An Insight into Differentiation and Integration

Differentiation

Differentiation is basically a task to find out how one variable is changing in relation to another variable, the latter is usually taken as a cause of the change.

For instance, \( y = x^2 \)

We might consider \( x \) taking values: 1, 2, 3, …… \( n, n+1 \) successively by increment 1, then \( y \) is increasing by an increment

\[
(n+1)^2 - n^2 = 2n + 1 \quad \text{at } x = n
\]

The average increment over the interval \((n, n+1)\) of \( x \) is

\[
\frac{(n+1)^2 - n^2}{1} = 2n + 1
\]

To be more generally,

the average increment over the interval \((x, x+1)\) of \( x \) is

\[
\frac{(x+1)^2 - x^2}{1} = 2x + 1
\]

By decreasing the amount of increment of \( x \) to be \( \frac{1}{2} \), the average increment over the interval \((x, x + \frac{1}{2})\) of \( x \) is

\[
\frac{(x+\frac{1}{2})^2 - x^2}{\frac{1}{2}} = 2x + (\frac{1}{2})^2
\]

Generalizing, the average increment over the interval \((x, x + \delta x)\) of \( x \) is

\[
\frac{(x+\delta x)^2 - x^2}{\delta x} = 2x + (\delta x)^2
\]

Going on this way, we then come to the idea that the instantaneous average increment at \( x \) is \( 2x \), when the interval becomes widthless. This is defined as the rate of change of \( x^2 \) with respective to \( x \) at any value of \( x \).

To express the process of work, which we define as differentiation, and the result, we write

\[
\frac{d}{dx} x^2 = \lim_{\delta x \to 0} \frac{(x + \delta x)^2 - x^2}{\delta x} = 2x
\]

We have three fundamental results of differentiation:

\[
\frac{d}{dx} x^n = nx^{n-1} \quad \text{for positive integer } n
\]

\[
\frac{d}{dx} \sin x = \cos x \quad \text{(Based on the fundamental limit result} \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1), \text{ when}
\]

\( x \) is in radians

\[
\frac{d}{dx} e^x = e^x
\]
From these, with the helps of a number of rules such as The Product Rules, The Quotient Rule, The Chain Rule, Implicit Function Rule… we deduce many standard results including
\[ \frac{d}{dx} x^n = nx^{n-1} \] for any rational number \( n \) (Finally, for any real number \( n \))

Of course, we should be able to establish all other results from first principles, which means a work relying only on the definition
\[ \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \]

**Integration**

Differentiation is important not only that it is a tool to track the change of one variable with respect to another, also, knowing of the derivative of a function leads us to the identification of the function itself.

For example, if we find that \( \frac{d}{dx} f(x) = \frac{d}{dx} g(x) = u(x) \), then \( f(x) \) and \( g(x) \) can only differ by a constant.

So, given \( \frac{d}{dx} f(x) = u(x) \), to find \( f(x) \), we first look for \( g(x) \) which has the known derivative \( u(x) \).

The job is \( \int u(x)dx \), which gives a result pending a constant.

We write \( f(x) = g(x) + c \).

If we know that \( f(a) = 0 \), then \( c = -g(a) \), \( f(x) = g(x) - g(a) \). This is a definite result, which gives value of \( f(x) \) at any value, for example, at \( x = b \),

\[ f(b) = g(b) - g(a) \], for which we write \( \int_a^b u(x)dx \)

The value is definite and the work is called a definite integration.

**Cases of using derivative to find the original functions**

A well-known case is finding distance traveled with knowledge of velocity \( v \). If \( s \) is the distance traveled in time \( t \), we have
\[ \frac{ds}{dt} = v \], based on the genuine definition of \( v \).

For some other cases, the derivative of a variable must be reasoned out, though they are usually taken intuitively. Take the case when we come to such work like finding area under a curve.
If we define $A(x)$ to be the area under the curve from $x = a$ to $x$

Intuitively, we take $\frac{dA}{dx} = y$, so $A = \int y \, dx$, Area $A(x)$ can then be found

To be a bit more rigorous
We first claim that $\delta A$ lies between $y \delta x$ and $(y + \delta y) \delta x$,

So, $\frac{\delta A}{\delta x}$ lies between $y$ and $(y + \delta y)$

$$\frac{dA}{dx} = \lim_{\delta x \to 0} \frac{\delta A}{\delta x} = y$$

Take another case, for finding the volume $V(h)$ of a right circular cone of height $h$
Denoting $V(h)$ to be the volume of the circular cone, of the same shape, with a height $h$, considered as a variable, $S$ to be the surface area of the circular cross-section at height $h,$
We claim that $\delta V$ lies between $S\delta h$ and $(S + \delta S)\delta h$

$\frac{\delta V}{\delta h}$ lies between $S$ and $(S + \delta S)$

so $\frac{dV}{dh} = \lim_{\delta h \to 0} \frac{\delta V}{\delta h} = S$.

By similarity, $r = kh$, for a constant $k$

$$V = \int S dh = \int \pi (kh)^2 dh = \pi k^2 \frac{h^3}{3} + c$$

$$= \frac{1}{3} \pi (kh)^2 h + c = \frac{1}{3} \pi r^2 h + c = \frac{1}{3} \pi r^2 h \text{ if we take } V = 0 \text{ for } h = 0$$

**Exercise (without solution attached)**

1. Differentiate $\tan x$ with respect to $x$ from first principles.
   (You may use the result $\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1$)

2. Base on the result $\frac{d}{dx} x^n = nx^{n-1}$ for positive integer $n$, and the rules of differentiation, prove that $\frac{d}{dx} x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}}$

   Prove also that $\frac{d}{dx} x^{\frac{2}{3}} = \frac{2}{3} x^{-\frac{1}{3}}$

3. A particle is moving vertically upwards with velocity $v = 50t - 5t^2$
   Show that it is moving downwards after $t = 10$
   Find the distance of the particle from its position at $t = 0$
(i) When \( t = 8 \)
(ii) When \( t = 12 \)

\[
s = \int_0^{12} (50t - 5t^2)dt = [25t^2 - \frac{5}{3}t^3]_0^{12} = 720
\]

4. Given that \( y = \sec x^0 \), find \( \frac{dy}{dx} \)

Given a reason why in Calculus, radian measure is preferred rather than degree measure.

5. A kite is at a horizontal distance \( l \) from the flyer and a vertical distance \( h \) above the same. When it is rising up at a velocity \( v \), the flyer let off more string to keep the horizontal distance constant.

Assuming the string to be straight, and the angle of elevation of the kite to be \( \theta \) radians, find the rate of change of \( \theta \) with respect to time \( t \)

6. Let \((e^x)^n = (1 + \frac{x}{n})^n\), where \( n \) is a large number

Show that \( \frac{d}{dx}(e^x)^n \approx (e^x)^n \)
Exercise (with solution attached)

1. **Differentiate** \( \tan x \) with respect to \( x \) from first principles.

   (You may use the result \( \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1 \))

   **Solution:**
   
   \[
   \frac{\tan(x + \delta x) - \tan x}{\delta x} = \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\delta x} = \frac{\delta x[\cos(x + \delta x) \cos x]}{\sin(x + \delta x - x)} = \frac{\sin \delta x}{\delta x} \frac{1}{\cos(x + \delta x) \cos x}
   \]

   So, \( \lim_{\delta x \to 0} \frac{\tan(x + \delta x) - \tan x}{\delta x} = \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \frac{1}{\cos(x + \delta x) \cos x} = \sec^2 x \)

2. **Base on the result** \( \frac{d}{dx} x^n = nx^{n-1} \) for positive integer \( n \), and the rules of differentiation, prove that \( \frac{d}{dx} x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}} \)

   Prove also that \( \frac{d}{dx} x^{\frac{2}{3}} = \frac{2}{3} x^{-\frac{1}{3}} \)

   **Solution:**
   
   Let \( y = x^\frac{1}{3} \), then \( y^3 = x \), \( 3y^2 \frac{dy}{dx} = 1 \), \( \frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3x^\frac{2}{3}} = \frac{1}{3} x^{-\frac{2}{3}} \)

   For proving of \( \frac{d}{dx} x^{\frac{2}{3}} = \frac{2}{3} x^{-\frac{1}{3}} \), let \( z^3 = x^2 \)

3. **A particle is moving vertically upwards with velocity** \( v = 50t - 5t^2 \)

   **Show that it is moving downwards after** \( t = 10 \)

   **Find the distance of the particle from its position at** \( t = 0 \)

   (iii) **When** \( t = 8 \)

   (iv) **When** \( t = 12 \)

   **Solution:**
   
   \( v = 50t - 5t^2 = 5t(10 - t) < 0 \) when \( t > 10 \)

   So, the particle is moving downwards after \( t = 10 \)

   (i) **If** \( s \) **is the distance moved upwards after** \( t \)

   \[
   \frac{ds}{dt} = v = 50t - 5t^2 \text{, noting that this is true for all} \ t, \text{ that is whether} \ t < 10 \text{ or} \ t \geq 10
   \]

   \[
   s = \int_0^8 (50t - 5t^2)dt = [25t^2 - \frac{5}{3}t^3]_0^8 = 746 \frac{2}{3}
   \]

   (ii) \( \frac{ds}{dt} = v = 50t - 5t^2 \text{, even though} \ t \geq 10 \)

   \[
   s = \int_0^{12} (50t - 5t^2)dt = [25t^2 - \frac{5}{3}t^3]_0^{12} = 720
   \]
4. Given that \( y = \sec x^\circ \), find \( \frac{dy}{dx} \)

Given a reason why in Calculus, radian measure is preferred rather than degree measure.

Solution

Let \( y = \sec x^\circ = \sec u \), where \( u \) radians = \( x \) degrees. Then \( 180u = x\pi \)

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\sec u \tan u) \cdot \frac{\pi}{180} = (\sec x^\circ \tan x^\circ) \cdot \frac{\pi}{180}
\]

As seen from the above, \( \frac{d}{dx} \sec x^\circ = (\sec x^\circ \tan x^\circ) \cdot \frac{\pi}{180} \)

Whereas \( \frac{d}{du} \sec u^\circ = \sec u \tan u \)

5. A kite is at a horizontal distance \( l \) from the flyer and a vertical distance \( h \) above the same. When it is rising up at a velocity \( v \), the flyer let off more string to keep the horizontal distance constant.

Assuming the string to be straight, and the angle of elevation of the kite to be \( \theta \) radians, find the rate of change of \( \theta \) with respective to time \( t \)

Solution

We have \( \tan \theta = \frac{h}{l} \), where \( h \) is a variable and \( l \) is a constant

Though an expression for \( \frac{d\theta}{dt} \) is what we are looking for, we need not write \( \theta = \tan^{-1}\left(\frac{h}{l}\right) \), and do \( \frac{d\theta}{dt} \). We can choose to do indirectly

Differentiating both sides of this equation, we get \( \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{l} \frac{dh}{dt} = \frac{1}{l} v \)

So \( \frac{d\theta}{dt} = (\cos^2 \theta) \frac{1}{l} v \), though the result is not totally in terms of the variable \( h \).

6. Let \( (e^\ast)^x = (1 + \frac{x}{n})^n \), where \( n \) is a large number

Show that \( \frac{d}{dx} (e^\ast)^x \approx (e^\ast)^x \)

Solution

\[
\frac{d}{dx} (e^\ast)^x = \frac{d}{dx} (1 + \frac{x}{n})^n = n(1 + \frac{x}{n})^{n-1} \cdot \frac{1}{n} = (1 + \frac{x}{n})^{n-1} \cdot (1 + \frac{x}{n}) = (1 + \frac{x}{n})^n \cdot \frac{1}{(1 + \frac{x}{n})}
\]

\[\approx (1 + \frac{x}{n})^n = (e^\ast)^x \text{, as } (1 + \frac{x}{n}) \approx 1\]