This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier’s archiving and manuscript policies are encouraged to visit:

http://www.elsevier.com/copyright
A new expression for matching polynomials

F.M. Dong
Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore 637616, Singapore

ARTICLE INFO

Article history:
Received 13 January 2011
Received in revised form 15 November 2011
Accepted 15 November 2011

Keywords:
Graph
Matrix
Matching
Matching polynomial

ABSTRACT

Let \( G \) be an arbitrary simple graph. Godsil and Gutman in 1978 and Yan et al. in 2005 established different expressions for the matching polynomial \( \mu(G, x) \) in terms of \( \det(xI_n - H) \) for some families of matrices \( H \). This paper improves their results and simplifies the computation of \( \mu(G, x) \).

1. Introduction

In this paper we consider simple graphs (i.e., a graph with no loops and parallel edges) only. For any graph \( G \), let \( V(G) \), \( E(G) \) and \( \nu(G) \) be its vertex set, edge set and order (i.e., \( \nu(G) = |V(G)| \)), respectively. If it is not mentioned elsewhere in this paper, we always assume that \( G \) is a simple graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{e_1, e_2, \ldots, e_{\epsilon}\} \), where \( \epsilon = |E| \). A matching of \( G \) is a subset \( M \) of \( E \) such that each vertex of \( G \) incident with at most one edge in \( M \). For any integer \( k \geq 0 \), let \( \phi_k(G) \) denote the number of matchings \( M \) of \( G \) with \( |M| = k \). It is clear that \( \phi_0(G) = 1 \) and \( \phi_1(G) = |E| \). One form of matching polynomial is \( \sum_{k \geq 0} \phi_k(G)x^k \) (see [1]). In this paper, we study another form of matching polynomial which is defined below:

\[
\mu(G, x) = \sum_{k=0}^{[n/2]} (-1)^k \phi_k(G)x^{n-2k}.
\]

(1.1)

This polynomial is also called the acyclic polynomial (see [4]). Throughout this paper, this polynomial \( \mu(G, x) \) will be referred to as the matching polynomial of \( G \).

Godsil and Gutman [2] showed that

\[
\mu(G, x) = 2^{-\epsilon} \sum_w \det(xI_n - A(w)),
\]

(1.2)

where the summation ranges over all \( 2^{\epsilon} \) distinct \( \epsilon \)-tuples \( w = (w_1, w_2, \ldots, w_\epsilon) \), \( w_j \in \{1, -1\} \) and the matrix \( A(w) = (a_{j,k}) \) with the tuple \( w = (w_1, w_2, \ldots, w_\epsilon) \) is defined as follows: \( a_{j,k} = w_j \) if \( v_jv_k \) is the edge \( e_s \) and \( a_{j,k} = 0 \) if \( v_jv_k \notin E \) for all \( j, k \).

Yan et al. [7] obtained a similar result that

\[
\mu(G, x) = 2^{-\epsilon} \sum_{G'} \det(xI_n + iA(G')),
\]

(1.3)
where the sum ranges over all $2^s$ orientations $G'$ of $G$, $i$ is the complex number with $i^2 = -1$ (i will be used to denote this number throughout this paper) and $\Lambda(G') = (a_{i,k})$ is the matrix defined as follows: $a_{i,k} = 1$ if $v_i v_k \in E$ and $a_{i,k} = 0$ otherwise.

This paper generalizes the above results by showing that if $F$ is a subset of $E$ such that every pair of cycles in $G - F$ (i.e., the subgraph obtained from $G$ by removing all edges in $F$) are edge-disjoint, then

$$\mu(G, x) = 2^{-|F|} \sum_B \det(xI_n - B),$$

where the sum ranges over all matrices in a set of $2^{|F|}$ matrices $B = (b_{i,j})$ with the property that $b_{i,j} \times b_{k,j} = 1$ when $v_i v_k \in E$ and $b_{i,j} = b_{k,j} = 0$ otherwise (see Corollary 2.2). When $F = E$, this result implies (1.2) and (1.3).

2. Main result

For any graph $G$, let $\mathcal{M}(G)$ be the set of matrices $(a_{i,j})_{n \times n}$ such that $a_{i,j}a_{k,j} = 1$ if $v_i v_k \in E$ and $a_{i,j} = a_{k,j} = 0$ otherwise. Note that $(a_{i,j}) \in \mathcal{M}(G)$ is an adjacency matrix of $G$ if $a_{i,j} = 1$ whenever $v_i v_k \in E$. It is well known (see [4–6]) that $\mu(G, x) = \det(xI_n - A)$ if $G$ is a forest and $A$ is an adjacency matrix of $G$. This result is actually a particular case of the following result due to Graovac and Polansky [3].

Theorem 2.1 ([3]). Let $G$ be a graph in which every pair of cycles are edge-disjoint and $A = (a_{i,k})$ be any matrix in $\mathcal{M}(G)$. Assume that for every cycle $C : v_1 v_2 \cdots v_i v_1$ in $G$, the following condition always holds:

$$a_{r_1,r_2} a_{r_2,r_3} \cdots a_{r_{n},r_1} \in \{1, -1, i, -i\} \text{ and } a_{r_1,r_2} a_{r_2,r_3} \cdots a_{r_{n},r_1} = -1.$$  

Then $\mu(G, x) = \det(xI_n - A)$. □

By Theorem 2.1, if $G$ is a forest, then $\mu(G, x) = \det(xI_n - A)$ holds for every matrix $A \in \mathcal{M}(G)$.

For $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$, assign every $e \in E$ a non-zero complex number $w_e$. We call $\{w_e\}_{e \in E}$ the weight-function of $E$, denoted by $w_G$ (or simply by $w$). Let $\mathcal{M}(G, w)$ be the set of $(n \times n)$-matrices $(a_{i,k})$ satisfying the condition below:

$$\begin{align*}
a_{i,k} &= \{w_e, -w_e\}, & \text{if } j < k \text{ and } v_i v_k = e \in E; \\
a_{i,k} &= 1/a_{k,j}, & \text{if } j > k \text{ and } v_i v_k = E; \\
a_{i,k} &= 0, & \text{otherwise.}
\end{align*}$$

(2.1)

Note that $\mathcal{M}(G, w)$ contains exactly $2^{|E|}$ matrices and $\mathcal{M}(G, w) \subseteq \mathcal{M}(G)$.

By the notation of a weight-function $w = \{w_e\}_{e \in E}$, the result of (1.2) due to Godsil and Gutman [2] is equivalent to the expression below with $w_e = i$ for all $e \in E$:

$$\mu(G, x) = 2^{-|E|} \sum_{A \in \mathcal{M}(G, w)} \det(xI_n - A),$$

(2.2)

The result of (1.3) due to Yan et al. [7] is also equivalent to (2.2) with $w_e = i$ for all $e \in E$. We shall show that (2.2) actually holds as long as $w_e \neq 0$ for all $e \in E$.

Let $G = (V, E)$ be any graph with $V = \{v_1, v_2, \ldots, v_n\}$ and weight-function $w = \{w_e\}_{e \in E}$. $F$ be a subset of $E$ and $A = (a_{i,k})$ be an $n \times n$ matrix with $a_{i,k} \neq 0$, whenever $v_i v_k \in E$. Let

$$g_F(G) = \{G - F : v_i, v_k : v_i v_k \in F : F' \subseteq F\},$$

(2.3)

where $G - F - V'$ is the subgraph of $G - F$ after deleting all vertices in $V'$, and $\mathcal{M}_F(A)$ be the set of matrices $(d_{i,k})_{n \times n}$ satisfying the following condition:

$$\begin{align*}
d_{i,k} &= \{a_{i,k}, -a_{i,k}\}, & \text{if } j < k \text{ and } v_i v_k \in F; \\
d_{i,k} &= 1/d_{k,j}, & \text{if } j < k \text{ and } v_i v_k \in F; \\
d_{i,k} &= a_{i,k}, & \text{otherwise.}
\end{align*}$$

(2.4)

Note that $G - F \in g_F(G)$ and every graph of $g_F(G)$ is a subgraph of $G - F$. It is also clear that $|\mathcal{M}_F(A)| = 2^{|F|}$. Note that if $A \in \mathcal{M}(G)$, then $A \in \mathcal{M}_F(A) \subseteq \mathcal{M}(G)$ for any $F \subseteq E$.

For any $n \times n$ matrix $A = (a_{i,k})$ and any non-empty subset $I$ of $\{1, 2, \ldots, n\}$, let $A[I]$ be the matrix obtained from $A$ by removing rows $s_1, s_2, \ldots, s_s$ and columns $s_1, s_2, \ldots, s_s$, where $\{s_1, s_2, \ldots, s_s\} = \{1, 2, \ldots, n\} - I$. 

---

1 This result was explained in [3]. It may have also appeared in some other articles.
For any subgraph $H$ of $G$ with $v(H) > 0$ and any $n \times n$ matrix $B = (b_{j,k})$ with the property that $b_{j,k} = 0$ whenever $v_jv_k \notin E$, we define a $(v(H) \times v(H))$ matrix $B_H$ corresponding to $H$. If $H$ is a spanning subgraph of $G$, let $B_H$ be the $n \times n$ matrix $(d_{j,k})$ such that

$$d_{j,k} = \begin{cases} b_{j,k}, & \text{if } v_jv_k \in E(H), \\ 0, & \text{otherwise}; \end{cases} \quad (2.5)$$

if $H$ is a subgraph of $G$ induced by $V' \subseteq V$, let $B_H = B[I]$, where $I = \{1 \leq t \leq n : v_t \in V'\}$; otherwise (i.e., $H$ is not a spanning subgraph nor an induced subgraph of $G$), let $B_H = C[I]$, where $C = B[I]$ for $I = \{1 \leq t \leq n : v_t \in V(H)\}$. By the definition of $B_H$, we have $B_H \in \mathcal{M}(H)$ if $B \in \mathcal{M}(G)$.

Our main purpose in this paper is to establish the following result.

**Theorem 2.2.** Let $G = (V, E)$ be any simple graph with $V = \{v_1, v_2, \ldots, v_n\}$, $F$ be any subset of $E$ and $A = (a_{j,k})$ be any matrix contained in $\mathcal{M}(G)$. Assume that

$$\mu(H, x) = \det(xI_{v(H)} - A_H) \quad (2.6)$$

holds for every $H \in \mathcal{F}(G)$ with $v(H) > 0$. Then

$$\mu(G, x) = 2^{-|F|} \sum_{B \in \mathcal{M}(A)} \det(xI_n - B). \quad (2.7)$$

We need to introduce some results which will be applied in the proof of Theorem 2.2.

**Lemma 2.1.** Let $A = (a_{j,k})$ be any $n \times n$-matrix, where $n \geq 3$. Let $A'$ be the matrix obtained from $A$ by replacing $a_{1,2}$ and $a_{2,1}$ by $-a_{1,2}$ and $-a_{2,1}$ respectively, $B$ be the matrix obtained from $A$ by replacing both $a_{1,2}$ and $a_{2,1}$ by 0, and let $C$ be the $(n-2) \times (n-2)$-matrix $A[I]$, where $I = \{3, 4, \ldots, n\}$. Then

$$\det(A) + \det(A') - 2 \det(B) = -2a_{1,2}a_{2,1} \det(C). \quad (2.8)$$

**Proof.** Let $D$ be the $n \times (n-2)$ matrix obtained from $A$ by deleting the last two columns. Observe that

$$\det(A) = \det\begin{pmatrix} a_{1,1} & 0 \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ \vdots & \vdots \\ a_{n,1} & a_{n,2} \end{pmatrix} D + \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & 0 \\ a_{3,1} & 0 \\ \vdots & \vdots \\ a_{n,1} & 0 \end{pmatrix} D + \det\begin{pmatrix} 0 & a_{1,2} \\ a_{2,1} & 0 \\ a_{3,1} & 0 \\ \vdots & \vdots \\ a_{n,1} & 0 \end{pmatrix} D + \det\begin{pmatrix} 0 & 0 \\ a_{2,1} & 0 \\ a_{3,1} & 0 \\ \vdots & \vdots \\ a_{n,1} & 0 \end{pmatrix} D + \det\begin{pmatrix} 0 & 0 \\ 0 & a_{2,1} \\ 0 & a_{3,1} \\ \vdots & \vdots \\ 0 & a_{n,1} \end{pmatrix} D + \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_{3,1} \\ \vdots & \vdots \\ 0 & a_{n,1} \end{pmatrix} D + \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_{3,1} \\ \vdots & \vdots \\ 0 & a_{n,1} \end{pmatrix} D + \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} D.$$

Note that $\det(A')$ has a similar expression, which can be obtained from the above expression by replacing $a_{1,2}$ and $a_{2,1}$ by $-a_{1,2}$ and $-a_{2,1}$ respectively. Thus the lemma holds. $\square$

By the matrix manipulations of exchanging rows and columns, Lemma 2.1 implies the following result.

**Corollary 2.1.** Let $A = (a_{j,k})_{n \times n}$ be any matrix and $s, t$ be integers with $1 \leq s < t \leq n$. Let $A'$ be the matrix obtained from $A$ by replacing $a_{s,t}$ and $a_{t,s}$ by $-a_{s,t}$ and $-a_{t,s}$ respectively, let $B$ be the matrix obtained from $A$ by replacing both $a_{s,t}$ and $a_{t,s}$ by 0, and let $C = A[I]$, where $I = \{1, 2, \ldots, n\} \setminus \{s, t\}$. Then

$$\det(A) + \det(A') - 2 \det(B) = -2a_{s,t}a_{t,s} \det(C). \quad (2.9)$$

Like many other polynomials of graphs, the matching polynomial also has a recursive expression which can be applied to compute the matching polynomial of any graph. It is clear that if $uv \in E(G)$, then $\varphi_k(G) = \varphi_k(G - uv) + \varphi_{k-1}(G - u - v)$ holds for any integer $k$ with $1 \leq k \leq v(G)/2$, where $G - uv$ and $G - u - v$ are the graphs obtained from $G$ by removing edge $uv$ and removing vertices $u, v$ respectively. Thus the next result follows (see [1,4]).
Lemma 2.2. For any edge $uv$ in $G$, $\mu(G, x) = \mu(G - uv, x) - \mu(G - u - v, x)$. □

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** We shall prove this result by induction on $|F|$. If $F = \emptyset$, then $\bar{g}_t(G) = |G|$ and $M_F(A) = \{A\}$, and so (2.6) implies (2.7). If $|E| = |F| = 1$, then $M_F(A) \subseteq M(G)$, and $\mu(G, x) = \det(xI_n - B)$ for every $B \in M_F(A)$ by Theorem 2.1, implying that (2.7) holds.

Assume that the result holds when $|F| < f$, where $f \geq 1$. Now consider the case that $|F| = f$. As the result holds when $|E| = |F| = 1$, we also assume that $n \geq 3$.

Choose any edge $e = v_1v_2$. For the convenience of writing, we can assume that $e = v_1v_2$. In fact, if $(j, k) \neq (1, 2)$, the only difference of the proof is at (2.19), which should follow from Corollary 2.1. Let $G_1 = G - v_1v_2$ and $G_2 = G - v_1 - v_2$. So $G_2$ is the graph with vertex set $V \setminus \{v_1, v_2\}$ and edge set $E \setminus E_{v_1v_2}$, where $E_{v_1v_2} = \{v_1v_2 \in E(G) : 1 \leq s \leq 2 \leq t \leq n, s < t\}$. It is clear that $v_1v_2 \in E_{v_1v_2}$. Let $F_1 = F \setminus \{v_1v_2\}$ and $F_2 = F \setminus E_{v_1v_2}$. Observe that $|F_1| = |F| - 1$ and $|F_2| = |F| - |E_{v_1v_2} \cap F| \leq |F| - 1$.

By the definition of $\bar{g}_t(G)$, we have $\bar{g}_t(G_1) \subseteq \bar{g}_t(G)$ for $s = 1, 2$, and so, by induction hypothesis,

$$\mu(G_1, x) = 2^{-|F_1|} \sum_{B \in M_{F_1}(A_1)} \det(xI_n - B)$$

and

$$\mu(G_2, x) = 2^{-|F_2|} \sum_{C \in M_{F_2}(A_2)} \det(xI_n - C),$$

where $A_s = A_{G_s}$ for $s = 1, 2$. Note that $A_1$ can be obtained from $A = (a_{i,j})$ by replacing $a_{1,2}$ and $a_{2,1}$ by 0, and $A_2$ is the matrix $A[I]$, where $I = \{3, 4, \ldots, n\}$.

By Lemma 2.2, we have

$$\mu(G, x) = \mu(G_1, x) - \mu(G_2, x).$$

Thus, to show that (2.7) holds for $G$, it suffices to show that

$$2^{-|F|} \sum_{D \in M_F(A)} \det(xI_n - D) = 2^{-|F_1|} \sum_{B \in M_{F_1}(A_1)} \det(xI_n - B) - 2^{-|F_2|} \sum_{C \in M_{F_2}(A_2)} \det(xI_n - C),$$

i.e.,

$$\sum_{D \in M_F(A)} \det(xI_n - D) - 2 \sum_{B \in M_{F_1}(A_1)} \det(xI_n - B) = -2^{2|E_{v_1v_2}|} \sum_{C \in M_{F_2}(A_2)} \det(xI_n - C).$$

Notice that for each $D = (d_{i,j}) \in M_F(A)$, the matrix obtained from $D$ by replacing both $d_{1,2}$ and $d_{2,1}$ by 0 is contained in $M_{F_1}(A_1)$, and the matrix $D[I]$, where $I = \{3, 4, \ldots, n\}$, belongs to $M_{F_2}(A_2)$. Thus

$$\sum_{D \in M_F(A)} \det(xI_n - D) = 2 \sum_{B \in M_{F_1}(A_1)} \det(xI_n - B) + \left( \sum_{C \in M_{F_2}(A_2)} \sum_{B \in M_{F_1}(A_1)} \det(xI_n - A) - 2 \sum_{B \in M_{F_1}(A_1)} \det(xI_n - B) \right),$$

where for each $C = (c_{i,j}) \in M_{F_2}(A_2)$, $\Psi(C)$ is the set of matrices $D = (d_{i,j})_{n \times n} \in M_F(A)$ defined below:

$$\begin{align*}
d_{i,j} &= c_{j-2,k-2}, & \text{if } j \geq 3, k \geq 3; \\
d_{i,j} &= a_{i,j}, & \text{if } 1 \leq j \leq k \leq n \text{ and } v_i \neq v_k \in F; \\
d_{i,j} &= 1/a_{i,j}, & \text{if } 1 \leq k \leq j \leq n \text{ and } v_i \neq v_k \in F; \\
d_{i,j} &= a_{i,j}, & \text{otherwise.}
\end{align*}$$

and $\Psi_1(C)$ is the set of matrices $B = (b_{i,j})_{n \times n} \in M_{F_1}(A_1)$ defined below:

$$\begin{align*}
b_{i,j} &= c_{j-2,k-2}, & \text{if } j \geq 3, k \geq 3; \\
b_{i,j} &= a_{i,j}, & \text{if } 1 \leq j \leq k \leq n \text{ and } v_i \neq v_k \in F_1; \\
b_{i,j} &= 1/b_{i,j}, & \text{if } 1 \leq k \leq j \leq n \text{ and } v_i \neq v_k \in F_1; \\
b_{i,j} &= a_{i,j}, & \text{otherwise.}
\end{align*}$$

So it remains to show that for each $C \in M_{F_2}(A_2)$,

$$\sum_{A \in \Psi(C)} \det(xI_n - A) - 2 \sum_{B \in \Psi_1(C)} \det(xI_n - B) = -2^{2|E_{v_1v_2} \cap F|} \det(xI_n - C).$$

Observe that $|\Psi(C)| = 2^{2|E_{v_1v_2}|}$ and $|\Psi_1(C)| = 2^{2|E_{v_1v_2}| - 1}$. For any $B \in \Psi_1(C)$, there exists exactly two distinct matrices $D' = (d'_{i,j})$ and $D'' = (d''_{i,j})$ in $\Psi(C)$ such that $B$ can be obtained from $D'$ by replacing both $d'_{1,2}$ and $d'_{2,1}$ by 0, and can be also
obtained from $D''$ by replacing both $d''_{1,2}$ and $d''_{2,1}$ by 0. It is clear that $d''_{1,2} = -d'_{1,2}$, $d''_{2,1} = -d'_{2,1}$ and $d''_{j,k} = d'_{j,k}$ whenever $[j, k] \neq \{1, 2\}$. Thus, by Lemma 2.1, we have

$$
\det(xI_n - D') + \det(xI_n - D'') - 2 \det(xI_n - B) = -2(-d'_{1,2})(-d'_{2,1}) \det(xI_{n-2} - C) = -2 \det(xI_{n-2} - C),
$$

implying that

$$
\sum_{A \in \Psi(C)} \det(xI_n - A) - 2 \sum_{B \in \Psi_1(C)} \det(xI_n - B) = |\Psi_1(C)|(-2 \det(xI_{n-2} - C))
$$

(2.19)

Thus we complete the proof. \Box

For any graph $G = (V, E)$ with $w = \{w_e\}_{e \in E}$ and any $F \subseteq E$, let $\mathcal{M}(G, w, F)$ be the set of matrices $(a_{j,k})_{n \times n}$ satisfying the following condition:

$$
a_{j,k} = \begin{cases} 
  w_e, & \text{if } j < k \text{ and } v_jv_k = e \in F; \\
  w_e, & \text{if } j < k \text{ and } v_jv_k = e \in E \setminus F; \\
  1/a_{j,k}, & \text{if } j > k \text{ and } v_jv_k = e \in E; \\
  0, & \text{otherwise.}
\end{cases}
$$

Note that $\mathcal{M}(G, w, F) \subseteq \mathcal{M}(G, w)$, $\mathcal{M}(G, w, E) = \mathcal{M}(G, w)$ and $|\mathcal{M}(G, w, F)| = 2^{|F|}$.

For any $A \in \mathcal{M}(G, w, F)$, we have $\mathcal{M}_F(A) = \mathcal{M}(G, w, F)$. Theorems 2.1 and 2.2 then imply the following result.

Corollary 2.2. Let $G = (V, E)$ be any simple graph with $w = \{w_e\}_{e \in E}$. Let $F$ be any subset of $E$ such that every pair of cycles in $G - F$ are edge-disjoint. If every cycle $C$ in $G - F$ satisfies the condition that $w_e \in \{1, -1, i, -i\}$ for all $e \in E(C)$ and

$$
\prod_{e \in E(C)} w_e^2 = -1,
$$

then

$$
\mu(G, x) = 2^{-|F|} \sum_{A \in \mathcal{M}(G, w, F)} \det(xI_n - A).
$$

(2.22)

In particular, if $G - F$ is a forest, we get the following result.

Corollary 2.3. Let $G = (V, E)$ be any simple graph with $w = \{w_e\}_{e \in E}$ and $F$ be any subset of $E$ such that $G - F$ is a forest. Then

$$
\mu(G, x) = 2^{-|F|} \sum_{A \in \mathcal{M}(G, w, F)} \det(xI_n - A).
$$

(2.23)

If $F = E$, then the results of (1.2) and (1.3) correspond to Corollary 2.3 for the two cases that $w_e = 1$ for all $e \in E$ and $w_e = i$ for all $e \in E$ respectively.

Acknowledgments

The author wishes to thank the referees for their very helpful suggestions.

References