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The vertex-cover polynomial of a graph

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Abstract

In this paper we define the vertex-cover polynomial $\Psi(G,\tau)$ for a graph G. The coefficient of τ^r in this polynomial is the number of vertex covers V' of G with $|V'|=r$. We develop a method to calculate $\Psi(G,\tau)$. Motivated by a problem in biological systematics, we also consider the mappings f from $\{1,2,\ldots,m\}$ into the vertex set $V(G)$ of a graph G, subject to $f^{-1}(x)\cup f^{-1}(y)\neq\emptyset$ for every edge xy in G. Let $F(G,m)$ be the number of such mappings f. We show that $F(G,m)$ can be determined from $\Psi(G,\tau)$.

Keywords: Graph; Vertex-cover; Graph-function; Graph-polynomial

1. Introduction

The graphs considered in this paper are undirected and contain no multi-edges, but may have loops. For a graph G, let $V(G)$, $E(G)$, $v(G)$ and $e(G)$ be the vertex set, edge set, order and size of G, respectively. The null graph is the graph G with $v(G)=0$. The reader is referred to [4] for any terminology not defined here.

For a graph G, $V'\subseteq V(G)$ is called an *r-vertex cover* in G if $|V'|=r$ and $V'\cap\{x,y\}\neq\emptyset$ for all $xy\in E(G)$. Let $\mathscr{CV}(G,r)$ be the set of r-vertex covers in G, and $\mathrm{cv}(G,r)=|\mathscr{CV}(G,r)|$. Observe that $\mathrm{cv}(G,r)=0$ if either $r<0$ or $r>v(G)$.

We define the following generating function:

$$\Psi(G,\tau)=\sum_{r=0}^{v(G)}\mathrm{cv}(G,r)\tau^r. \quad (1)$$

For example, let K_n be the complete graph on $n\geqslant 1$ vertices. Then $\Psi(K_n,\tau)=\tau^n+n\tau^{n-1}$ since $\mathrm{cv}(K_n,n)=1$, $\mathrm{cv}(K_n,n-1)=n$ and $\mathrm{cv}(K_n,r)=0$ if $0\leqslant r<n-1$.

It is natural to call $\Psi(G,\tau)$ the *vertex-cover polynomial* of G. In this paper, we shall develop a method to calculate $\Psi(G,\tau)$. By definition, $\mathrm{cv}(G,r)$ can be obtained

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if $\Psi(G,\tau)$ is determined. Recall that the vertex-cover number of G is the minimum number r such that G has an r-vertex cover. Thus, the vertex-cover number can be determined from $\Psi(G,\tau)$.

Now we define another graph function $F(G,m)$ for any graph G and nonnegative integer m. The definition of $F(G,m)$ is motivated by a problem in biology [2], where it was necessary to calculate the number of mappings f, from a given finite set to the vertex set $V(G)$ of a graph G, such that $\bigcup_{v\in V'} f^{-1}(v)\neq\emptyset$ for each member V' of a given set $\mathscr{S}$ of subsets of $V(G)$. In this paper, we consider the case $\mathscr{S}=\{\{x,y\}\,|\,xy\in E(G)\}$. For a graph G and an integer $m\geqslant 0$, define $\mathscr{F}(G,m)$ to be the set of mappings

$$f:\{1,2,\ldots,m\}\to V(G), \tag{2}$$

subject to $f^{-1}(x)\cup f^{-1}(y)\neq\emptyset$ for every $xy\in E(G)$. Note that for $v(G)=0$ or $m=0$, we have

$$\mathscr{F}(G,m)=\begin{cases}\{\emptyset\} & \text{if } e(G)=0 \text{ and } m=0,\\ \emptyset & \text{otherwise.}\end{cases} \tag{3}$$

Let $F(G,m)=|\mathscr{F}(G,m)|$. By the definition of $\mathscr{F}(G,m)$, we observe that $F(G,m)$ is a graph-function.

We shall show that $F(G,m)$ can be expressed in terms of $\mathrm{cv}(G,r)$ for $r\geqslant 0$. Thus $F(G,m)$ can be obtained from $\Psi(G,\tau)$.

2. Vertex-cover polynomials

In this section, we shall develop a method to calculate $\Psi(G,\tau)$. Observe that $\Psi(G,\tau)$ is independent of the multiplicity of any edge in G. Hence we assume that G contains no multi-edges.

We first consider some special types of graphs. By definition,

Lemma 2.1. *For the null graph G, $\Psi(G,\tau)=1$.* □

For integer $n\geqslant 1$, let N_n be the graph with n vertices and no edges.

Lemma 2.2. *For any integer $n\geqslant 1$, we have*

$$\Psi(N_n,\tau)=\sum_{r=0}^{n}\binom{n}{r}\tau^r=(1+\tau)^n. \tag{4}$$

Proof. By definition, we have $\mathrm{cv}(N_n,r)=\binom{n}{r}$ for any integer r with $0\leqslant r\leqslant n$. Thus the result is obtained by (1). □

Lemma 2.3. *For a graph G, if there is a loop at each vertex of G, then*

$$\Psi(G,\tau)=\tau^{v(G)}. \tag{5}$$

Proof. By definition, we have $\mathrm{cv}(G,r)=0$ when $r<v(G)$ and $\mathrm{cv}(G,v(G))=1$. Thus the result is obtained by (1). □

We now give a reduction method for computing $\Psi(G,\tau)$ for general graphs.

For $S\subseteq V(G)$, let $G-S$ denote the graph obtained from G by deleting all vertices in S and all edges incident with any vertices in S. For simplicity, for $x\in V(G)$, we let $G-x$ denote the graph $G-\{x\}$ and let $N_G(x)=\{y\in V(G)\mid y\neq x, xy\in E(G)\}$.

Theorem 2.1. *Let G be a graph and $L=\{x\in V(G)\mid xx\in E(G)\}$. Then*

$$\Psi(G,\tau)=\tau^{|L|}\Psi(G-L,\tau). \tag{6}$$

Proof. Observe that for any $S\subset V(G)$, S is an r-vertex cover of G iff $L\subseteq S$ and $S-L$ is an $(r-|L|)$-vertex cover of $G-L$. Thus

$$\mathrm{cv}(G,r)=\mathrm{cv}(G-L,r-|L|)$$

for $r=1,2,\ldots,v(G)$. Hence the result follows. □

Theorem 2.2. *Let G be a graph with no loops and $v(G)\geqslant 2$. Let $x\in V(G)$ and $d=|N_G(x)|$. Then*

$$\Psi(G,\tau)=\tau\Psi(G-x,\tau)+\tau^{d}\Psi(G-x-N_G(x),\tau). \tag{7}$$

Proof. We first show that

$$\mathrm{cv}(G,r)=\mathrm{cv}(G-x,r-1)+\mathrm{cv}(G-x-N_G(x),r-d). \tag{8}$$

Let S be an r-vertex cover of G. There are two cases: $x\in S$ or $x\notin S$.

We observe that S is an r-vertex cover with $x\in S$ iff $x\in S$ and $S-\{x\}$ is an $(r-1)$-vertex cover of $G-x$. Thus the number of such r-vertex covers S is $\mathrm{cv}(G-x,r-1)$.

If $x\notin S$, then by definition, $N_G(x)\subseteq S$ and $S-N_G(x)$ is an $(r-d)$-vertex cover of $G-x-N_G(x)$. On the other hand, for any $(r-d)$-vertex cover S' of $G-x-N_G(x)$, $S'\cup N_G(x)$ is an r-vertex cover of G. Hence the number of r-vertex covers S with $x\notin S$ is $\mathrm{cv}(G-x-N_G(x),r-d)$. Thus (8) holds. Hence

$$\Psi(G,\tau)=\sum_{r=0}^{v(G)}\mathrm{cv}(G,r)\tau^{r}$$

$$=\sum_{r=0}^{v(G)}(\mathrm{cv}(G-x,r-1)+\mathrm{cv}(G-x-N_G(x),r-d))\tau^{r}$$

$$= \sum_{r=1}^{v(G)} \mathrm{cv}(G-x, r-1)\tau^r + \sum_{r=d}^{v(G)} \mathrm{cv}(G-x-N_G(x), r-d)\tau^r$$

$$= \tau\Psi(G-x,\tau) + \tau^d\Psi(G-x-N_G(x),\tau). \qquad \square$$

Theorems 2.1 and 2.2 together with Lemmas 2.1–2.3 give a reduction method to calculate $\Psi(G,\tau)$. Our next theorem shows that $\Psi(G,\tau)$ is multiplicative on a disjoint union of graphs. In this theorem $G_1 \cup G_2$ denotes the graph G with two disjoint subgraphs G_1 and G_2 such that $V(G)=V(G_1)\cup V(G_2)$ and $E(G)=E(G_1)\cup E(G_2)$.

Theorem 2.3. *Let $G=G_1 \cup G_2$ for two graphs G_1 and G_2. Then $\Psi(G,\tau)=\Psi(G_1,\tau)\Psi(G_2,\tau)$.*

Proof. We proceed by induction on $v(G_1)$. If G_1 or G_2 is null, the result follows from Lemma 2.1. Suppose $v(G_1)\geqslant 1$ and $v(G_2)\geqslant 1$. If G contains a loop, the result follows from Theorem 2.1. Suppose therefore that G has no loop. If $G_1=N_1$ then, by Theorem 2.2, we have

$$\Psi(G,\tau) = \tau\Psi(G_2,\tau) + \Psi(G_2,\tau)$$
$$= (\tau+1)\Psi(G_2,\tau) = \Psi(G_1,\tau)\Psi(G_2,\tau).$$

Otherwise we choose $x\in V(G_1)$ and let $d=|N_{G_1}(x)|$. Then

$$\Psi(G,\tau) = \tau\Psi(G_1-x,\tau)\Psi(G_2,\tau) + \tau^d\Psi(G_1-x-N_{G_1}(x),\tau)\Psi(G_2,\tau)$$
$$= (\tau\Psi(G_1-x,\tau) + \tau^d\Psi(G_1-x-N_{G_1}(x),\tau))\Psi(G_2,\tau)$$
$$= \Psi(G_1,\tau)\Psi(G_2,\tau). \qquad \square$$

In the following, we shall determine $\Psi(G,\tau)$ for some special graphs G.

Theorem 2.4. *Let G be a graph with no loops and $v(G)\geqslant 2$. For $x\in V(G)$,* if *$N_G(x)=V(G)-\{x\}$, then*

$$\Psi(G,\tau) = \tau\Psi(G-x,\tau) + \tau^{v(G)-1}.$$

Proof. It follows from Theorem 2.2 and Lemma 2.1. $\square$

Lemma 2.4. *For the path graph P_n with n vertices, where $n\geqslant 1$, we have*

$$\Psi(P_n,\tau) = \sum_{i=0}^{n} \binom{i+1}{n-i}\tau^i = \sum_{i=\lceil (n-1)/2\rceil}^{n} \binom{i+1}{n-i}\tau^i.$$

Proof. The result holds for $n\leqslant 2$, since $\Psi(P_1,\tau)=1+\tau$ and $\Psi(P_2,\tau)=2\tau+\tau^2$. Suppose that the result holds for $n<k$, where $k\geqslant 3$. Now let $n=k$. By Theorem 2.2 and by

induction on n, we have

$$\begin{aligned}\Psi(P_n,\tau) &= \tau\Psi(P_{n-1},\tau)+\tau\Psi(P_{n-2},\tau)\\ &= \tau\sum_{i=0}^{n-1}\binom{i+1}{n-1-i}\tau^i+\tau\sum_{i=0}^{n-2}\binom{i+1}{n-2-i}\tau^i\\ &= \tau\sum_{i=0}^{n-1}\left(\binom{i+1}{n-1-i}+\binom{i+1}{n-2-i}\right)\tau^i\\ &= \tau\sum_{i=0}^{n-1}\binom{i+2}{n-1-i}\tau^i\\ &= \sum_{i=1}^{n}\binom{i+1}{n-i}\tau^i\\ &= \sum_{i=0}^{n}\binom{i+1}{n-i}\tau^i,\end{aligned}$$

where the last equality holds since $\binom{i+1}{n-i}=0$ when $n\geqslant 2$ and $i=0$. □

Lemma 2.5. *For the cycle graph C_n, where $n\geqslant 3$, we have*

$$\Psi(C_n,\tau)=\sum_{i=1}^{n}\frac{n}{i}\binom{i}{n-i}\tau^i=\sum_{i=\lceil n/2\rceil}^{n}\frac{n}{i}\binom{i}{n-i}\tau^i.$$

Proof. The result holds for $n=3$, since $\Psi(C_3,\tau)=\tau^3+3\tau$. By Theorem 2.2, we have

$$\Psi(C_n,\tau)=\tau\Psi(P_{n-1},\tau)+\tau^2\Psi(P_{n-3},\tau)$$

for $n\geqslant 4$. Then the result follows by using Lemma 2.4. □

Lemma 2.6. *For the wheel graph W_n of n vertices, where $n\geqslant 4$, we have*

$$\Psi(W_n,\tau)=\tau^{n-1}+\sum_{i=\lceil (n+1)/2\rceil}^{n}\frac{n-1}{i-1}\binom{i-1}{n-i}\tau^i.$$

Proof. By Theorem 2.4,

$$\Psi(W_n,\tau)=\tau\Psi(C_{n-1},\tau)+\tau^{n-1}$$

and the result follows from Lemma 2.5. □

Lemma 2.7. *For the complete bipartite graph $K_{p,q}$, where $p\geqslant 1$ and $q\geqslant 1$,*

$$\Psi(K_{p,q},\tau)=\tau^p(1+\tau)^q+\tau^q(1+\tau)^p-\tau^{p+q}.$$

Proof. First consider the case when $p=1$. By Theorem 2.2,

$$\Psi(K_{1,q},\tau)=\tau\Psi(N_q,\tau)+\tau^q=\tau(1+\tau)^q+\tau^q.$$

Hence the lemma holds for $p=1$. By Theorem 2.2 we have

$$\Psi(K_{p,q},\tau)=\tau\Psi(K_{p-1,q},\tau)+\tau^q\Psi(N_{p-1},\tau).$$

Then the result follows by induction and Lemma 2.2. □

We recursively define the balanced tree B_r with a root vertex for $r\geqslant 0$. When $r=0$, B_r is the graph with one vertex, which is the root vertex. When $r\geqslant 1$, let B_r be the tree obtained from two disjoint copies of B_{r-1} by adding a new vertex x_r and two new edges joining x_r to the root vertices of the two copies of B_{r-1}. The root vertex of B_r is x_r. The following result is obtained by Theorem 2.2.

Lemma 2.8. $\Psi(B_0,\tau)=1+\tau$, $\Psi(B_1,\tau)=\tau+3\tau^2+\tau^3$ *and for* $r\geqslant 2$,

$$\Psi(B_r,\tau)=\tau\Psi^2(B_{r-1},\tau)+\tau^2\Psi^4(B_{r-2},\tau). \qquad \square$$

3. Properties of vertex-cover polynomials

In this section, we shall consider only simple graphs. Let $\bar{G}$ denote the complement of a simple graph G, and let $k_r(G)$ be the number of subgraphs in G isomorphic to the complete graph K_r for any non-negative integer r. We always assume that $k_0(G)=1$. By definition, we have $\mathrm{cv}(G,r)=k_{n-r}(\bar{G})$ for all r with $0\leqslant r\leqslant n$, where $n=v(G)$. Thus

Lemma 3.1. *Let G be a simple graph of order n. Then*

$$\Psi(G,\tau)=\sum_{r=0}^{n}k_{n-r}(\bar{G})\tau^r. \qquad \square \tag{9}$$

An interesting problem is to decide whether two given graphs G and H have the same vertex-cover polynomial. By (9), we observe that $\Psi(G,\tau)=\Psi(H,\tau)$ iff $v(G)=v(H)$ and $k_i(\bar{G})=k_i(\bar{H})$ for $i=0,1,\dots,v(G)$. A result is immediately obtained.

Lemma 3.2. *For two graphs G and H, if $\bar{G}$ is K_3-free, then $\Psi(G,\tau)=\Psi(H,\tau)$ iff $v(G)=v(H)$, $e(G)=e(H)$ and $\bar{H}$ is also K_3-free.* □

Another problem is to study whether a given polynomial can be the vertex-cover polynomial of some graph. A necessary condition is obtained from (9).

Lemma 3.3. *For a simple graph G of order n, we have*

$$\tau^n + n\tau^{n-1} + \left(\binom{n}{2} - e(G)\right)\tau^{n-2} \leqslant \Psi(G,\tau) \leqslant (1+\tau)^n \tag{10}$$

for $\tau \geqslant 0$. Moreover, $\Psi(G,\tau) = \tau^n + n\tau^{n-1} + (\binom{n}{2} - e(G))\tau^{n-2}$ iff $\bar{G}$ is K_3-free, and $\Psi(G,\tau) = (1+\tau)^n$ iff G is empty. □

From Lemma 3.3, we observe that the polynomial $\Psi(G,\tau)$ has no positive real roots. It is also an interesting problem to study the roots of $\Psi(G,\tau)$.

4. Application in biological systematics

We have defined $F(G,m)$ in the first section. The graph function $F(G,m)$ is used in biological systematics to determine the order, size and dimension of a Buneman graph. To derive the relation between $F(G,m)$ and $\mathrm{cv}(G,r)$, we need the following two results.

Lemma 4.1. *Let G be any graph and m any nonnegative integer. For any $f \in \mathscr{F}(G,m)$, we have*

$$V' = \{f(1), f(2), \ldots, f(m)\} \in \mathscr{CV}(G,r),$$

where $r = |V'|$.

Proof. For every $uv \in E(G)$, we have $f^{-1}(u) \cup f^{-1}(v) \neq \emptyset$ and thus $V' \cap \{u,v\} \neq \emptyset$. By the definition of $\mathscr{CV}(G,r)$, $V' \in \mathscr{CV}(G,r)$. □

In the following, $S(m,r)$ denotes a Stirling number of the second kind.

Lemma 4.2. *Let G be any graph and m any nonnegative integer. For any $V' \in \mathscr{CV}(G,r)$, where $r \geqslant 0$, there are exactly $r!S(m,r)$ mappings $f \in \mathscr{F}(G,m)$ such that*

$$\{f(1), f(2), \ldots, f(m)\} = V'.$$

Proof. There are exactly $r!S(m,r)$ surjections

$$f : \{1,2,\ldots,m\} \to V'.$$

Since V' is a vertex cover of G, we have $f \in \mathscr{F}(G,m)$ for every such f. □

Theorem 4.1. *For any graph G and $m \geqslant 0$, we have*

$$F(G,m) = \sum_{r=0}^{v(G)} \mathrm{cv}(G,r) r! S(m,r). \tag{11}$$

Proof. The result follows from Lemmas 4.1 and 4.2. □

By this result, we can get another relation between $F(G,m)$ and $\Psi(G,\tau)$.

Theorem 4.2. *For any graph G, we have*

$$\Psi(G, e^{\tau} - 1) = \sum_{m=0}^{\infty} \frac{F(G,m)}{m!} \tau^m. \tag{12}$$

In fact, Theorem 4.2 is a special case of the following result.

Theorem 4.3. *Let $P(x) = \sum_{r=0}^{k} a_r x^r$ be any polynomial. If $b_n = \sum_{r=0}^{k} a_r r! S(n,r)$ for all $n \geqslant 0$, then*

$$P(e^y - 1) = \sum_{n=0}^{\infty} \frac{b_n y^n}{n!}. \tag{13}$$

Proof. Observe that

$$\sum_{n=0}^{\infty} \frac{b_n y^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{k} a_r r! S(n,r) \frac{y^n}{n!} = \sum_{r=0}^{k} a_r \sum_{n=0}^{\infty} \frac{r! S(n,r) y^n}{n!}.$$

By (3.6.2) in [5],

$$\sum_{n=0}^{\infty} \frac{r! S(n,r) y^n}{n!} = (e^y - 1)^r. \tag{14}$$

The result thus follows. □

5. For further reading

The following references may also be of interest to the reader: [1,3].

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