cos q + \cos p \sin q
\end{align*}
\]

\[
\begin{align*}
sin(p - q) &= -\cos(\frac{\pi}{2} + p - q) = -[\cos(\frac{\pi}{2} + p) \cos q - \sin(\frac{\pi}{2} + p) \sin q] \\
&= \sin p \cos q - \cos p \sin q
\end{align*}
\]
\[
\cos \theta = a, \quad \sin \theta = b, \quad |a| \leq 1
\]

State a relation between \(\alpha\) and \(\beta\) if

(i) \(\alpha\) and \(\beta\) are in the same quadrant

(ii) \(\alpha\) and \(\beta\) are in different quadrant

(iv) there is no information about the quadrants of \(\alpha\) and \(\beta\).

Answers:

(i) \(\beta = 2n\pi + \alpha\), where \(n \in \mathbb{Z}\)

(ii) \(\beta = 2n\pi + \pi - \alpha\), where \(n \in \mathbb{Z}\)

(iii) \(\beta = \pi - \alpha\), \(2\pi + \alpha\), \(3\pi - \alpha\), \(\vdots\)

So, \(\beta = n\pi + (-1)^n\alpha\), where \(n \in \mathbb{Z}\).

6. \(\alpha, \beta\) are two different solutions of the equation

\(\cos \theta = a, \quad |a| \leq 1\)

State a relation between \(\alpha\) and \(\beta\)

Answer

\(\beta = 2n\pi \pm \alpha\), where \(n \in \mathbb{Z}\)

7. \(\alpha, \beta\) are two different solutions of the equation

\(\tan \theta = a\)

What is the values of \(\alpha - \beta\)?

Answer

\(\alpha - \beta = n\pi\), for \(n \in \mathbb{Z}\)

8. In the solution for the equation \(\cos(2x - 70^\circ) = 0.5\), values of \(x\) required for

are in the interval \((-150^\circ, 570^\circ)\)

How many possible values of \(x\) are there?

Answer

The interval length for \(x\) is \(570^\circ - (-150^\circ) = 720^\circ = 2 \times 360^\circ\)

So, the interval length for \((2x - 70^\circ)\) is \(4 \times 360^\circ\)

So, there are 8 possible values for \((2x - 70^\circ)\), and therefore also 8 possible values for \(x\).