Some open problems on Chromatic polynomials

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1. The chromatic polynomial was introduced by Birkhoff in 1912 as a way to attack the four-colour problem.

2. Whitney (1932) established many fundamental results.

3. Birkhoff and Lewis in 1946 conjectured that the chromatic of any planar graph has no zeros larger than 4.
4. **R.C. Read in 1968** published an well referenced introductory article on chromatic polynomials.
Colourings of graphs

A 3-colouring, satisfying the condition:

Every two adjacent vertices are assigned different colours.
Chromatic polynomials

For any graph $G$ and any positive integer $\lambda$,

$P(G, \lambda)$ denotes the number of $\lambda$-colourings in $G$.

Or

$P(G, \lambda)$ denotes the number of mappings

$$f: V(G) \rightarrow \{1, 2, \ldots, \lambda\}$$

such that $f(x) \neq f(y)$ whenever $xy$ is an edge in $G$. 
Examples

1. $K_3$

$$P(K_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2).$$
Examples

2. $K_n$

$$P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1).$$
Examples

Empty graph $O_n$

$P(O_n, \lambda) = \lambda^n$. 
Fundamental Theorem

• For any graph $G$ and non-adjacent vertices $x,y$ in $G$,

$$P(G, \lambda) = P(G + xy, \lambda) + P(G/xy, \lambda),$$

where $G/xy$ is the graph obtained from $G$ by identifying $x$ and $y$ and removing all multi-edges but one.
By Maple

Find chromatic polynomials by Maple.
Chromatic polynomials

1. \( P(G,\lambda) \) is a polynomial in \( \lambda \) of degree \( n \), where \( n \) is the order of \( G \).

2. Actually,

\[
P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)(\lambda)_k,
\]

where \( \alpha(G,k) \) is the number of \( k \)-independent partitions of \( V(G) \), and \( (\lambda)_k = \lambda(\lambda-1) \ldots (\lambda-k+1) \).
Another expression

\[ P(G, \lambda) = \sum_{F \subseteq E} (-1)^{|F|} \lambda^{c(F)}, \]

where \( c(F) \) is the number of components in the subgraph of \( G \) induced by \( F \).
A graph-function

Graphs $G$ → $P(G, \lambda)$

It is a polynomial of $\lambda$. So $\lambda$ is regarded as a variable
Research directions

1. On the coefficients
2. Determine chromatic polynomials
3. Inequalities
4. Characterize chromatic polynomials
5. Locate zeros
Problem 1

By Read and Tutte (1988).

Problem: Find a general formula for the chromatic polynomial of $X_{m,n}$ below:

\[ X_{8,4} \]
Problem 2

By Welsh (1970’s) and Brenti (1992):

Conjecture: For every graph $G$ and $k \in \mathbb{N}$,

$$\left(P(G, k)\right)^2 \geq P(G, k + 1)P(G, k - 1).$$

Disproved by Paul Seymour in 1997.
Conjecture:
For every graph $G$ and $\lambda \in \mathbb{R}$ with $\lambda \geq n$, where $n$ is the order of $G$:

\[
(P(G,k))^2 \geq P(G,k+1)P(G,k-1).
\]
A known result:

Let $G$ be a connected $(n,m)$-graph.

If

$$\lambda \geq \max \{n - 1, \sqrt{2}(m - n + 2.5)\},$$

then

$$P(G, \lambda)^2 \geq P(G, \lambda + 1)P(G, \lambda - 1).$$
Problem 3

Unimodal Conjecture: by Read (1968).

For any graph $G$ of order $n$, if

$$P ( G , \lambda ) = \sum_{i = 1}^{n} a_i \lambda^i ,$$

then there are no $j$ such that

$$|a_j| > |a_{j+1}| \quad \text{and} \quad |a_{j+1}| < |a_{j+2}| .$$
Known Results

   It is known that if $G$ is connected, then
   \[ |a_n| \leq |a_{n-1}| \leq \cdots \leq |a_{\left\lfloor n/2 \right\rfloor}|. \]

2. Unimodel conjecture holds for some graphs:
   Chordal graphs, broken wheels, etc.
Strong Log Concave Conjecture by Hoggar (1974):

For any graph $G$ of order $n$, if

$$P(G, \lambda) = \sum_{i=1}^{n} a_i \lambda^i,$$

then there are no $j$ such that

$$a_{j+1}^2 > a_j a_{j+2}.$$
Problem 4

By Read (1968).

Is it possible to find a set of necessary and sufficient conditions for a polynomial to be the chromatic polynomial of some graph?

Some necessary conditions have been found.
Problem 5

By Read (1968)

Is there a necessary and sufficient condition for two graphs to have the same chromatic polynomial?
Problem 6

By Read (1968)

1. G is called **chromatically unique** if \( P(G, \lambda) = P(H, \lambda) \) implies that G is isomorphic to H for any graph H.

2. Is there a **necessary and sufficient condition** for a graph G to be chromatically unique?
Problem 7

By Xu and Li (1984):
For any even $n$ with $n \geq 12$,
is the wheel $W_n$ chromatically unique?
Chordless cycles

A cycle $C$ is called a chordless cycle if

1. $|C| > 3$ and
2. every two non-consecutive vertices on $C$ are not adjacent.
Chordal graphs

If $G$ contains no any chordless cycle, $G$ is called a chordal graph.
For any chordal graph $G$, 

$$P(G, \lambda) = \prod_{i=1}^{n} (\lambda - p_i)$$

for some integers $p_1, p_2, \ldots, p_n$.

In other words, $P(G, \lambda)$ has only integral roots.
Problem:
Is there a non-chordal graph $H$ such that $P(H, \lambda)$ has only integral roots?
An example Discovered by Read

1. Non-chordal
2. its chromatic polynomial is

\[(\lambda - 3)^2 (\lambda)^5.\]
Problem 8 proposed by Dmitriev (1980)

For any integer $p \geq 5$, does there exist a non-chordal graph $G$ such that

(1) $G$ has a chordless cycle of length $p$ and

(2) $P(G, \lambda)$ has only integer roots?

This problem has been solved for $p \leq 12$. 
An algebraic problem:

For any integer $k > 2$, do there exist positive integers $p_1, p_2, \cdots, p_k$ such that the following polynomial has only integer roots:

$$\prod_{i=1}^{k} (x - p_i) + (-1)^{k+1} \prod_{i=1}^{k} p_i = 0?$$
This algebraic problem

If it is settled

Dmitriev Problem
Known results on chromatic zeros

• (Sokal) Chromatic zeros are dense on the whole complex plane, i.e., for any complex $z_0$ and $\varepsilon > 0$, there exist chromatic zeros in $|z-z_0|< \varepsilon$.

• There are no chromatic zeros in the three intervals $(-\infty,0)$, $(0,1)$ and $(1,32/27]$ (by Jackson).

• For any graph $G$ of order $n$, $P(G, \lambda)$ has no zeros in the interval $(n-1, \infty)$. 
Problem 9

Beraha numbers:

\[ B_n = 2 + 2 \cos\left(\frac{2\pi}{n}\right), \quad n = 1, 2, 3, \ldots. \]

\[ \lim_{n \to \infty} B_n = 4. \]
By Beraha (1975):

Is it true that for every $\varepsilon > 0$ that there exists a plane triangulation $G$ such that $P(G, \lambda)$ has a zero in the interval $(B_n - \varepsilon, B_n + \varepsilon)$?

This problem is still open for $n=8$ and $n \geq 11$. 
Problem 10

Is there a planar graph $G$ such that $P(G, \lambda)$ has a zero in $(4,5)$?

or

$P(G, \lambda)>0$ for every loopless planar graph $G$ and every $\lambda$ in $(4,5)$?
Problem 11

Conjecture by Jackson (1993):

If $G$ is a 3-connected non-bipartite graph, then $P(G, \lambda)$ has no zeros in $(1,2)$. 
Problem 12

• **Known Result:**

  If $G$ is connected, then the multiplicity of zero $\lambda = 1$ of $P(G, \lambda)$ is equal to the block number of $G$.

• So if $G$ is 2-connected, then $(\lambda - 1)^2$ is not a factor of $P(G, \lambda)$.

• **Conjecture:** if $G$ is 3-connected, then $(\lambda - 2)^2$ is not a factor of $P(G, \lambda)$.
Problem 13

Conjecture by Sokal (2002):

\[ P(G, \lambda) > 0 \]

for every graph \( G \) and real \( \lambda > \Delta(G) \).

It is still open for \( n > 3\Delta(G) + 1 \).
Special Case: \( \Delta(G) = 3 \).

If \( \Delta(G) = 3 \), then

\[
P(G, \lambda) > 0 \text{ for all real } \lambda > 3.
\]
Problem 14

Let $G$ be a graph. For any two vertices $x, y$, 

$$\lambda(x, y) = \text{maximum number of edge-disjoint paths from } x \text{ to } y.$$ 

$$\Lambda \left( G \right) = \max_{x \neq y} \lambda \left( x, y \right).$$
By Sokal (2002)

Conjecture:

\[ P(G, \lambda) > 0 \text{ for every graph } G \text{ and real } \lambda > \Lambda(G) \]
Problem 15

Conjecture:

for any graph G, if z is a complex zero of $P(G, \lambda)$, then

$$\text{Re}(z) \leq n - 1,$$

where $n$ is the order of $G$. 
Problem 16

Conjecture:

for any graph $G$, $P(G, \lambda)$ has no zero $z$ such that

$$\text{Re}(z) = 0 \quad \text{and} \quad \text{Im}(z) \neq 0.$$
The end