

An Insight into Division Algorithm, Remainder and Factor Theorem

Division Algorithm

Recall division of a positive integer by another positive integer

For example, $78 \div 7$, we get 11 and remainder 1

We confine the quotient to be a positive integer and allow it to be as large as possible, but that the remainder must be positive. The restriction results in the remainder being smaller than the divisor, otherwise the division is not complete, we should increase the quotient.

Division in Polynomials

A polynomial is an algebraic expression of the form

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, where a_0, a_1, \dots, a_n are real numbers and n is a positive integer.

When a polynomial $P(x)$ is divided by another polynomial $D(x)$, we stop the division when the remainder has degree less than the divisor $D(x)$

So we see at once that division is needed only if degree of $D(x)$ is not more than that of $P(x)$. At any step of the division, we try to clear the term with the highest degree left from the dividend

One case of division is of great interest to us, it is when the divisor is of degree 1, that is when $D(x)$ is of the form $ax - b$, then the remainder is a constant, that is, a term not involving x

The result is usually written in the simple form

$P(x) = (x - \alpha)Q(x) + R$, where the divisor appears as $x - \alpha = x - \frac{b}{a}$ instead of $ax - b$

This can be written also as $P(x) = (ax - b)\left[\frac{Q(x)}{a}\right] + R$. (We must have already implied that $a \neq 0$ when we say the divisor is $ax - b$)

We see that the remainder is the same whether we take the divisor as $ax - b$ or $x - \alpha$ as long as $\frac{b}{a} = \alpha$

Looking at $P(x) = (x - \alpha)Q(x) + R$ again, we see that the LHS and RHS are actually identical expression, in fact the equality holds for any value of x

We may write $P(x) \equiv (x - \alpha)Q(x) + R$, meaning that both side are equal for all values of x

The result $P(\alpha) = R$ follows

This result states that $P(\alpha)$ can be found by division or by substitution.

If $P(\alpha) = 0$, we find a root α of the equation $P(x) = 0$ and at the same time a factor $x - \alpha$ of $P(x)$

Our interest is generally the solving of the equation $P(x) = 0$, of degree 2 or more

The algorithm is:

Work out $P(\alpha_1), P(\alpha_2), P(\alpha_3), \dots$ until we get $P(\alpha) = 0$

Use division or some other method to obtain $P(x) = (x - \alpha)Q(x)$

Repeat now with $Q(x)$

When we get $Q(\beta) = 0$, we shall get $Q(x) = (x - \beta)Q_2(x)$

So, $P(x) = (x - \alpha)(x - \beta)Q_2(x)$

The above process looks tedious. In practice, we will take $\alpha_1, \alpha_2, \alpha_3, \dots$ to be the easy choices such as 0, 1, 2, -1, ... then the rational numbers made out from factors of a_0 and a_n if all coefficients of $P(x)$ are integers

After successfully putting $P(x) = (x - \alpha)Q(x)$, try for the other roots, which are now in $Q(x) = 0$, save repeating the failure cases as we try for $P(x) = 0$, but do include the successful cases again.

For enhanced effectiveness, leave out writing $x^n, x^{n-1}, x^{n-2}, \dots, x^2, x$ in the work for long division, but ensuring that the coefficients going to the correct positions. We do the long division, to see that the remainder (the value of $P(x)$) is zero. If so, we have found, at the same time, the linear factor and its corresponding quotient.

Example

Solve the cubic equation $x^3 - 7x^2 + 4x + 12 = 0$

Solution

Let $f(x) = x^3 - 7x^2 + 4x + 12$

We find that

$$f(0) = 12, f(1) = 10, f(-1) = -1 - 7 - 4 + 12 = 0$$

So, $x + 1$ is a factor, or say $x = -1$ is a root

By long division,

$$\begin{array}{r} \overline{1-8+12} \\ 1+1 \overline{)1-7+4+12} \\ \underline{1+1} \\ -8+4 \\ \underline{-8-8} \\ 12+12 \\ \underline{12+12} \\ 0 \end{array}$$

$$f(x) = (x+1)(x^2 - 8x + 12)$$

Let $q(x) = x^2 - 8x + 12$

By inspection, $q(x) = (x-2)(x-6)$

We have $f(x) = (x+1)(x-2)(x-6) = 0 \Rightarrow x = -1, 2, 6$

Exercise (without solution attached)

1. Factorize the expression $x^4 - 3x^2 - 6x + 8$
2. Given $x^2 - 3x + 2$ is a factor of $x^4 + ax^2 + bx + 8$, find the values of a and b .
3. Find the remainder when $x^4 - x^2 + 1$ is divided by $2x - 2$
4. Find the remainder when $x^3 - 2x^2 + x + 1$ is divided by $2x - 3$
5. Factorize the polynomial $x^4 + x^3 - 3x^2 - x + 2$ as far as possible
6. Factorize the polynomial $x^3 - 6ax^2 + 11a^2x - 6a^3$ as far as possible
7. Show that $x - a$ is a factor of $x^3 + (1 - a)x^2 + (3 - a)x - 3a$. Find the other factor
8. Show by Remainder Theorem that $x - \frac{-1 + \sqrt{13}}{2}$ is a factor of $x^2 + x - 3$
9. Show that $x - 2\sqrt{3}$ is a factor of $x^2 - \sqrt{3}x - 6$. What is the other factor?
10. A polynomial leaves remainder 2 when it is divided by $x - 1$ or $x^2 + 1$. If the coefficient of x^3 is 1, what is the polynomial?

Exercise (with solution attached)

1. Factorize the expression $x^4 - 3x^2 - 6x + 8$

Answer

$$(x-1)(x-2)(x^2 + 3x + 4)$$

2. Given $x^2 - 3x + 2$ is a factor of $x^4 + ax^2 + bx + 8$, find the values of a and b .

Solution

$$x^2 - 3x + 2 = (x-1)(x-2), \text{ so } x-1 \text{ and } x-2 \text{ are factors of } x^4 + ax^2 + bx + 8$$

.....

$$a = -3, b = -6$$

3. Find the remainder when $x^4 - x^2 + 1$ is divided by $2x - 2$

Answer

$$1$$

4. Find the remainder when $x^3 - 2x^2 + x + 1$ is divided by $2x - 3$

Answer

$$\frac{11}{8}$$

5. Factorize the polynomial $x^4 + x^3 - 3x^2 - x + 2$ as far as possible

Answer

$$(x+1)(x-1)^2(x+2)$$

6. Factorize the polynomial $x^3 - 6ax^2 + 11a^2x - 6a^3$ as far as possible

Answer

Try $x = a, x = 2a, x = 3a$

$$(x-a)(x-2a)(x-3a)$$

7. Show that $x - a$ is a factor of $x^3 + (1-a)x^2 + (3-a)x - 3a$. Find the other factor

Answer

$$(x-a)(x^2 + x - 3)$$

8. Show by Remainder Theorem that $x - \frac{-1 + \sqrt{13}}{2}$ is a factor of $x^2 + x - 3$

Answer

By substituting $x = \frac{-1 + \sqrt{13}}{2}$, we get remainder zero

9. Show that $x - 2\sqrt{3}$ is a factor of $x^2 - \sqrt{3}x - 6$. What is the other factor?

Answer

By substituting $x = 2\sqrt{3}$, we get remainder zero

By division, we get the other factor $x + \sqrt{3}$

$$(x - 2\sqrt{3})(x + \sqrt{3})$$

10. A polynomial leaves remainder 2 when it is divided by $x-1$ or x^2+1 . If the coefficient of x^3 is 1, what is the polynomial?

Answer

Let the polynomial be $f(x)$, then $f(x)-2$ is divisible by $x-1$ and x^2+1

So $f(x)-2 = (x-1)(x^2+1)$, $f(x) = (x-1)(x^2+1)+2 = x^3 - x^2 + x + 1$