The 3-Connectivity of a Graph and the Multiplicity of Zero “2” of Its Chromatic Polynomial

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Abstract: Let G be a graph of order n, maximum degree Δ, and minimum degree δ. Let P(G, λ) be the chromatic polynomial of G. It is known that the multiplicity of zero “0” of P(G, λ) is one if G is connected, and the multiplicity of zero “1” of P(G, λ) is one if G is 2-connected. Is the multiplicity of zero “2” of P(G, λ) at most one if G is 3-connected? In this article, we first construct an infinite family of 3-connected graphs G such that the multiplicity of zero “2” of P(G, λ) is more than one, and then characterize 3-connected graphs G with Δ + δ ≥ n such that the multiplicity of zero “2” of P(G, λ) is at most one. In particular, we show that for a 3-connected

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graph $G$, if $\Delta + \delta \geq n$ and $(\Delta, \delta) \neq (n-3,3)$, where $\delta$ is the third minimum degree of $G$, then the multiplicity of zero “2” of $P(G, \lambda)$ is at most one. © 2011 Wiley Periodicals, Inc. J Graph Theory 70: 262–283, 2012

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1. INTRODUCTION

Let $G$ be a simple graph (i.e., there are no multiedges and loops). Let $V(G)$, $E(G)$, $v(G)$, $e(G)$, $c(G)$, $\Delta(G)$, $\delta(G)$, and $P(G, \lambda)$ be, respectively, the vertex set, edge set, order, size, number of components, maximum degree, minimum degree, and chromatic polynomial of $G$. For any $x \in V(G)$, let $N(x) = \{y : xy \in E(G)\}$ and $N[x] = N(x) \cup \{x\}$. For any $S \subseteq V(G)$, let $N(S) = \bigcup_{x \in S} N(x) \cup \{x\}$, $G[S]$ (or simply $[S]$) be the subgraph of $G$ induced by $S$. $G - S$ be the induced subgraph $G[V(G) \setminus S]$, and $G/S$ be the graph with vertex set $(V(G) \setminus S) \cup \{w\}$, where $w \notin V(G)$, and edge set $E(G - S) \cup \{wx : x \in N(S)\}$. Thus, $G/S$ can be obtained from $G$ by first removing all edges in $G[S]$ and then identifying all vertices in $S$. If $S = \{x\}$, $G - S$ is also written as $G - x$. If $S = \{u, v\}$, we also write $G/uv$ for $G/S$ regardless if $uv \in E(G)$. Let $I_S$ be the family of independent sets $A$ of $G$ with $A \subseteq S$.

For a connected graph $G$, a vertex $x$ in $G$ is called a cut-vertex of $G$ if $G - x$ is disconnected. A connected graph is said to be separable if it contains a cut-vertex; and non-separable otherwise. Non-separable graphs consist of $K_2$ and all 2-connected graphs.

A zero of $P(G, \lambda)$ is called a chromatic zero of $G$. It is well known that the multiplicity of the chromatic zero “0” of $G$ is equal to $c(G)$. The multiplicity of the chromatic zero “1” of $G$ provides also a nice piece of structural information of $G$. For any connected graph $G$, let $b(G)$ be the number of blocks of $G$ if $v(G) \geq 2$ and $b(G) = 0$ if $v(G) = 1$. For any disconnected graph $G$, let

$$b(G) = \sum_{i=1}^{k} b(G_i),$$

where $G_1, G_2, \ldots, G_k$ are the (connected) components of $G$.

Theorem 1.1 (Whitehead and Zhao [7] and Woodall [8]). For any graph $G$, the multiplicity of the chromatic zero “1” of $G$ is equal to $b(G)$.

Wakelin [6] showed that flow polynomials have the similar property: if $G$ is bridgeless and loopless, then 1 is a zero of its flow polynomial $F(G, \lambda)$ of multiplicity equal to $b(G)$. Thus, 1 is a simple zero of $F(G, \lambda)$ if $G$ is 2-connected. Very recently, Jackson [3] showed that 2 is a simple zero of $F(G, \lambda)$ if $G$ is 3-connected and near-cubic (i.e., at most one vertex of $G$ has degree not equal to 3).

For the chromatic polynomials, it is an attractive problem (see [1], p. 36) to study whether 2 is a simple zero of $P(G, \lambda)$ if $G$ is 3-connected and non-bipartite. Woodall [8] showed that 2 is a simple zero of $P(G, \lambda)$ for any plane triangulation $G$. It is also not difficult to verify that 2 is a simple zero of $P(G, \lambda)$ for any 3-connected chordal

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graph $G$, where a chordal graph is a graph which does not contain any cycle $C$ with $|V(C)| \geq 4$ and $|V(C)| \leq 3$.

The main results we establish in this article are as follows:

(i) there exists an infinite family $\mathcal{J}$ of 3-connected graphs $G$ such that $(\lambda - 2)^2 |P(G, \lambda)|$ (see Section 4);

(ii) for any 3-connected graph $G$ with $\Delta(G) + \delta(G) \geq v(G)$, $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if $G \in \mathcal{J}$.

In Section 2, we shall show that for any non-bipartite graph $G$ and $w \in V(G)$, $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if

$$\sum_{A \in \mathcal{I}_S} (-1)^{|A|} \tau(G - A) = 0,$$  \hfill (2)

where $S = V(G) \setminus N[w]$ and $\tau(H)$ is the value of the polynomial $(-1)^{|H|} P(H, \lambda)/(\lambda - 1)$ when $\lambda = 1$ for a non-empty graph $H$ (i.e., $E(H) \neq \emptyset$). In Section 3, we prove some results on the function $\tau(G)$ which will be used in Sections 4 and 5. In Section 4, we exhibit an infinite family $\mathcal{J}$ of 3-connected graphs $G$ such that $(\lambda - 2)^2 |P(G, \lambda)|$. In Section 5, we apply the results in Sections 2 and 3 to show that for any non-bipartite 3-connected graph $G$, if there exists $w \in V(G)$ such that $\delta(G - w - A) > 0$ for every $A \in \mathcal{I}_S \setminus \{S\}$ and

$$\sum_{x \in S} \frac{1}{d(x) - 1} \leq 1,$$  \hfill (3)

where $S = V(G) \setminus N[w]$, then $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if $G \in \mathcal{J}$. It thus follows immediately that for any 3-connected graph $G$ with $\Delta(G) + \delta(G) \geq v(G)$, $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if $G \in \mathcal{J}$. Since $\Delta(H) = v(H) - 3$ and $\delta(H) = \delta_3(H) = 3$ for every $H \in \mathcal{J}$, we conclude that $(\lambda - 2)^2 |P(G, \lambda)|$ for any 3-connected graph $G$ with $\Delta(G) + \delta(G) \geq v(G)$ and $(\Delta(G), \delta_3(G)) \neq (v(G) - 3, 3)$.

\section{AN EQUIVALENT CONDITION}

In this section, we shall find an equivalent condition for $(\lambda - 2)^2$ to be a factor of $P(G, \lambda)$. We first state two well-known fundamental results for evaluating $P(G, \lambda)$, which can be found in [1, 4, 5].

**Theorem 2.1.** Let $G$ be a graph with two non-adjacent vertices $x$ and $y$. Then

$$P(G, \lambda) = P(G + xy, \lambda) + P(G/xy, \lambda),$$  \hfill (4)

where $G + xy$ is the graph obtained from $G$ by adding an edge joining $x$ and $y$.

**Lemma 2.1.** Let $G$ be a graph with $v(G) \geq 2$. If $w$ is a vertex in $G$ with $d(w) = v(G) - 1$, then

$$P(G, \lambda) = \lambda P(G - w, \lambda - 1).$$  \hfill (5)

Lemma 2.1 can be generalized for an arbitrary vertex $w$ in the graph $G$. 

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Lemma 2.2. For any graph $G$ with $v(G) \geq 2$ and $w \in V(G)$,
\[
P(G, \lambda) = \lambda \sum_{A \in \mathcal{I}_S} P(G - (A \cup \{w\}), \lambda - 1),
\]
where $S = V(G) \setminus N[w]$.

**Proof.** If $S = \emptyset$, (6) follows from Lemma 2.1. Now assume that $|S| \geq 1$ and $y \in S$. Since $wy \notin E(G)$, by Theorem 2.1,
\[
P(G, \lambda) = P(G + wy, \lambda) + P(G/wy, \lambda).
\]
Let $S' = S \setminus \{y\}$ and $A$ be the family of $A \subseteq S \setminus \{y\}$ such that $A \cup \{y\}$ is independent. Then, by induction,
\[
P(G + wy, \lambda) = \lambda \sum_{A \in \mathcal{I}_{S'}} P(G - (A \cup \{w\}), \lambda - 1)
\]
and
\[
P(G/wy, \lambda) = \lambda \sum_{A \in A} P(G - (A \cup \{y\}), \lambda - 1).
\]
Now (6) follows from (7), (8), and (9). \hfill \Box

For any non-empty graph $G$, we have $b(G) \geq 1$ and so $f(G, \lambda)$ defined below is a polynomial in $\lambda$ by Theorem 1.1:
\[
f(G, \lambda) = (-1)^{v(G)} P(G, \lambda) / (\lambda - 1).
\]
Now let $\tau(G) = f(G, 1)$ for any non-empty graph $G$.

For any family $A$ of some subsets of $V(G)$ such that $G - A$ is a non-empty graph for every $A \in A$, define
\[
g(G, A) = \sum_{A \in A} (-1)^{|A|} \tau(G - A).
\]
It is easy to verify the following result.

Lemma 2.3. Let $G$ be any graph and $w$ be any vertex in $G$. Then there exists $A \in \mathcal{I}_S$, where $S = V(G) \setminus N[w]$, such that $G - (A \cup \{w\})$ is an empty graph if and only if $G$ is bipartite.

We can now prove the main result in this section.

**Theorem 2.2.** Let $G$ be any non-bipartite graph and $w \in V(G)$. Then $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if $g(G - w, \mathcal{I}_S) = 0$, where $S = V(G) \setminus N[w]$.

**Proof.** By Lemma 2.2, $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if $(\lambda - 1)^2$ is a factor of
\[
\sum_{A \in \mathcal{I}_S} P(G - (A \cup \{w\}), \lambda) = (\lambda - 1) \sum_{A \in \mathcal{I}_S} (-1)^{(G) - 1 - |A|} f(G - (A \cup \{w\}), \lambda)
\]
\[
= (\lambda - 1)(-1)^{(G) - 1} \sum_{A \in \mathcal{I}_S} (-1)^{|A|} f(G - (A \cup \{w\}), \lambda). \quad (12)
\]
By Lemma 2.3, for each $A \in \mathcal{I}_S$, $G-(A \cup \{w\})$ is a non-empty graph, implying that $f(G-(A \cup \{w\}), \lambda)$ is a polynomial by Theorem 1.1. Thus, $(\lambda-2)^2|P(G, \lambda)$ if and only if

$$\sum_{A \in \mathcal{I}_S} (-1)^{|A|}f(G-(A \cup \{w\}), 1)=0. \quad (13)$$

The result thus follows from (11). \[\blacksquare\]

3. RESULTS ON $\tau(G)$

In this section, we shall study some basic properties on $\tau(G)$ which will be applied in the following sections of this article. Let $G$ be a non-separable graph. We will show that $\tau(G) \geq 0$, $\tau(G) \geq \tau(H)$ for any non-empty subgraph $H$ of $G$, and $\tau(G) \geq (d(w)-1)\tau(G-w)$ for any $w \in V(G)$. Some other results on $\tau(G)$ will also be developed.

Lemma 3.1. Let $G$ be any non-empty graph.

(i) $\tau(G)=0$ if and only if $b(G) \geq 2$.

(ii) Assume that $e(G) \geq 2$. For any $xy \in E(G)$,

$$\tau(G)=\tau(G-xy)+\tau(G/xy), \quad \text{(14)}$$

where $G-xy$ is the graph obtained from $G$ by removing $xy$.

(iii) If $T$ is a set of isolated vertices in $G$, then

$$\tau(G)=(-1)^{|T|}\tau(G-T). \quad \text{(15)}$$

Proof. (i) follows directly from Theorem 1.1 and (ii) follows directly from the definition of $\tau(G)$ and Theorem 2.1. We just prove (iii). Since $T$ is a set of isolated vertices in $G$, we have

$$P(G, \lambda)=\lambda^{|T|}P(G-T, \lambda), \quad \text{(16)}$$

and so

$$f(G, \lambda)=\frac{(-1)^{v(G)}\lambda^{|T|}P(G-T, \lambda)}{\lambda-1}=(-1)^{|T|}\lambda^{|T|}f(G-T, \lambda), \quad \text{(17)}$$

and thus $\tau(G)=f(G, 1)=(-1)^{|T|}f(G-T, 1)=(-1)^{|T|}\tau(G-T). \quad \blacksquare$

Lemma 3.2. Let $G$ be a 2-connected graph. Then either $G-uv$ or $G/uv$ is non-separable for every $uv \in E(G)$.

It is not difficult to prove this result by showing that every two vertices of $V(G)$ are contained in a cycle of $G-uv$ if $b(G/uv) \geq 2$. Here, we give a proof by applying Lemma 3.1(i) and (ii).

Proof of Lemma 3.2. Suppose that $b(G-uv) \geq 2$ and $b(G/uv) \geq 2$. Then $\tau(G-uv)=0$ and $\tau(G/uv)=0$ by Lemma 3.1(i), implying that $\tau(G)=0$ by Lemma 3.1(ii). But $\tau(G)=0$ implies that $b(G) \geq 2$ by Lemma 3.1(i), contradicting the condition that $G$ is 2-connected. Hence $b(G-uv)=1$ or $b(G/uv)=1$. \[\blacksquare\]
Lemma 3.3. Let $G$ be any connected graph with $v(G) \geq 2$. Then

(i) $\tau(G) \geq 0$, where the equality holds if and only if $G$ is separable;
(ii) $\tau(G) \geq \tau(H)$ for any connected spanning subgraph $H$;
(iii) if $H = G - E_0$ is connected, where $E_0 = \{uv_i : v_i \in N(u) \text{ for } i = 1, \ldots, k \}$ for some $k \geq 1$ and $u \in V(G)$, then

$$\tau(G) \geq \tau(H) + \sum_{i=1}^{k} \tau(H/uv_i), \tag{18}$$

and furthermore, if $H$ is non-separable, then $\tau(G) = \tau(H)$ if and only if $\{u,v_i\}$ is a cut-set of $H$ for all $i = 1, \ldots, k$.

Proof. (i) If $G$ is separable, then $\tau(G) = 0$ by Lemma 3.1(i). Now assume that $b(G) = 1$. It is easy to verify that $\tau(G) > 0$ if $v(G) = 2$. If $v(G) \geq 3$ and $xy \in E(G)$, then both $G - xy$ and $G/xy$ are connected. So $b(G - xy) = 1$ or $b(G/xy) = 1$ by Lemma 3.2. Thus, we can show that $\tau(G) > 0$ by induction on $e(G)$.

(ii) The result follows from (i) of this lemma and Lemma 3.1(ii).

(iii) By Lemma 3.1(ii),

$$\tau(G) = \tau(H) + \sum_{i=1}^{k} \tau(G_i/uv_i), \tag{19}$$

where $G_i = G - \{uv_j : 1 \leq j < i\}$. As $H/uv_i$ is a connected spanning subgraph of $G_i/uv_i$, we have $\tau(G_i/uv_i) \geq \tau(H/uv_i)$ by (ii) of this lemma. Thus (18) holds.

Since $G_i/uv_i$ is connected, $\tau(G_i/uv_i) \geq 0$ by (i) of this lemma for each $i$. Thus, by (19), $\tau(G) = \tau(H)$ if and only if $\tau(G_i/uv_i) = 0$ for all $i: 1 \leq i \leq k$. Assume that $H$ is non-separable. Then each $G_i$ is non-separable, implying that $\tau(G_i/uv_i) = 0$ if and only if $\{u,v_i\}$ is a cut-set of $G_i$. Since $G_i - \{u,v_i\} \supseteq H - \{u,v_i\}$, $\{u,v_i\}$ is a cut-set of $G_i$ if and only if $\{u,v_i\}$ is a cut-set of $H$ for all $1 \leq i \leq k$. Hence, the result holds.

For a non-separable graph $G$, let $Q(G)$ be the set of vertices $y \in V(G)$ such that either $uv \in E(G)$ or $\{y,u,v\}$ is a cut-set of $G$ for every pair $u,v \in N(y)$. In the following, we shall show that $\tau(G) \geq (d(w) - 1)\tau(G - w)$ for any $w \in V(G)$ and the equality holds if and only if $w \in Q(G)$.

Lemma 3.4. Let $G$ be any connected graph with $v(G) \geq 3$ and $w$ be a vertex in $G$ such that $w$ is not a cut-vertex. Then

$$\tau(G) \geq (d(w) - 1)\tau(G - w) + \sum_{u,v \in N(w)} \tau((G - w)/uv), \tag{20}$$

and furthermore, if $G$ is 2-connected, then $\tau(G) \geq (d(w) - 1)\tau(G - w)$, where the equality holds if and only if $w \in Q(G)$.

We need to apply the following result due to Zykov [9].
Theorem 3.1 (Zykov [9]). Let \( G_1 \) and \( G_2 \) be two subgraphs of \( G \) such that \( E(G_1) \cup E(G_2) = E(G) \), \( V(G_1) \cup V(G_2) = V(G) \), and \([V(G_1) \cap V(G_2)] \) is isomorphic to \( K_r \). Then

\[
P(G, \lambda) = \frac{P(G_1, \lambda)P(G_2, \lambda)}{P(K_r, \lambda)}.
\]  

(21)

For any \( u \in V(G) \), if \( d(u) = 0 \) or \([N(u)] \) is a complete subgraph of \( G \), then \( u \) is called a simplicial vertex of \( G \). By Theorem 3.1, we have:

Corollary 3.1. Let \( u \) be a simplicial vertex of \( G \).

(i) \( P(G, \lambda) = (\lambda - d(u))P(G - u, \lambda) \).

(ii) If \( G - u \) is a non-empty graph, then \( \tau(G) = (d(u) - 1)\tau(G - u) \).

Proof. If \( d(u) = 0 \), (i) follows from the definition of \( P(G, \lambda) \); if \( d(u) \geq 1 \), (i) follows from Theorem 3.1 directly. By (i), we have

\[
f(G, \lambda) = \frac{(-1)^{v(G)}(\lambda - d(u))P(G - u, \lambda)}{\lambda - 1} = - (\lambda - d(u))f(G - u, \lambda).\]

(22)

Thus, (ii) is obtained.

Proof of Lemma 3.4. We shall prove this result by induction on \( d(w) \). It is clear that \( d(w) \geq 1 \).

Case 1. \( d(w) \leq 2 \).

If \( d(w) = 1 \), then \( \tau(G) = 0 \) by Corollary 3.1(ii), and so the equality of (20) holds. Now let \( d(w) = 2 \) and \( N(w) = \{u, v\} \). If \( uv \in E(G) \), Corollary 3.1(ii) implies that \( \tau(G) = \tau(G - w) \). If \( uv \notin E(G) \), then Lemma 3.1(ii) implies that

\[
\tau(G) = \tau(G - wu) + \tau(G/wu).\]

(23)

As \( b(G - wu) \geq 2 \), we have \( \tau(G - wu) = 0 \) by Lemma 3.1(i). Thus \( \tau(G) = \tau(G/wu) \). Note that \( G/wu \) is the graph \( (G - w) + uv \). Then, by Lemma 3.1(ii) and Lemma 3.3(i), we have

\[
\tau(G) = \tau((G - w) + uv) = \tau(G - w) + \tau((G - w)/uv).\]

(24)

If \( G \) is 2-connected, then \( \tau((G - w)/uv) \geq 0 \) and so \( \tau(G) \geq \tau(G - w) \), where the equality holds if and only if \( b(G - w) = 0 \), i.e., \( b((G - w)/uv) \geq 2 \). As \( b((G - w)/uv) \geq 2 \) if and only if \( \{w, u, v\} \) is a cut-set of \( G \), the lemma holds when \( d(w) = 2 \).

Case 2. \( d(w) \geq 3 \).

Let \( u \in N(w) \). By Lemma 3.1(ii),

\[
\tau(G) = \tau(G - wu) + \tau(G/wu).\]

(25)

It is clear that \( G - wu \) is connected and \( w \) is not a cut-vertex of \( G - wu \). By induction, (20) holds for \( G - wu \). By Lemma 3.3(iii),

\[
\tau(G/wu) \geq \tau(G - w) + \sum_{v \in N(w) \setminus \{w\}} \tau((G - w)/uv).\]

(26)

By (25) and (26), (20) holds for \( G \).
Now assume that $G$ is 2-connected. If $G - w$ is non-separable, then $G - wu$ is 2-connected and by induction, $\tau(G - wu) \geq (d(w) - 2)\tau(G - w)$, where the equality holds if and only if $w \in Q(G - wu)$, and by (26), $\tau(G/wu) \geq \tau(G - w)$, where the equality holds if and only if $\tau((G - w)/uv) = 0$ (i.e., $\{w, u, v\}$ is a cut-set of $G$) for every $v \in N(w) \setminus \{u\}$ with $uv \notin E(G)$. Hence, by (25), if $G - w$ is non-separable, then $\tau(G) \geq (d(w) - 1)\tau(G - w)$, where the equality holds if and only if $w \in Q(G)$.

Finally, consider the case that $G - w$ is separable. Then $\tau(G - w) = 0$. But $\tau(G) > 0 = (d(w) - 1)\tau(G - w)$. As $G - w$ is separable, there exist two blocks $B_1$ and $B_2$ in $G - w$ such that $B_i$ contains only one cut-vertex of $G - w$ for both $i = 1, 2$. Since $G$ is 2-connected, there exists $v_i \in N(w) \cap V(B_i)$ such that $v_i$ is a cut-vertex of $G - w$ for $i = 1, 2$. Observe that $\{w, v_1, v_2\}$ is not a cut-set of $G$ and so $w \notin Q(G)$. Actually Lemma 3.8 immediately implies that $w \notin Q(G)$ if $G$ is 2-connected and $G - w$ is separable.

Therefore, this lemma holds.

**Corollary 3.2.** Let $G$ be any 2-connected graph and $S \subseteq V(G)$. Then

$$\sum_{y \in S} \tau(G - y) \leq \tau(G) \sum_{y \in S} \frac{1}{d(y) - 1},$$

where the equality holds if and only if $S \subseteq Q(G)$.

**Proof.** By Lemma 3.4, for every $y \in S$,

$$\frac{\tau(G)}{d(y) - 1} \geq \tau(G - y),$$

where the equality holds if and only if $y \in Q(G)$. Hence, the result holds.

Lemma 3.3(ii) shows that $\tau(G) \geq \tau(H)$ if $G$ is connected and $H$ is a connected spanning subgraph of $G$. In the following we provide some conditions for the inequality of $\tau(G) \geq \tau(H)$ to be strict, where $H$ is a non-empty subgraph of $G$.

**Lemma 3.5.** Let $G$ be any non-separable graph and $H$ a non-empty subgraph of $G$. Then $\tau(G) \geq \tau(H)$, where the inequality is strict if one of the following conditions holds:

(i) $b(H) \geq 2$ or $c(H)$ is even;
(ii) $G - x$ is non-separable and $x \notin Q(G)$ for some $x \in V(G) \setminus V(H)$;
(iii) there exists $uv \in E(G) \setminus E(H)$ such that both $G - uv$ and $G/uv$ are non-separable.

**Proof.** We first prove that $\tau(G) \geq \tau(H)$ by induction on $|E(G)|$. If $H$ is a connected spanning subgraph of $G$, then $\tau(G) \geq \tau(H)$ by Lemma 3.3(ii). If $b(H) \geq 2$, then $\tau(G) - 0 = \tau(H)$ by Lemma 3.3(i). By Lemma 3.1(iii), it remains to consider the case that $H$ is non-separable and $V(H) \neq V(G)$. Then there exists $uv \in E(G)$ such that $u \notin V(H)$. Note that $H$ is a subgraph of both $G - uv$ and $G/uv$, and either $G - uv$ or $G/uv$ is non-separable. Suppose that $G - uv$ is non-separable for example. Then $\tau(G - uv) \geq \tau(H)$ by induction. It is clear that $\tau(G/uv) \geq 0$ by Lemma 3.3(i). Thus, $\tau(G) \geq \tau(H)$ by Lemma 3.1(ii).

If condition (i) holds, i.e., $b(H) \geq 2$ or $c(H)$ is even, then Lemma 3.1(i) and (iii) and Lemma 3.3(i) imply that $\tau(H) \leq 0$ and so $\tau(G) > \tau(H)$.

If condition (ii) holds, i.e., there exists $x \in V(G) \setminus V(H)$ such that $G - x$ is non-separable and $x \notin Q(G)$, then $\tau(G) > \tau(G - x)$ by Lemma 3.4 and $\tau(G - x) \geq \tau(H)$, implying that $\tau(G) > \tau(H)$.
Now assume that condition (iii) holds, i.e., both $G - uv$ and $G/uv$ are non-separable. Lemmas 3.1(ii) and 3.3(i) yield that $\tau(G) > \tau(G - uv)$. As $H$ is a subgraph of $G - uv$ and $G - uv$ is non-separable, we have $\tau(G - uv) \geq \tau(H)$. Hence $\tau(G) > \tau(H)$. 

In the following, we shall characterize the structure of a 2-connected graph $G$ such that $G - \{y_1, y_2\}$ is connected and $\tau(G) = \tau(G - \{y_1, y_2\})$ for two adjacent vertices $y_1$ and $y_2$.

**Lemma 3.6.** Let $G$ be a 2-connected graph and $M = G - \{y_1, y_2\}$, where $y_1$ and $y_2$ are two adjacent vertices in $G$. Assume that $M$ is connected and $e(M) \geq 1$. Then $\tau(G) \geq \tau(M)$, where the equality holds if and only if $M$ is non-separable and there exist $u, v \in V(M)$ such that

(i) $N(y_i) \cap V(M) \subseteq \{u, v\}$ for $i = 1, 2$ and $4 \leq d(y_1) + d(y_2) \leq 5$; and

(ii) either $uv \in E(G)$ or $b(M/uv) \geq 2$.

**Proof.** By Lemma 3.5, $\tau(G) \geq \tau(M)$. If $M$ is separable, then $\tau(G) > \tau(M)$ by Lemma 3.1(i). Thus, we now assume that $M$ is non-separable.

Since $G$ is 2-connected, there exist distinct $u, v \in V(M)$ such that $y_1 u, y_2 v \in E(G)$. If $N(y_1) \subseteq \{u, v\}$, then both $G - y_1 u$ and $G/y_1 u$ are non-separable, and so $\tau(G) > \tau(M)$ by Lemma 3.5. Similarly, if $N(y_2) \subseteq \{u, v\}$, then $\tau(G) > \tau(M)$.

If $N(y_i) \cap V(M) = \{u, v\}$ for both $i = 1, 2$, then both $G - y_1 y_2$ and $G/y_1 y_2$ are non-separable and so $\tau(G') > \tau(M)$ by Lemma 3.5.

Thus, we may now assume that

$$N(y_1) \cap V(M) = \{u\}, \quad \{v\} \subseteq N(y_2) \cap V(M) \subseteq \{u, v\},$$

as shown in Figure 1. We first show that $\tau(G) = \tau(G + uv)$. It is clear that $\tau(G) = \tau(G + uv)$ if $uv \in E(G)$. If $uv \notin E(G)$, then by Lemma 3.1(ii), we also have

$$\tau(G) = \tau(G + uv) - \tau(G/uv) = \tau(G + uv),$$

as $b(G/uv) \geq 2$ implies that $\tau(G/uv) = 0$ by Lemma 3.1(i).
We now show that $\tau(G+uv) = \tau(M+uv)$. By Theorem 3.1,

$$P(G+uv, \lambda) = \frac{P(L, \lambda)P(M+uv, \lambda)}{\lambda(\lambda-1)}, \quad (31)$$

where $L$ is the subgraph of $G+uv$ induced by $\{y_1,y_2,u,v\}$. Thus, either $L \cong C_4$ or $L \cong K_4-e$, where $K_4-e$ is the graph obtained from $K_4$ by removing one edge. So

$$P(L, \lambda) = (\lambda - 1)^4 + (\lambda - 1) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3) \quad (32)$$

or

$$P(L, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2. \quad (33)$$

Hence

$$f(G+uv, \lambda) = \frac{(-1)^{\chi(G)}P(G+uv, \lambda)}{\lambda-1}$$

$$= \frac{(-1)^{\chi(G)}P(L, \lambda)P(M+uv, \lambda)}{\lambda(\lambda-1)^2}$$

$$= \frac{(-1)^2P(L, \lambda)}{\lambda(\lambda-1)}f(M+uv, \lambda)$$

$$= \begin{cases} (\lambda^2 - 3\lambda + 3)f(M+uv, \lambda) & \text{if } L \cong C_4, \\ (\lambda - 2)^2f(M+uv, \lambda) & \text{if } L \cong K_4-e, \end{cases} \quad (34)$$

implying that $\tau(G+uv) = \tau(M+uv)$ no matter whether $L \cong C_4$ or $L \cong K_4-e$.

If $uv \in E(G)$, then $\tau(M+uv) = \tau(M)$ and so the result holds for this case. Now assume that $uv \notin E(G)$. Then, by Lemma 3.1(ii), $\tau(M+uv) = \tau(M)$ if and only if $b(M/uv) \geq 2$. Hence, $\tau(G) = \tau(M)$ if and only if $b(M/uv) \geq 2$. The proof is thus completed.

Finally, in this section, we establish two results on $Q(G)$ which will also be used in the next section.

**Lemma 3.7.** Let $G$ be a 2-connected graph and $y \in V(G)$. If $b(G-y) \geq 2$, then $G-\{y,u_1,u_2\}$ is connected for some non-adjacent vertices $u_1,u_2 \in N(y)$.

**Proof.** Suppose that $b(G-y) \geq 2$. Then $G-y$ has two blocks, say $B_1$ and $B_2$, such that $B_i$ contains only one cut-vertex of $G-y$, denoted by $z_i$, for $i=1,2$. It is possible that $z_1$ and $z_2$ are the same vertex of $G$. Since $G$ is 2-connected, there exists $u_i \in N(y) \cap (V(B_i) \setminus \{z_i\})$ for $i=1,2$. Observe that $B_i-u_i$ is connected for $i=1,2$, as $B_i$ is a block of $G-y$. Thus, $G-\{y,u_1,u_2\}$ is connected.

**Lemma 3.8.** If $G$ is a 2-connected graph, then $G-y$ is non-separable for every $y \in Q(G)$.

**Proof.** Let $y \in Q(G)$. Suppose that $G-y$ is separable. By Lemma 3.7, there exists non-adjacent vertices $u_1,u_2 \in N(y)$ such that $G-\{y,u_1,u_2\}$ is connected. Then the definition of $Q(G)$ implies that $y \notin Q(G)$, a contradiction.
Lemma 3.9. Let G be a 2-connected graph and \( y_1, y_2 \in Q(G) \).

(i) If \( b(G - \{y_1, y_2\}) \geq 2 \), then \( N(y_1) \setminus \{y_2\} = N(y_2) \setminus \{y_1\} = \{u, v\} \) for some \( u, v \in V(G) \setminus \{y_1, y_2\} \).

(ii) If \( y_1 y_2 \in E(G) \), then \( N[y_1] = N[y_2] \).

(iii) Each component of \( G[Q(G)] \) is a complete graph.

Proof. (i) Assume that \( b(G - \{y_1, y_2\}) \geq 2 \). By Lemma 3.8, \( G - y_2 \) is non-separable, and so \( G - y_2 \) is 2-connected as \( b(G - \{y_1, y_2\}) \geq 2 \). By Lemma 3.7, there exists non-adjacent vertices \( u_1, u_2 \) in \( N(y_1) \setminus \{y_2\} \) such that \( G - \{y_1, y_2, u_1, u_2\} \) is connected. However, as \( u_1, u_2 \in N(y_1) \) and \( u_1 u_2 \notin E(G) \), the condition that \( y_1 \in Q(G) \) implies that \( \{y_1, u_1, u_2\} \) is a cut-set of \( G \). Thus \( y_2 \) is an isolated vertex of \( G - \{y_1, u_1, u_2\} \), i.e., \( N(y_2) \subseteq \{y_1, u_1, u_2\} \). It means that \( N(y_2) \setminus \{y_1\} \subseteq \{u_1, u_2\} \subseteq N(y_1) \setminus \{y_2\} \). By the same argument, we can show that \( N(y_1) \setminus \{y_2\} = N(y_2) \setminus \{y_1\} = \{u_1, u_2\} \).

(ii) Suppose that \( y_1 y_2 \in E(G) \) and \( N[y_1] \nsubseteq N[y_2] \). Let \( x \in N(y_1) \setminus N(y_2) \). As \( y_1 \in Q(G) \), \( \{y_1, y_2, x\} \) is a cut-set of \( G \), implying that \( G - \{y_1, y_2\} \) is separable. By (i), we have \( N(y_1) \setminus \{y_2\} = N(y_2) \setminus \{y_1\} \), contradicting the assumption that \( N[y_1] \nsubseteq N[y_2] \).

(iii) follows from (ii) directly.

4. A FAMILY OF 3-CONNECTED GRAPHS

For any graph \( H \) and \( S = \{y_1, y_2\} \subseteq V(H) \), \( H \) is said to have property \( v \) with respect to \( S \) if \( H \) satisfies the following conditions illustrated in Figure 2:

(a) \( y_1 y_2 \notin E(H) \);

(b) \( N(x_i) = \{y_1, y_2\} \) for every \( x_i \in T \), where \( T = \{x_i : i = 1, 2, \ldots, r\} \) is the set of isolated vertices of \( H - \{y_1, y_2\} \);

(c) there exist \( u, v \in V(M) \) such that \( N(y_1) \cap V(M) = \{u\} \) and \( \{v\} \subseteq N(y_2) \cap V(M) \subseteq \{u, v\} \), where \( M = H - (S \cup T) \);

(d) \( uv \in E(H) \) or \( b(M/uv) \geq 2 \).

Theorem 4.1. Assume that \( G \) is a graph with \( w \in V(G) \) such that \( d(w) = v(G) - 3 \) and \( G - w \) has property \( v \) with respect to \( S \), where \( S = \{y_1, y_2\} = V(G) \setminus N[w] \). Let \( T \) be the set of isolated vertices of \( G - (\{w\} \cup S) \).

(i) If \( |T| \geq 2 \) and \( G - w \) is non-separable, then \( G \) is 3-connected.

(ii) If \( |T| \) is odd, then \( (\lambda - 2)^2 |P(G, \lambda)| \).

![Figure 2](image-url)
**Proof.** Since $G-w$ has property $v$ with respect to $S$, we can assume that $G-w$ is as the graph shown in Figure 2. Let $T=\{x_1,x_2,\ldots,x_r\}$. Notice that if $r=1$, then $N(y_1)=\{x_1,u\}$ and so $G$ is not 3-connected.

(i) Assume that $|T| \geq 2$ and $G-w$ is non-separable. Suppose on the contrary that $G$ is not 3-connected. Then $G$ has a cut-set $C$ with $|C| \leq 2$. Since $G-w$ is non-separable, we have $w \notin C$, implying that $N[w] \setminus C \subseteq V(G')$ for some component $G'$ of $G-C$. Let $G''$ be another component of $G-C$. Since $V(G) \setminus N[w] = \{y_1,y_2\}$ and $N[w] \setminus C \subseteq V(G')$, we have $V(G'') \subseteq \{y_1,y_2\}$. As $y_1y_2 \notin E(G)$, $V(G'') = \{y_i\}$ for some $i=1,2$, implying that $|C| \geq d(y_i) \geq 1 + |T| \geq 3$, a contradiction.

(ii) Assume that $r=|T|$ is odd. We first show that $\tau(G-w) = \tau(M)$. By Lemma 3.1(ii), we have

$$\tau(G-w) = \tau((G-w)+y_1y_2) - \tau((G-w)/y_1y_2).$$

(35)

Since $b((G-w)/y_1y_2) \geq 2$ when $r \geq 1$, we have $\tau((G-w)/y_1y_2) = 0$ by Lemma 3.1(i), and so $\tau(G-w) = \tau((G-w)+y_1y_2)$ by (35). As every vertex in $T$ is a simplicial vertex of degree 2 of $(G-w)+y_1y_2$, by Corollary 3.1(ii), we have

$$\tau((G-w)+y_1y_2) = (2-1)^r \tau(\tilde{H},\tilde{\lambda}) = \tau(\tilde{H},\tilde{\lambda}),$$

(36)

where $H = (G-(\{w\} \cup T))+y_1y_2$, i.e., the graph obtained from $G-(\{w\} \cup T)$ by adding an edge $y_1y_2$. Since $G-w$ has property $v$ with respect to $S$, $H$ satisfies conditions in Lemma 3.6. Thus it implies that $\tau(H) = \tau(M)$, where $M = G-(\{w\} \cup S \cup T)$. Hence $\tau(G-w) = \tau(M)$.

Since $|T|$ is the set of isolated vertices of $G-\{w,y_1,y_2\}$, Lemma 3.1(iii) implies that

$$\tau(G-\{w,y_1,y_2\}) = (-1)^{|T|} \tau(M) = (-1)^r \tau(M).$$

(37)

Since $b(G-\{w,y_i\}) \geq 2$, Lemma 3.1(i) implies that $\tau(G-\{w,y_i\}) = 0$ for $i=1,2$. Hence

$$g(G-w,\mathcal{I}_S) = \sum_{A \subseteq \{y_1,y_2\}} \tau((G-w)-A)$$

$$= \tau(G-w) + \tau(G-\{w,y_1,y_2\})$$

$$= \tau(G-w) + (-1)^r \tau(M)$$

$$= 0,$$

(38)

where the last equality follows from the condition that $r$ is odd and the result that $\tau(G-w) = \tau(M)$.

Again, as $uv \in E(G)$ or $b(M/uv) \geq 2$, $G-(A \cup \{w\})$ is a non-empty graph for every $A \in \mathcal{I}_S$, and so Theorem 2.2 implies that $(\tilde{\lambda}-2)^2 |P(G,\tilde{\lambda})$. ■

Let $\mathcal{J}$ be the family of $G$ with a vertex $w$ of degree $v(G)-3$ such that $G-w$ is non-separable, $G-w$ has property $v$ with respect to $V(G) \setminus N[w]$ and the subgraph $[N(w)]$ has exactly an odd number larger than 1 of isolated vertices. The graph shown in Figure 3 is an example of graphs in $\mathcal{J}$ for every odd number $r \geq 3$, where $u$ and $y_2$ may be or may not be adjacent.

The definition of $\mathcal{J}$ and Theorem 4.1 yield the following result.

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Corollary 4.1. For any $G \in \mathcal{J}$, $G$ is 3-connected, $\Delta(G) = v(G) - 3$, $\delta(G) = \delta_3(G) = 3$, and $(\lambda - 2)^2 |P(G, \lambda)| = v(G) - 3$, $(G) \neq 3$, $(G) = 3$, and $(\lambda - 2)^2 |P(G, \lambda)| = v(G) - 3$.

5. GRAPHS G WITH $\Delta(G) + \delta(G) \geq v(G)$

Theorem 2.2 shows that for a non-bipartite graph $G$, $(\lambda - 2)^2 |P(G, \lambda)|$ if and only if $G$ contains a vertex $w$ such that $g(G - w, IS) \neq 0$, where $S = V(G) \setminus N[w]$. In this section, we first find some sufficient conditions for $g(H, IS)$ to be non-zero for a given graph $H$ and $S \subseteq V(H)$. This result will be applied to prove our main result.

A family $\mathcal{A}$ of sets is said to be downward closed if $A_1 \subseteq A_2 \in \mathcal{A}$ implies that $A_1 \in \mathcal{A}$. For any graph $G$ and $S \subseteq V(G)$, let $\mathcal{A}(S)$ be the family of $A \subseteq IS$ such that $A$ is downward closed.

Let $G$ be any graph and $A \subseteq \{A : A \subseteq V(G)\}$ such that $G - A$ is a non-empty graph for every $A \in \mathcal{A}$. For integers $s, t : 0 \leq s \leq t$, let $A_{s,t} = \{A \in \mathcal{A} : s \leq |A| \leq t\}$, $A_s = \{A \in \mathcal{A} : |A| \geq s\}$, and

$$g_s(G, A) = \sum_{A \in A_s} (-1)^{|A| - s} e(G - A).$$

Thus $g_0(G, A) = g(G, A)$.

We first prove the following result.

Lemma 5.1. Let $G$ be any graph and $B \subseteq V(G)$. Assume that $b(G - B) \geq \min\{2, c(G - B)\}$ and $|N(x) \cap B| \geq 2$ for every $x \in B$. If there exists $B_1 \subseteq B$ such that $G - B_1$ is not 2-connected, then $b(G - B) \geq 2$.

Proof. Let $H = G - B$. Since $b(H) \geq \min\{2, c(H)\}$, we have $v(H) \geq 2$ and $b(H) \geq 1$. Suppose that $b(H) = 1$. Then $H$ is non-separable as $b(H) \geq \min\{2, c(H)\}$.

Assume that there exists $B_1 \subseteq B$ such that $G - B_1$ is not 2-connected. For every $x \in B \setminus B_1$, as $H$ is non-separable and $|N(x) \cap V(H)| = |N(x) \setminus B| \geq 2$, $G[V(H) \cup \{x\}]$ is 2-connected. Hence $G[V(H) \cup (B \setminus B_1)]$ (i.e., $G - B_1$) is 2-connected, a contradiction. ■
Lemma 5.2. Let $G$ be any non-separable graph, $S \subseteq V(G)$ and $A \in \mathcal{Y}(S)$ with $A \neq \emptyset$. Assume that $b(G-A) \geq \min\{2,c(G-A)\}$ for every $A \in A$ and

$$\sum_{y \in S} \frac{1}{d(y)-1} \leq 1.$$  \hspace{1cm} (40)

Then $g_s(G,A) \geq 0$ for any non-negative integer $s$, where the equality holds if and only if one of the following conditions is satisfied:

(i) $s = 0$, the equality of (40) holds, $A = \emptyset \cup \{y \in S\}$, and $S \subseteq Q(G)$;

(ii) $s \geq 1$ and $b(G-A) \geq 2$ for every $A \in A_{s,s}$.

**Proof.** First, consider the case that $G \cong K_2$. We have $S = \emptyset$ by (40). Since $A \neq \emptyset$, we have $A = \emptyset$, and thus $g_0(G,A) = \tau(K_2) = 1$ and $g_s(G,A) = 0$ for $s \geq 1$. Thus, the result holds for this case. Now we assume that $G$ is 2-connected.

We will prove this lemma by induction on $|A|$. If $|A| = 1$, then $A = \emptyset$ as $A$ is downward closed, and so it is clear that $g_0(G,A) = 0$ for $s \geq 1$ and $g_0(G,A) = \tau(G) > 0$ by Lemma 3.3(i). It is also obvious that condition (i) does not hold, but condition (ii) holds since $A_{s,s} = \emptyset$ for $s \geq 1$. Thus, the lemma is proven when $|A| = 1$. Assume that the lemma holds when $|A| < m$, where $m \geq 2$. Now suppose that $|A| = m$.

We first prove two claims below.

**Claim A.** For any $A \in A$, we have $\tau(G-A) \geq 0$, where the equality holds if $G-A_0$ is not 2-connected for some $A_0 \subset A$.

Let $A \in A$. If $\tau(G-A) < 0$, then $b(G-A) = 1$ by Lemma 3.1(i) and $c(G-A) \geq 2$ by Lemma 3.3(i), contradicting the given condition that $b(G-A) \geq \min\{2,c(G-A)\}$ for all $A \in A$. Thus, $\tau(G-A) \geq 0$.

For every $x \in A$, we have $d(x) \geq 2$ as $G$ is 2-connected. Since $A$ is an independent set of $G$, $|N(x) \setminus A| = d(x) \geq 2$. If $G-A_0$ is not 2-connected for some $A_0 \subset A$, then $b(G-A) \geq 2$ by Lemma 5.1 and so $\tau(G-A) = 0$ by Lemma 3.1(i). Claim A holds.

**Claim B.** For any integer $r \geq 0$, if $\tau(G-A) = 0$ for all $A \in A_{r,r}$, then $\tau(G-A) = 0$ for all $A \in A_r$ and therefore $g_r(G,A) = 0$.

Suppose that $\tau(G-A) = 0$ for all $A \in A_{r,r}$. Let $A' \in A_{r+1}$. As $A$ is downward closed, there exists $A_0 \in A_{r,r}$ such that $A_0 \subset A'$. So $\tau(G-A_0) = 0$, implying that $b(G-A_0) \geq 2$ by Lemma 3.1(i). Thus, $\tau(G-A') = 0$ by Claim A. Claim B holds.

Now we consider the two cases that $s \geq 1$ and $s = 0$.

**Case 1.** $s \geq 1$.

If $\tau(G-A_0) = 0$ for all $A_0 \in A_{s,s}$, then $g_s(G,A) = 0$ by Claim B. Now assume that $\tau(G-A_0) \neq 0$ for some $A_0 \in A_{s,s}$. Claim A implies that $\tau(G-A_0) > 0$. We shall show that $g_s(G,A) > 0$ under this condition.

Choose $A' \subset A$ such that $|A'|$ is maximum under the conditions that $\tau(G-A') > 0$ and $|A'|-s$ is even. Such $A'$ exists as $\tau(G-A_0) > 0$ and $|A_0|-s = 0$. By the selection of $A'$ and Claim A, we have $\tau(G-A) = 0$ for all $A \in A$ with $|A| = |A'| + 2$, and thus Claim B implies that $\tau(G-A) = 0$ for all $A \in A$ with $|A| = |A'| + 2$. Hence

$$g_s(G,A) = g_{s,s}(G,A),$$  \hspace{1cm} (41)

where $t = |A'| + 1$. Let $B = \{A \in A : A' \not\subset A\}$. Note that $B \in \mathcal{Y}(S)$ and $b(G-A) \geq \min\{2,c(G-A)\}$ for every $A \in B \subset A$. Since $A' \in A \setminus B$, we have $0 < |B| < |A|$. By
induction, we have $g_s(G, B) \geq 0$. Thus, by (41), we have

$$
geq g_s(G, A) = g_s(G, B) + (-1)^{|A| - s} \tau(G - A') + \sum_{y \in S'} (-1)^{|A'| + 1 - s} \tau(G - (A' \cup \{y\}))$$

$$
\geq \tau(G - A') - \sum_{y \in S'} \tau(G - (A' \cup \{y\}))$$

$$= \tau(H) - \sum_{y \in S'} \tau(H - \{y\}),$$

where $S' = \{y \in S \setminus A' : A' \cup \{y\} \in A\}$ and $H = G - A'$. For every $y \in S'$, as $A' \cup \{y\}$ is independent of $G$, we have $d_H(y) = d_G(y)$. Thus

$$
\sum_{y \in S'} \frac{1}{d_H(y) - 1} \leq \sum_{y \in S'} \frac{1}{d_G(y) - 1} < \sum_{y \in S} \frac{1}{d_G(y) - 1} \leq 1,
$$

where the strict inequality follows from the fact that $A' \subseteq S$ and $|A'| \geq s \geq 1$. By (42), (43), and Corollary 3.2, we have

$$g_s(G, A) \geq \tau(H) - \sum_{y \in S'} \tau(H - y) > 0.$$  

**Case 2.** $s = 0$.

Note that

$$g_0(G, A) = \tau(G) - \sum_{y \in S, \{y\} \in A} \tau(G - y) + g_2(G, A).$$

By the result in case 1, we have $g_2(G, A) \geq 0$, where the equality holds if and only if $\tau(G - A) = 0$ for every $A \in \mathcal{A}_{2,2}$. By (40) and Corollary 3.2, we have

$$\tau(G) \geq \sum_{\{y\} \in A} \tau(G - y).$$

Thus $g_0(G, A) \geq 0$. Now assume that $g_0(G, A) = 0$. Then $g_2(G, A) = 0$ and the equality of (46) holds, where the latter implies that the equality of (40) holds, $\{y\} \in A$ for all $y \in S$, and $S \subseteq Q(G)$ by Corollary 3.2. It remains to show that $\mathcal{A}_2 = \emptyset$.

Suppose that $\mathcal{A}_2 \neq \emptyset$. Then there exists $\{y_1, y_2\} \in \mathcal{A}$. We can show that $G - \{y_1, y_2\}$ is non-separable. If $G - \{y_1, y_2\}$ is separable, by Lemma 3.9, we have $N(y_1) \setminus \{y_2\} = N(y_2) \setminus \{y_1\} = \{u, v\}$ for some $u, v \in V(G)$. As $\{y_1, y_2\}$ is an independent set of $G$, we have $N(y_1) = N(y_2) = \{u, v\}$, implying that $d(y_1) = d(y_2) = 2$, contradicting (40). Thus, $G - \{y_1, y_2\}$ is non-separable. So $\tau(G - \{y_1, y_2\}) > 0$ by Lemma 3.3(i), which implies that $g_2(G, A) > 0$ by the result of (ii) in this lemma (i.e., case 1 in the proof), contradicting the fact that $g_2(G, A) = 0$. Hence, $\mathcal{A}_{2,2} = \emptyset$ and therefore $\mathcal{A}_2 = \emptyset$.

Therefore, this lemma holds for both cases that $s \geq 1$ and $s = 0$ when $|A| = m$. The proof is thus completed.

In the following, we shall apply Lemma 5.2 to establish another result for $g(G, \mathcal{I}_S)$ to be non-negative, where $S \subseteq V(G)$. The condition “$b(G - A) \geq \min\{2, c(G - A)\}$” for all $A \in \mathcal{A}$ in Lemma 5.2 will be replaced by “$\delta(G - A) > 0$” for all $A \in \mathcal{I}_S \setminus \{S\}$.
Lemma 5.3. Let G be a non-separable graph and S ⊂ V(G) such that E([S]) ∪ E(G − S) ≠ ∅. Assume that δ(G − A) > 0 for every A ∈ ℐS \ [S] and

\[ \sum_{y \in S} \frac{1}{d(y) - 1} \leq 1. \]  

(47)

Then g(G, ℐS) ≥ 0, where the equality holds if and only if one of the following conditions is satisfied:

(i) \( S = \{y\}, d(y) = 2 \) and \( y \in Q(G) \);

(ii) \( |S| = 2, G \) has property \( v \) with respect to \( S \) and \( |T| \) is odd, where \( T \) is the set of isolated vertices of \( G - S \);

(iii) \( |S| \geq 2 \) and there exist \( u, v \in V(G) \) \( \setminus S \) such that \( N[y] = \{u, v\} \cup S \) for all \( y \in S \).

Proof. We shall prove this result by induction on \( |S| \). If \( |S| = 0 \), then \( g(G, ℐS) = \tau(G) > 0 \) by Lemma 3.3(i). If \( |S| = 1 \) (say \( S = \{y\} \)), then

\[ g(G, ℐS) = \tau(G) - \tau(G - y) \]  

(48)

and as \( d(y) \geq 2 \), by Corollary 3.2, we have \( g(G, ℐS) \geq 0 \), where the equality holds if and only if condition (i) is satisfied. Assume that the result holds if \( |S| < m \), where \( m \geq 2 \). Now consider the case that \( |S| = m \).

We first prove the following claim:

Claim A. If \( G - \{y_1, y_2\} \) is non-separable for some \( \{y_1, y_2\} \in ℐS \), then \( g(G, ℐS) > 0 \).

Note that

\[ g(G, ℐS) = g(G, A) + g(H, ℐS') \]  

(49)

where \( A = \{A \in ℐS : \{y_1, y_2\} \not\subset A\} \), \( H = G - \{y_1, y_2\} \), and \( S' = \{y \in S \setminus \{y_1, y_2\} : \{y, y_1, y_2\} \in ℐS\} \). Since \( \{y_1, y_2\} \subset S \not\subset A \), we have \( \delta(G - A) > 0 \) for every \( A \in A \). Thus, by Lemma 5.2, we have \( g(G, A) \geq 0 \), where the equality holds if and only if the equality of (47) holds, \( S \subset Q(G) \), and \( A_2 = \emptyset \).

We now show that \( g(H, ℐS') \geq 0 \). If we can show that \( H \) and \( S' \) satisfy the conditions in the lemma, then \( g(H, ℐS') \geq 0 \) by induction as \( |S'| < |S| \).

Note that \( G - \{y_1, y_2\} \) (i.e., \( H \)) is non-separable. It can be shown that \( E([S']) \cup E(H - S') \neq \emptyset \). Suppose that \( S' \) is an independent set of \( G \). Let \( S_1 = S' \cup \{y_1, y_2\} \). Then \( S_1 \in ℐS \). If \( S_1 = S \), then the given condition in the lemma implies that \( G - S_1 \) (i.e., \( H - S' \)) is a non-empty graph; otherwise, it implies that \( \delta(G - S_1) > 0 \). Hence \( E([S']) \cup E(H - S') \neq \emptyset \). For any \( B \in ℐS \setminus \{S'\} \), we have \( B \cup \{y_1, y_2\} \in ℐS \setminus \{S\} \) and so \( \delta(H - B) = \delta(G - (B \cup \{y_1, y_2\})) > 0 \) by the given condition in the lemma. For each \( y \in S' \), we have \( d_H(y) = d_G(y) \). So

\[ \sum_{y \in S'} \frac{1}{d_H(y) - 1} < \sum_{y \in S} \frac{1}{d_G(y) - 1} \leq 1. \]  

(50)

Since \( |S'| < |S| \), by induction, we have \( g(H, ℐS') \geq 0 \), where the inequality is strict if \( S' = \emptyset \).

Hence, \( g(G, ℐS) \geq 0 \) by the results \( g(G, A) \geq 0 \) and \( g(H, ℐS') \geq 0 \) and (49). Suppose that \( g(G, ℐS) = 0 \). Then \( g(H, ℐS') = 0 \) and \( g(G, A) = 0 \), where the latter implies that \( A_2 = \emptyset \) by Lemma 5.2. However, \( A_2 = \emptyset \) implies that \( S' = \emptyset \), which further yields that \( g(H, ℐS') = \tau(H) > 0 \), a contradiction. Hence, \( g(G, ℐS) > 0 \) and the claim holds.
Let $r=|T|$, where $T$ is the set of isolated vertices of $G-S$. We now consider in two cases.

**Case 1.** $S$ is not independent or $b(G-S) \geq 2$ or $r+|S|$ is even.

Observe that if $S$ is not independent, then $T_S = T_S \setminus \{S\}$; if $b(G-S) \geq 2$, then $\tau(G-S) = 0$ by Lemma 3.1(i); if $r+|S|$ is even and $S$ is independent, then $G-S$ is a non-empty graph by the given condition and so

$$(-1)^{|S|} \tau(G-S) = (-1)^r \tau(G-S) = \tau(G-(S \cup T)) \geq 0,$$

where the last inequality is strict if and only if $G-(S \cup T)$ is non-separable (i.e., $b(G-S)=1$) by Lemmas 3.1(i) and 3.3(i). Thus, in Case 1, we have

$$g(G, T_S) = \sum_{A \in T_S} (-1)^{|A|} \tau(G-A) \geq \sum_{A \in T_S \setminus \{S\}} (-1)^{|A|} \tau(G-A),$$

where the last equality holds if and only if $S \notin T_S$ or $b(G-S) \geq 2$ by (51).

Since $T_S \setminus \{S\} \in Y(S)$, by Lemma 5.2(i), we have

$$\sum_{A \in T_S \setminus \{S\}} (-1)^{|A|} \tau(G-A) \geq 0,$$

where the equality holds if and only if the equality of (47) holds, $S \subseteq Q(G)$, and $T_S \setminus \{S\} = \emptyset \cup \{y : y \in S\}$.

Assume that $g(G, T_S) = 0$. Then the equality of (47) holds, $S \subseteq Q(G)$, $T_S \setminus \{S\} = \emptyset \cup \{y : y \in S\}$, and $S \notin T_S$ or $b(G-S) \geq 2$.

If $|S| = 2$ and $S \notin T_S$, then $b(G-S) \geq 2$. Since $S \subseteq Q(G)$, we have $d(y_i) = 2$ for each $y_i \in S$ by Lemma 3.9(i), which contradicts (47). Thus, $|S| \geq 3$ or $S \notin T_S$. Note that $T_S \setminus \{S\} = \emptyset \cup \{y : y \in S\}$. Hence $G[S]$ is complete. By Lemma 3.9(ii), $N[y_i] = N[y_j]$ for every pair $y_i, y_j \in S$, and therefore there exists $D \subseteq V(G-S)$ such that $N[y_i] = S \cup D$ for all $y_i \in S$. As the equality of (47) holds, we have

$$\frac{1}{\sum_{y \in S} d(y) - 1} = \frac{|S|}{|S| + |D| - 2} = 1,$$

implying that $|D| = 2$. Thus, condition (iii) is satisfied.

**Case 2.** $S$ is independent, $b(G-S) = 1$ and $r+|S|$ is odd.

Let $M = G-(S \cup T)$. As $b(M) = b(G-S) = 1$ and $M$ has no isolated vertices, $M$ is non-separable.

As $|S| = m \geq 2$, (47) implies that $d(y) \geq 3$ for every $y \in S$. If $r = 0$ (i.e., $T = \emptyset$), then $N(y) \subseteq V(M)$ and so $G[V(M) \cup \{y\}]$ is 2-connected for every $y \in S$. Thus, $G-\{y_1, y_2\}$ is non-separable for any pair $y_1, y_2 \in S$, implying that $g(G, T_S) > 0$ by the result of Claim A. Hence, we now assume that $r \geq 1$.

We then prove some claims which hold within Case 2.

**Claim 1.** $G[S \cup T]$ is a complete bipartite graph with bipartition $\{S, T\}$, as shown in Figure 4.

For each $x \in T$, we have $N(x) \subseteq S$. If $N(x) \not\subseteq S$, then $N(x) \in T_S \setminus \{S\}$ as $S$ is independent, implying that $d(G-N(x)) > 0$ by the given condition in the lemma, contradicting the fact that $x$ is an isolated vertex in the subgraph $G-N(x)$. Thus, $N(x) = S$ for each $x \in T$. As $S$ is independent, Claim 1 holds.
**Claim 2.** There exist distinct $y_1, y_2 \in S$ such that $G - A$ is 2-connected for each $A \subseteq S \setminus \{y_1, y_2\}$.

Since $|S| \geq 2$ and $G$ is non-separable, there exist two distinct vertices in $S$, say $y_1$ and $y_2$, and two distinct vertices in $M$, say $u$ and $v$, such that $y_1 u \in E(G)$ and $y_2 v \in E(G)$. Thus the subgraph $G[V(M) \cup T \cup \{y_1, y_2\}]$ is 2-connected. Since $|N(y) \cap (T \cup V(M))| = d(y) \geq 3$ for each $y \in S$, Claim 2 holds.

**Claim 3.** $\tau(G - S) = (-1)^r \tau(M)$.

Since $G - S$ consists of a non-separable component (i.e., $M$) and a set of isolated vertices (i.e., $T$), we have $\tau(G - S) = (-1)^{|T|} \tau(M) = (-1)^r \tau(M)$ by Lemma 3.1(iii). So Claim 3 holds.

Note that Claim 3 actually holds in the whole lemma. In the following, we shall complete the proof of Case 2 in three subcases.

**Subcase (2.1).** $|S| \geq 4$.

Choose any two distinct vertices $y_3, y_4 \in S \setminus \{y_1, y_2\}$. Thus $G - \{y_3, y_4\}$ is 2-connected by Claim 2, implying that $g(G, I_S) > 0$ by the result of Claim A.

**Subcase (2.2).** $|S| = 3$.

We shall show that $g(G, I_S) > 0$ in this subcase. Since $|S| = 3$ and $|S| + r$ is odd, $r$ is even. Note that $G - y_3$ is 2-connected by Claim 2, where $y_3 \in S \setminus \{y_1, y_2\}$.

For every pair $z_1, z_2 \in N(y_3)$, if $\{z_1, z_2\} \not\subseteq V(M)$, then $(G - y_3)/z_1z_2$ is 2-connected by Claim 1 and the fact that $G - y_3$ is 2-connected, and thus Lemma 3.5 implies that $\tau((G - y_3)/\{z_1, z_2\}) \geq \tau(M)$. Let $p$ be the number of such pairs $z_1, z_2 \in N(y_3)$ with $\{z_1, z_2\} \not\subseteq V(M)$. So

$$p = \binom{r}{2} + r(d(y_3) - r). \quad (55)$$

It is clear that $d(y_3) \geq r \geq 2$. By (47), $d(y_3) \geq 3$. Thus, it is not difficult to show that $p > d(y_3) - 1$. By Lemma 3.4,

$$\frac{\tau(G)}{d(y_3) - 1} \geq \tau(G - y_3) + \frac{p \tau(M)}{d(y_3) - 1} > \tau(G - y_3) + \tau(M). \quad (56)$$
Lemma 3.4 also implies that \( \tau(G)/d(y_i) - 1 \geq \tau(G - y_i) \) for \( i = 1, 2 \). Then the following inequality follows from (47) and (56):

\[
\tau(G) \geq \frac{3}{d(y_i) - 1} \left( \sum_{i=1}^{3} \tau(G - y_i) \right) + \tau(M). \tag{57}
\]

Since \( b(G - \{y_i, y_j\}) \geq 2 \), we have \( \tau(G - \{y_i, y_j\}) = 0 \) by Lemma 3.1(i) for all \( 1 \leq i < j \leq 3 \). Thus

\[
g(G, I_S) = \sum_{A \in I_S} (-1)^{|A|} \tau(G - A) = \tau(G) - \left( \sum_{i=1}^{3} \tau(G - y_i) \right) - \tau(G - S)
\]

\[
\geq \tau(M) - (-1)^r \tau(M) = 0,
\]

where the last two steps follow from (57), Claim 3, and the fact that \( r \) is even.

**Subcase (2.3).** \(|S| = 2\).

Since \(|S| + r \) is odd, \( r \) is odd in this subcase. As \( r \geq 1 \), \( y_i \) is a cut-vertex of \( G - y_{3-i} \) for \( i = 1, 2 \). Thus

\[
g(G, I_S) = \tau(G) + \tau(G - \{y_1, y_2\}) = \tau(G) + (-1)^r \tau(M) = \tau(G) - \tau(M),
\]

where the second last equality follows from Claim 3. By Lemma 3.1 and the fact that \( b(G/y_1y_2) = 0 \), we have

\[
\tau(G) = \tau(G+y_1y_2) - \tau(G/y_1y_2) = \tau(G + y_1y_2).
\]

Let \( H \) be the graph obtained from \( G - T \) by adding the edge \( y_1y_2 \). As \( P(G + y_1y_2, \lambda) = (\lambda - 2)^r P(H, \lambda) \), we have \( \tau(G + y_1y_2) = \tau(H) \). Hence \( \tau(G) = \tau(H) \). By Lemma 3.6, \( \tau(H) \geq \tau(M) \) where the equality holds if and only if

(a) \( N_H(y_i) \cap V(M) \subseteq \{u, v\} \) and \( 4 \leq d_H(y_1) + d_H(y_2) \leq 5 \); and

(b) either \( uv \in E(H) \) or \( b(M/uv) \geq 2 \).

Assume that \( d_H(y_1) \leq d_H(y_2) \). Then Condition (a) is equivalent to that \( N_G(y_1) \cap V(M) = \{u\} \) and \( \{v\} \subseteq N_G(y_2) \cap V(M) \subseteq \{u, v\} \). In condition (b), \( uv \in E(H) \) is equivalent to that \( uv \in E(G) \).

Therefore, in subcase (2.3), \( g(G, I_S) \geq 0 \), where the equality holds if and only if \( G \) has property \( v \) with respect to \( S \), i.e., condition (ii) is satisfied.

Thus, the lemma holds when \(|S| = m\), and the proof is then completed.

We are now in a position to establish the main result in this section.

**Theorem 5.1.** Let \( G \) be a 3-connected graph. If \( G \) contains a vertex \( w \) such that \( \delta((G - w) - A) > 0 \) for every \( A \in I_S \setminus \{S\} \) and

\[
\sum_{x \in S} \frac{1}{d(x) - 1} \leq 1,
\]

where \( S = V(G) \setminus N[w] \), then \( (\lambda - 2)^2 |P(G, \lambda) \) if and only if \( G \in \mathcal{J} \).

**Proof.** If \( G \) is bipartite, then it is clear that \( G \notin \mathcal{J} \) and \( (\lambda - 2)^2 |P(G, \lambda) \). So it suffices to consider the case that \( G \) is not bipartite. Note that \( S \) is independent if and only if

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Thus, \((G-w)-S\) is independent. Thus, \(d_{G-w}(x) = d_G(x)\) for every \(x \in S\). Applying Lemma 5.3 to \(G-w\) yields that \(g(G-w, I_S) \geq 0\), where the equality holds if and only if one of conditions (i), (ii), and (iii) in Lemma 5.3 is satisfied for \(G-w\). Only condition (ii) can be satisfied, as condition (i) or (iii) implies that \(G\) is not 3-connected, a contradiction. Hence, \(g(G-w, I_S) = 0\) if and only if \(|S| = 2\), \(G-w\) has property \(v\) with respect to \(S\), and the subgraph \([N(w)]\) has exactly an odd number of isolated vertices (as \(G\) is 3-connected, \([N(w)]\) has at least three isolated vertices). Thus, \(g(G-w, I_S) = 0\) if and only if \(G \in J\). The result of the theorem then follows directly from Theorem 2.2.

**Corollary 5.1.** Let \(G\) be a 3-connected graph of order \(n\). If \(G\) contains a vertex \(w\) such that \(d(w)+d(y) \geq n\) for all \(y \in S \cup T\), where \(S = V(G) \setminus N[w]\) and \(T\) is the set of isolated vertices in \([N(w)]\), then \((\lambda-2)^2|P(G, \lambda)|\) if and only if \(G \in J\).

**Proof.** Note that \(|S| = n-1-d(w)|S|\). Since \(d(w)+d(y) \geq n\) for all \(y \in S \cup T\), we have \(d(y) \geq n-d(w) = |S|+1\) for all \(y \in S \cup T\), implying that \(\delta(G-w-A) > 0\) for each \(A \in I_S \setminus \{S\}\) and

\[
\sum_{x \in S} \frac{1}{d(x)-1} = \sum_{x \in S} \frac{1}{|S|} = 1.
\]

Then the result follows directly from Theorem 5.1.

Let \(\delta_3(G)\) be the third minimum degree among the degrees of all vertices in \(G\).

**Corollary 5.2.** Let \(G\) be a 3-connected graph of order \(n\) with \(\Delta(G) + \delta(G) \geq n\). We have

(i) \((\lambda-2)^2|P(G, \lambda)|\) if and only if \(G \in J\);

(ii) if \(\Delta(G) + \delta_3(G) \neq (n-3, 3)\), then \((\lambda-2)^2|P(G, \lambda)|\); and

(iii) if \(\Delta(G) + \delta(G) \geq n+1\), then \((\lambda-2)^2|P(G, \lambda)|\).

**Proof.** Let \(w \in V(G)\) with \(d(w) = \Delta(G)\). If \(\Delta(G) + \delta(G) \geq n\), then \(d(w)+d(y) \geq n\) for all \(y \in V(G)\), and so Corollary 5.1 implies that (i) holds. If \(G \in J\), we have \(\Delta(G) = n-3\) and \(\delta(G) = \delta_3(G) = 3\) by the definition of \(J\) by Corollary 4.1. Thus, (ii) and (iii) also hold.

Note that the minimum value among the orders of all graphs in \(J\) is 8.

**Corollary 5.3.** If \(G\) is a 3-connected graph of order at most 7, then \((\lambda-2)^2\) is not a factor of \(P(G, \lambda)\).

**Proof.** Let \(n\) be the order of \(G\). It is clear that \(\delta(G) \geq 3\) and if \(n\) is odd, then \(\Delta(G) \geq 4\). Thus, \(\Delta(G) + \delta(G) \geq n\) if \(n \leq 7\). By Corollary 5.2, we have \(G \in J\) if \((\lambda-2)^2|P(G, \lambda)|\). As every graph in \(J\) is of order at least 8, the result holds.

6. PROBLEMS

To end this article, we would like to propose two related problems for further study. Corollary 5.2 characterizes all 3-connected graphs \(G\) with \(\Delta(G) + \delta(G) \geq v(G)\) such
FIGURE 5. (A) The Hershel graph and (B) a graph obtained from the Hershel graph.

that \((\lambda - 2)^2 | P(G, \lambda)\). However, not much information on 3-connected graphs \(G\) with \(\Delta(G) + \delta(G) < v(G)\) and \((\lambda - 2)^2 | P(G, \lambda)\) has been found.

Very recently such a graph was discovered by Jackson [2]. The Hershel graph is the smallest non-Hamiltonian polyhedral graph, as shown in Figure 5(A). Let \(H\) be a graph obtained from the Hershel graph by replacing a vertex of degree 3 by a \(K_3\), as shown in Figure 5(B). Note that \(H\) is 3-connected with \(v(H) = 13\), \(\Delta(H) = 4\), and \(\delta(H) = 3\). Jackson [2] observed that \((\lambda - 2)^2 | P(H, \lambda)\):

\[
P(H, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2(\lambda^9 - 16\lambda^8 + 121\lambda^7 - 568\lambda^6 + 1831\lambda^5 - 4227\lambda^4 + 7034\lambda^3 - 8196\lambda^2 + 6113\lambda - 2239).
\]

(63)

**Problem 6.1.** Characterize 3-connected graphs \(G\) with \(\Delta(G) + \delta(G) < v(G)\) such that \((\lambda - 2)^2 | P(G, \lambda)\).

On the other hand, we also want to know whether there exists a 3-connected graph \(G\) whose chromatic zero “2” has multiplicity larger than 2 (or even very large). In what follows, we shall introduce a method of generating 3-connected graphs \(G\) such that its chromatic zero “2” of \(G\) has sufficiently large multiplicity and \(v(G) - (\Delta(G) + \delta(G))\) is also sufficiently large.

Let \(k\) be any integer with \(k \geq 2\). For \(i = 1, 2, \ldots, k\), let \(G_i\) be a graph isomorphic to the graph shown in Figure 5(B). Now we construct a graph \(G\) from \(G_1, G_2, \ldots, G_k\). Note that \(v(G_i) = 13\), \(\Delta(G_i) = 4\), and \(\delta(G_i) = 3\) for all \(i\). Also notice that \(G_i\) contains only one triangle with vertices of degree 3. Let \(x_i, y_i, z_i\) represent the three vertices in the only triangle of \(G_i\) for all \(i = 2, \ldots, k\). Let \(G\) be the graph obtained from \(G_1, G_2, \ldots, G_k\) by identifying all vertices in \(\{x_1, x_2, \ldots, x_k\}\), all vertices in \(\{y_1, y_2, \ldots, y_k\}\), and all vertices in \(\{z_1, z_2, \ldots, z_k\}\), respectively. Observe that \(G\) is 3-connected, \(\delta(G) = 3\),

\[
\Delta(G) = 3 + (k - 1) = k + 2
\]

(64)

and

\[
v(G) = \sum_{i=1}^{k} v(G_i) - 3(k - 1) = 10k + 3.
\]

(65)

As \((\lambda - 2)^2 | P(G_i, \lambda)\) for all \(i = 1, 2, \ldots, k\), Theorem 3.1 implies that \((\lambda - 2)^{k+1} | P(G, \lambda)\).

It is noted that, in the above construction, \(G\) contains a complete cut-set for any \(k \geq 2\) (a complete cut-set is a cut-set whose induced subgraph is a complete graph).
Although some graphs in $\mathcal{J}$ do not contain complete cut-set, we are not sure whether $(\lambda - 2)^3$ can be a factor of their chromatic polynomials. Now we propose the second problem to end this article.

**Problem 6.2.** Does there exist a 3-connected graph $G$ such that $G$ contains no complete cut-set and the multiplicity of its chromatic zero “2” is at least three?

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