An Insight into Quadratic Equations and Cubic Equations with Real Coefficients

Quadratic Equations

A quadratic equation is an equation of the form
$ax^2+bx+c=0$, where $a \neq o$
It can be solved quickly if we can factorize the expression ax^2+bx+c
If we find $ax^2+bx+c=$ $a(x-\alpha)(x-\beta)$, then $a(x-\alpha)(x-\beta)=0$ leads to the solutions $x=\alpha,$ or $x=\beta$
In the case of encountering some difficulty in the factorization, we have the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The formula ensures the solution of the equation, if there is any, or otherwise reveals its situation.
The important component in the formula is the discriminant $\Delta=b^2-4ac$

Relations between roots and coefficients of a quadratic equation

Assuming the roots of

$ax^2+bx+c=0$ to be $\alpha,\ \beta$

We can re-write the equation as

$$a(x-\alpha)(x-\beta)=0$$

It is clear then that

$$ax^2+bx+c\equiv a(x-\alpha)(x-\beta)\equiv a[x^2-(\alpha+\beta)x+\alpha\beta]$$

So $\quad \alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

These results can also be deduced by equating $\alpha,\ \beta$ to

$\frac{-b+\sqrt{b^2-4ac}}{2a}$, $\frac{-b-\sqrt{b^2-4ac}}{2a}$ respectively

and then work out $\alpha+\beta$ and $\alpha\beta$

If there is no real root for the equation, the expression $f(x)=ax^2+bx+c$ cannot change sign as we vary the values of x through all numbers, that is, it must remain either positive or negative throughout, otherwise according to continuity, for $f(x)$ to change from a positive value to a negative value, it must be zero somewhere, say at $x=\alpha$, and that is the root of $f(x)=0$.
Hence $f(x)=ax^2+bx+c$ is either positive or negative for all values of x if the discriminant $\Delta=b^2-4ac$ <0, that is, when the equation $f(x)=0$ has no real root.

Cubic Equation

A cubic equation is an equation of the form
$ax^3+bx^2+cx+d=0$, where $a \neq o$

Let $g(x) = ax^3 + bx^2 + cx + d$

While we say that a quadratic (real) equation may not have real root, a cubic (real) equation has at least one real root.
For if L is a large value,
the sign of $g(L)$ is different from the sign of $g(-L)$
(Full discussion is taken as an exercise)

So, given a cubic (real) equation, it is wise to find by trail one root first, for there is certainly one
With one root found, the corresponding factor for the other roots will be a quadratic expression. The second step is then solving a quadratic equation.

Relations between roots and coefficients of a cubic equation

Assuming the roots of

$ax^3 + bx^2 + cx + d = 0$ to be α, β and γ

We can re-write the equation as

$$a(x-\alpha)(x-\beta)(x-\gamma) = 0$$

Thus $ax^3 + bx^2 + cx + d \equiv a(x-\alpha)(x-\beta)(x-\gamma)$

We see that

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

If one non-zero root γ is found,

We will have $\alpha + \beta = -\frac{b}{a} - \gamma$

and $\alpha\beta = -\frac{d}{a\gamma}$

With these, we can solve the remaining quadratic equation, perhaps, as effective as by inspection

If one root γ is found to be zero, then $d = 0$, the equation is actually

$$ax^3 + bx^2 + cx = 0$$

which is $x(ax^2 + bx + c) = 0$

The other root is given by the quadratic equation $ax^2 + bx + c = 0$

Exercise (without solution attached)

1. Given the quadratic equation $ax^2 + bx + c = 0$
   Show that if a and c have different signs, the equation will have real roots, but the converse of this statement is not true

2. Given the quadratic equation $ax^2 + bx + c = 0$
   State the conditions for it to have
   (i) two roots of different signs
   (ii) two roots of the same sign

3. Given two equations $u + v = h,\ uv = k$. Show how to solve for u and v

4. Show that when L is a large value, $g(L) = aL^3 + bL^2 + cL + d$ has the same sign as aL^3, $g(L)$ and $g(-L)$ have different signs

5. Show that $x = 1$ is a root of the equation $f(x) = 4x^3 - 8x^2 + x + 3 = 0$
   Show that the sum of the other two roots is 1 and that their product is $-\dfrac{3}{4}$
   Hence write down the quadratic equation for the other two roots

6. Given that $p - q = 7$ and $pq = -10$. Solve for p, q by using the idea of sum of roots and product of roots. (two sets of answers)

Exercise (with solution attached)

1 Given the quadratic equation $ax^2+bx+c=0$
Show that if a and c have different signs, the equation will have real roots, but the converse of this statement is not true

Solution

Consider the discriminant $\Delta=b^2-4ac$. If ac is negative, then $\Delta=b^2-4ac$ is positive. On the other hand, $\Delta=b^2-4ac$ is positive does not require that ac is negative

2 Given the quadratic equation $ax^2+bx+c=0$
State the conditions for it to have
(i) two roots of different signs
(ii) two roots of the same sign

Solution

(i) Assume $\alpha,\ \beta$ to be the two roots, then $\alpha\beta=\dfrac{c}{a}<0$
if a and c have different signs, this in turn ensures real roots for the equation
So the condition is that a and c have different signs.

(ii) Assume $\alpha,\ \beta$ to be the two roots, then $\alpha\beta=\dfrac{c}{a}>0$
Further, to ensure real roots, $\Delta=b^2-4ac\geq 0$
So the conditions are that a and c have same sign, and that $\Delta=b^2-4ac\geq 0$

3 Given two equations $u+v=h,\ uv=k$. Show how to solve for u and v

Solution

(I) We may view u,v as two roots, and so we have
Sum of roots = h, product of roots= k,
We then solve $x^2-hx+k=0$

(II) By substitution $v=\dfrac{k}{u}$, $u+\dfrac{k}{u}=h$, so $u^2-hu+k=0$

4. Show that when L is a large value, $g(L)=aL^3+bL^2+cL+d$ has the same sign as aL^3, $g(L)$ and $g(-L)$ have different signs

Solution

$g(L)=L^3(a+\dfrac{b}{L}+\dfrac{c}{L^2}+\dfrac{d}{L^3})$, when L is large, $\dfrac{b}{L}+\dfrac{c}{L^2}+\dfrac{d}{L^3}$ is very small as compared to a, so the sign of $a+\dfrac{b}{L}+\dfrac{c}{L^2}+\dfrac{d}{L^3}$ is the sign of a,
the sign of $g(L)$ is the sign of aL^3
Similarly, the sign of $g(-L)$ is the sign of $a(-L)^3=-aL^3$
So $g(L)$ and $g(-L)$ have different sign, when L is large

5. Show that $x=1$ is a root of the equation $f(x)=4x^3-8x^2+x+3=0$

Show that the sum of the other two roots is 1 and that their product is $-\frac{3}{4}$

Hence write down the quadratic equation for the other two roots

Solution

We find that $f(1)=0$,

So, $x=1$ is a root of the equation $f(x)=4x^3-8x^2+x+3=0$

Let the three roots be $\alpha,\ \beta$ and γ, with $\gamma=1$

Then $\alpha+\beta+\gamma=2$

And $\alpha\beta\gamma=-\frac{3}{4}$

Therefore $\alpha+\beta=1$, $\alpha\beta=-\frac{3}{4}$

The quadratic equation for the remaining two roots is

$x^2-x-\frac{3}{4}=0$, which is the same as $4x^2-4x-3=0$

6. Given that $p-q=7$ and $pq=-10$. Solve for p,q by using the idea of sum of roots and product of roots. (two sets of answers)

Solution

Let $\alpha = p$ and $\beta=-q$, then we have $\alpha+\beta=7$ and $\alpha\beta=10$

The quadratic equation for $\alpha,\ \beta$ is $x^2-7x+10=0$

We get $(\alpha,\ \beta) =$ (2, 5) or (5, 2), so $(p,$ q$)=$(2, -5) or (5, -2)