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Proof of a Chromatic Polynomial Conjecture

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Let $P(G, \lambda)$ denote the chromatic polynomial of a graph G. It is proved in this paper that for every connected graph G of order n and real number $\lambda \geqslant n$, $(\lambda-2)^{n-1} P(G, \lambda) - \lambda(\lambda-1)^{n-2} P(G, \lambda-1) \geqslant 0$. By this result, the following conjecture proposed by Bartels and Welsh is proved: $P(G, n)(P(G, n-1))^{-1} > e$ for every graph G of order n.

1. INTRODUCTION

In this paper, all graphs considered are simple graphs. We always suppose that G is a graph. Let $V(G)$, $E(G)$, $v(G)$, and $e(G)$ be the vertex set, edge set, order and number of edges of G. For a positive integer λ, a *λ-colouring* of G is a mapping $f\colon V(G) \to \{1, ..., \lambda\}$ such that $f(x) \neq f(y)$ whenever x and y are adjacent in G. Let $P(G, \lambda)$ denote the number of λ-colourings in G. It is well known that $P(G, \lambda)$ is a polynomial in λ, called the *chromatic polynomial* of G. In this paper, $P(G, \lambda)$ is considered to be a polynomial in the real number λ.

We are concerned with the following conjecture about $P(G, \lambda)$,

$$P(G, n)(P(G, n-1))^{-1} \geqslant e, \tag{1}$$

where n is the order of G and $e(=2.7182818...)$ is the base of natural logarithms. This conjecture was proposed by Bartels and Welsh [1]. Paul Seymour [4] obtained the following result, which is very close to the conjecture,

$$P(G, n)(P(G, n-1))^{-1} \geqslant \tfrac{685}{252} (=2.7182539...). \tag{2}$$

We first prove that for every connected graph G of order n and real number $\lambda \geqslant n$,

$$(\lambda-2)^{n-1} P(G, \lambda) - \lambda(\lambda-1)^{n-2} P(G, \lambda-1) \geqslant 0. \tag{3}$$

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We then deduce from (3) that

$$(\lambda-1)^n P(G,\lambda)-\lambda^n P(G,\lambda-1)\geqslant 0, \tag{4}$$

for every graph G of order n and $\lambda\geqslant n$. In particular, taking $\lambda=n$, we obtain $P(G,n)(P(G,n-1))^{-1}\geqslant(n/(n-1))^n>e$.

2. APPROACH

For $x\in V(G)$, let $d_G(x)$, or simply $d(x)$, denote the degree of x in G, and let $G-x$ denote the graph obtained from G by deleting x and all edges incident with x. Define

$$\psi(G,\lambda)=(\lambda-2)^{v(G)-1}P(G,\lambda)-\lambda(\lambda-1)^{v(G)-2}P(G,\lambda-1), \tag{5}$$

$$\phi(G,x,\lambda)=(\lambda-1)(2\lambda-d(x)-2)\,P(G-x,\lambda)-(2\lambda-3)\,P(G,\lambda), \tag{6}$$

where $x\in V(G)$.

We shall prove inequality (3) by induction. This proof also shows that $\phi(G,x,\lambda)\geqslant 0$ for $\lambda\geqslant v(G)\geqslant 2$ if both G and $G-x$ are connected.

THEOREM 2.1. (i) *$\psi(G,\lambda)\geqslant 0$ if G is connected and $\lambda\geqslant v(G)$;*

(ii) *$\phi(G,x,\lambda)\geqslant 0$ if both G and $G-x$ are connected and $\lambda\geqslant v(G)\geqslant 2$.*

The proof of Theorem 2.1 is divided into three parts, which correspond to the following three results.

LEMMA 2.1. *Theorem* 2.1 *holds for connected graphs with order at most* 3.

Proof. Let G be a connected graph. We just verify the result for $v(G)=3$. When $v(G)=3$, either $G=K_3$ or $G=P_3$, where P_3 is the path with 3 vertices. By (5), we have

$$\psi(G,\lambda)=\begin{cases}\lambda(\lambda-1)(\lambda-2), & \text{if } G=K_3,\\ 0, & \text{if } G=P_3.\end{cases}$$

If $G=K_3$ and $x\in V(G)$, then by (6),

$$\phi(G,x,\lambda)=(2\lambda-4)\,\lambda(\lambda-1)^2-(2\lambda-3)\,\lambda(\lambda-1)(\lambda-2)=\lambda(\lambda-1)(\lambda-2).$$

For $G=P_3$, $G-x$ is connected iff x is an end-vertex of G. If $G=P_3$ and x is an end-vertex of P_3, then by (6),

$$\phi(G,x,\lambda)=(2\lambda-3)\,\lambda(\lambda-1)^2-(2\lambda-3)\,\lambda(\lambda-1)^2=0. \quad \blacksquare$$

LEMMA 2.2. *Let n be any integer with $n \geqslant 4$. Suppose that Theorem* 2.1(i) *and* (ii) *holds for all connected graphs with order at most $n-1$. Then Theorem* 2.1(ii) *holds for connected graphs with order n.*

LEMMA 2.3. *Let n be any integer with $n \geqslant 4$. Suppose that Theorem* 2.1(i) *holds for all connected graphs with order at most $n-1$ and that Theorem* 2.1(ii) *holds for all connected graphs with order at most n. Then Theorem* 2.1(i) *holds for all connected graphs with order n.*

The proofs of Lemmas 2.2 and 2.3 will be given in Sections 4 and 5, respectively. After establishing Theorem 2.1, we can prove inequality (4).

THEOREM 2.2. *For every graph G of order n, where $n \geqslant 1$, when $\lambda \geqslant n$, we have*

$$(\lambda-1)^n P(G, \lambda) - \lambda^n P(G, \lambda-1) \geqslant 0. \tag{7}$$

COROLLARY. *For any graph G of order n, where $n \geqslant 1$, we have*

$$\frac{P(G, n)}{P(G, n-1)} \geqslant \frac{n^n}{(n-1)^n} > e. \tag{8}$$

3. RECURSIVE EXPRESSIONS FOR $P(G, \lambda)$

In this section, the chromatic polynomial of a graph is expressed in terms of chromatic polynomials of graphs with lower orders.

For nonadjacent vertices x and y in G, let $G \cdot xy$ be the graph obtained from G by identifying x and y and replacing the double edges by single ones, and let $G + xy$ be the graph obtained from G by adding the edge (x, y) if (x, y) is not an edge in G, and $G + xy = G$ otherwise.

LEMMA 3.1 [2]. *For $x, y \in V(G)$, if $(x, y) \notin E(G)$, then*

$$P(G, \lambda) = P(G + xy, \lambda) + P(G \cdot xy, \lambda). \tag{9}$$

For $x \in V(G)$, let $N_G(x)$, or simply $N(x)$, denote the set of vertices in G adjacent to x.

LEMMA 3.2 [3]. *Let $v(G) \geqslant 2$ and $x \in V(G)$. If $d(x) = 0$ or $N(x)$ is a clique in G, then*

$$P(G, \lambda) = (\lambda - d(x))\, P(G - x, \lambda). \tag{10}$$

For distinct vertices $x_1, x_2, ..., x_i$ in G, define the graph $G_{x_1, x_2, ..., x_i}$,

$$G_{x_1, x_2, ..., x_i} = \begin{cases} G, & \text{if} \quad i = 1, \\ G + x_1 x_i + x_2 x_i + \cdots + x_{i-1} x_i, & \text{if} \quad i \geqslant 2. \end{cases} \tag{11}$$

Applying Lemma 3.1, we have

LEMMA 3.3. *For a sequence of distinct vertices $x_1, x_2, ..., x_i$ in G, we have*

$$P(G, \lambda) = P(G_{x_1, ..., x_i}, \lambda) + \sum_{\substack{1 \leqslant j < i \\ (x_j, x_i) \notin E(G)}} P(G_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i, \lambda). \tag{12}$$

LEMMA 3.4. *Let $x \in V(G)$ with $d(x) = d \geqslant 1$, $N(x) = \{x_1, x_2, ..., x_d\}$ and $G^* = G - x$. Then*

$$P(G, \lambda) = (\lambda - 1)\, P(G^*, \lambda) - \sum_{i=2}^{d} P(G^*_{x_1, ..., x_i}, \lambda). \tag{13}$$

Proof. When $d = 1$, it follows from Lemma 3.2. Now let $d \geqslant 2$. By Lemma 3.1,

$$P(G, \lambda) = P(G - x_d x, \lambda) - P(G^*_{x_1, ..., x_d}, \lambda),$$

where $G - x_d x$ is the graph obtained from G by deleting the edge (x_d, x). The result is then obtained by induction. ∎

LEMMA 3.5. *Let $x \in V(G)$ with $d(x) = d \geqslant 1$, $N(x) = \{x_1, x_2, ..., x_d\}$ and $G^* = G - x$. Then*

$$P(G, \lambda) = (\lambda - d)\, P(G^*, \lambda) + \sum_{i=2}^{d} \sum_{\substack{1 \leqslant j < i \\ (x_j, x_i) \notin E(G)}} P(G^*_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i, \lambda). \tag{14}$$

Proof. By Lemma 3.4, we have

$$P(G, \lambda) = (\lambda - d)\, P(G^*, \lambda) + \sum_{i=2}^{d} (P(G^*, \lambda) - P(G^*_{x_1, ..., x_i}, \lambda)).$$

The result then follows from Lemma 3.3. ∎

Remark. Lemmas 3.4 and 3.5 give methods to find chromatic polynomials of graphs recursively.

4. PROOF OF LEMMA 2.2

LEMMA 4.1 [3]. *Let* $v(G) \geqslant 2$ *and* $x \in V(G)$. *If* $N(x) = V(G) - \{x\}$, *then*

$$P(G, \lambda) = \lambda P(G - x, \lambda - 1). \tag{15}$$

Before proving Lemma 2.2, we define two functions and find their relations with $\psi(G, \lambda)$ and $\phi(G, x, \lambda)$.

For any graph G and real number λ, define

$$\psi_1(G, \lambda) = (\lambda - 1)(2\lambda - v(G) - 2)\, P(G, \lambda) - \lambda(2\lambda - 3)\, P(G, \lambda - 1), \tag{16}$$

$$\psi_2(G, \lambda) = ((2\lambda - v(G) - 3)(\lambda - 1) + 1)\, P(G, \lambda) - \lambda(2\lambda - 3)\, P(G, \lambda - 1). \tag{17}$$

LEMMA 4.2. *Let* $v(G) \geqslant 2$ *and* $x \in V(G)$. *If* $N(x) = V(G) - \{x\}$, *then*

$$\phi(G, x, \lambda) = \psi_1(G - x, \lambda). \tag{18}$$

Proof. By Lemma 4.1, $P(G, \lambda) = \lambda P(G - x, \lambda - 1)$. Let $v(G) = n$. We have $d_G(x) = n - 1$, $v(G - x) = n - 1$ and

$$\begin{aligned}\phi(G, x, \lambda) &= (\lambda - 1)(2\lambda - n - 1)\, P(G - x, \lambda) - (2\lambda - 3)\, \lambda P(G - x, \lambda - 1)\\ &= \psi_1(G - x, \lambda). \quad \blacksquare\end{aligned}$$

LEMMA 4.3. *Let* $v(G) \geqslant 2$ *and* $x \in V(G)$. *If* $N(x) = N(y) = V(G) - \{x, y\}$ *for some* $y \in V(G)$, *then*

$$\phi(G, x, \lambda) = \psi_2(G - x, \lambda). \tag{19}$$

Proof. From Lemmas 3.1 and 4.1,

$$P(G, \lambda) = P(G + xy, \lambda) + P(G \cdot xy, \lambda) = \lambda P(G - x, \lambda - 1) + P(G - x, \lambda).$$

The results then follows from (6) and (17). ▮

LEMMA 4.4. $P(G, \lambda) \geqslant 0$ *for every graph* G *and real number* $\lambda \geqslant v(G) - 1$.

Proof. It is easy to verify the result for $v(G) \leqslant 2$. When $v(G) \geqslant 3$, it follows from Lemma 3.5 by induction. ▮

LEMMA 4.5. *Let* $v(G) \geqslant 2$ *and* $\lambda \geqslant v(G) + 1$. *Then* $\psi(G, \lambda) \geqslant 0$ *implies that* $\psi_1(G, \lambda) \geqslant 0$ *and* $\psi_2(G, \lambda) \geqslant 0$.

Proof. Let $v(G)=n\geqslant 2$ and $\lambda\geqslant n+1$. By Lemma 4.4, $P(G,\lambda)\geqslant 0$. Then

$$\psi_1(G,\lambda)-\psi_2(G,\lambda)=(\lambda-2)\,P(G,\lambda)\geqslant 0$$

and

$$\begin{aligned}
(\lambda-1)^{n-2}\,\psi_2(G,\lambda)-(2\lambda-3)\,\psi(G,\lambda)&\\
&=((\lambda-1)^{n-2}\,(2\lambda^2-(n+5)\,\lambda+n+4)-(\lambda-2)^{n-1}\,(2\lambda-3))\,P(G,\lambda)\\
&=P(G,\lambda)\sum_{i=3}^{n}((\lambda-1)^{i-2}\,(\lambda-2)^{n-i}\,(2\lambda^2-(i+5)\,\lambda+i+4)\\
&\quad-(\lambda-1)^{i-3}\,(\lambda-2)^{n-i+1}\,(2\lambda^2-(i+4)\,\lambda+i+3))\\
&=P(G,\lambda)\sum_{i=3}^{n}(\lambda-1)^{i-3}\,(\lambda-2)^{n-i}\,((\lambda-1)(\lambda-i-1)+1)\\
&\geqslant 0.
\end{aligned}$$

The results follows immediately. ∎

Proof of Lemma 2.2. We shall first show that Theorem 2.1(ii) holds for any complete graph K_r, where $r\geqslant 2$. Observe that for any $x\in V(K_r)$, $d(x)=r-1$ and by Lemma 3.2, $P(K_r,\lambda)=(\lambda-r+1)\,P(K_r-x,\lambda)$. When $\lambda\geqslant r$,

$$\begin{aligned}
\phi(K_r,x,\lambda)&=(\lambda-1)(2\lambda-r-1)\,P(K_r-x,\lambda)\\
&\quad-(2\lambda-3)(\lambda-r+1)\,P(K_r-x,\lambda)\\
&=(r-2)(\lambda-2)\,P(K_{r-1},\lambda)\\
&\geqslant 0.
\end{aligned}$$

Let q be any integer with $n-1\leqslant q<\binom{n}{2}$. Suppose that for every connected graph H of order n and every $x\in V(H)$ such that $H-x$ is connected, if $e(H)>q$, then $\phi(H,x,\lambda)\geqslant 0$ for $\lambda\geqslant v(H)$.

Now let Q be any connected graph with $v(Q)=n$ and $e(Q)=q$, and $w\in V(Q)$ such that $Q-w$ is connected. We shall show that $\phi(Q,w,\lambda)\geqslant 0$ for $\lambda\geqslant n$. Let $d_Q(w)=d$. It is clear that $d\geqslant 1$.

Case 1. $V(Q)=\{w\}\cup N_Q(w)$.

By Lemma 4.2, $\phi(Q,w,\lambda)=\psi_1(Q-w,\lambda)$. Since $Q-w$ is connected and $v(Q-w)=n-1$, we have $\psi(Q-w,\lambda)\geqslant 0$ for $\lambda\geqslant n-1$ by the condition in Lemma 2.2. By Lemma 4.5, we have $\psi_1(Q-w,\lambda)\geqslant 0$ for $\lambda\geqslant n$. Thus $\phi(Q,w,\lambda)\geqslant 0$ for $\lambda\geqslant n$.

Case 2. $N_Q(u)\neq V(Q)-\{w,u\}$ for some $u\in V(Q)-\{w\}-N_Q(w)$.

There exists a vertex $v\in V(Q)-\{w,u\}$ such that $(u,v)\notin E(Q)$. Let $Q'=Q+uv$ and $Q''=Q\cdot uv$. Since both Q and $Q-w$ are connected, the four graphs Q', $Q'-w$, Q'' and $Q''-w$ are all connected. It is clear that $d_{Q'}(w)=d_{Q''}(w)=d$. Since $v(Q')=n$, $e(Q')=q+1$ and $v(Q'')=n-1$, by induction and by the condition in Lemma 2.2, respectively, we have

$$\begin{cases}\phi(Q', w, \lambda)=(\lambda-1)(2\lambda-d-2)\,P(Q'-w, \lambda)-(2\lambda-3)\,P(Q', \lambda)\geqslant 0,\\ \phi(Q'', w, \lambda)=(\lambda-1)(2\lambda-d-2)\,P(Q''-w, \lambda)-(2\lambda-3)\,P(Q'', \lambda)\geqslant 0,\end{cases}$$

where $\lambda\geqslant n$. By Lemma 3.1, we have

$$\begin{cases}P(Q, \lambda)=P(Q', \lambda)+P(Q'', \lambda),\\ P(Q-w, \lambda)=P((Q-w)+uv, \lambda)+P((Q-w)\cdot uv, \lambda).\end{cases}$$

Since $(Q-w)+uv=Q'-w$ and $(Q-w)\cdot uv=Q''-w$, we have

$$\phi(Q, w, \lambda)=\phi(Q', w, \lambda)+\phi(Q'', w, \lambda).$$

Hence $\phi(G, w, \lambda)\geqslant 0$ when $\lambda\geqslant n$.

Case 3. $N_Q(u)=V(Q)-\{w, u\}$ for all $u\in V(Q)-\{w\}-N_Q(w)$. Observe that $d\leqslant n-2$. We now study two subcases.

Case 3.1. $d=n-2$. Let u be the unique vertex in $V(Q)-\{w\}-N_Q(w)$. Since $N_Q(u)=N_Q(w)=V(Q)-\{w, u\}$, we have $\phi(Q, w, \lambda)=\psi_2(Q-w, \lambda)$ by Lemma 4.3. By the same argument as Case 1, we have $\phi(Q, w, \lambda)\geqslant 0$ for $\lambda\geqslant n$.

Case 3.2. $d\leqslant n-3$. Let $u, v\in V(Q)-\{w\}-N_Q(w)$ and $Q^*=Q-u$. By Lemmas 3.1 and 4.1,

$$P(Q, \lambda)=P(Q+uw, \lambda)+P(Q\cdot uw, \lambda)=\lambda P(Q^*, \lambda-1)+\lambda P(Q^*-w, \lambda-1),$$

and

$$P(Q-w, \lambda)=\lambda P(Q^*-w, \lambda-1).$$

It is clear that $d_{Q^*}(w)=d$. Hence

$$\begin{aligned}\phi(Q, w, \lambda)&=(\lambda-1)(2\lambda-d-2)\,P(Q-w, \lambda)-(2\lambda-3)\,P(Q, \lambda)\\ &=(\lambda-1)(2\lambda-d-2)\,\lambda P(Q^*-w, \lambda-1)\\ &\quad-\lambda(2\lambda-3)(P(Q^*, \lambda-1)+P(Q^*-w, \lambda-1))\\ &=\lambda(2\lambda^2-(d+6)\,\lambda+d+5)\,P(Q^*-w, \lambda-1)\\ &\quad-\lambda(2\lambda-3)\,P(Q^*, \lambda-1)\\ &=\frac{\lambda(2\lambda-3)}{2\lambda-5}\,\phi(Q^*, w, \lambda-1)+\frac{(d-1)\,\lambda}{2\lambda-5}\,P(Q^*-w, \lambda-1).\end{aligned}$$

Since $N_Q(v)=V(Q)-\{w, v\}$ and $d\geqslant 1$, both Q^* and Q^*-w are connected. Since $v(Q^*)=n-1$, by the condition in Lemma 2.2, $\phi(Q^*, w, \lambda-1)\geqslant 0$ for $\lambda\geqslant n-1$. By Lemma 4.4, $P(Q^*-w, \lambda-1)\geqslant 0$ for $\lambda\geqslant n-2$. Thus $\phi(Q, w, \lambda)\geqslant 0$ when $\lambda\geqslant n$.

Hence Theorem 2.1(ii) holds for the graph Q. This completes the proof. ▮

5. PROOF OF LEMMA 2.3

LEMMA 5.1. *Let G be a graph with $v(G)=k$ and $x\in V(G)$, where $k\geqslant 4$ and $d(x)=d\geqslant 1$. Let $N_G(x)=\{x_1, x_2, ..., x_d\}$ and $G^*=G-x$. We have*

$$\psi(G,\lambda)=(\lambda-1)(\lambda-d-1)\,\psi(G^*,\lambda)+(\lambda-2)^{k-3}\,\phi(G,x,\lambda)$$
$$+(\lambda-1)^2\sum_{i=2}^{d}\sum_{\substack{1\leqslant j<i\\ (x_j,x_i)\notin E(G)}}\psi(G^*_{x_1,...,x_{j-1},x_i}\cdot x_jx_i,\lambda). \tag{20}$$

Proof. By the definition of $\psi(G,\lambda)$ and $\phi(G,x,\lambda)$, we have

$$\begin{aligned}\psi(G,\lambda)&-(\lambda-2)^{k-3}\,\phi(G,x,\lambda)\\ &=(\lambda-1)^2\,(\lambda-2)^{k-3}\,P(G,\lambda)-\lambda(\lambda-1)^{k-2}\,P(G,\lambda-1)\\ &-(\lambda-2)^{k-3}\,(\lambda-1)(2\lambda-d-2)\,P(G^*,\lambda).\end{aligned}$$

Then by Lemma 3.5,

$$\begin{aligned}\psi(G,\lambda)&-(\lambda-2)^{k-3}\,\phi(G,x,\lambda)\\ &=(\lambda-1)^2\,(\lambda-2)^{k-3}\,(\lambda-d)\,P(G^*,\lambda)\\ &+(\lambda-1)^2\,(\lambda-2)^{k-3}\sum_{i=2}^{d}\sum_{\substack{1\leqslant j<i\\ (x_j,x_i)\notin E(G)}}P(G^*_{x_1,...,x_{j-1},x_i}\cdot x_jx_i,\lambda)\\ &-\lambda(\lambda-1)^{k-2}\,(\lambda-d-1)\,P(G^*,\lambda-1)\\ &-\lambda(\lambda-1)^{k-2}\sum_{i=2}^{d}\sum_{\substack{1\leqslant j<i\\ (x_j,x_i)\notin E(G)}}P(G^*_{x_1,...,x_{j-1},x_i}\cdot x_jx_i,\lambda-1)\\ &-(\lambda-2)^{k-3}\,(\lambda-1)(2\lambda-d-2)\,P(G^*,\lambda)\end{aligned}$$

$$= (\lambda-1)(\lambda-d-1)((\lambda-2)^{k-2} P(G^*, \lambda) - \lambda(\lambda-1)^{k-3} P(G^*, \lambda-1))$$

$$+ (\lambda-1)^2 \sum_{i=2}^{d} \sum_{\substack{1 \leqslant j < i \\ (x_j, x_i) \notin E(G)}} ((\lambda-2)^{k-3} P(G^*_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i, \lambda)$$

$$- \lambda(\lambda-1)^{k-4} P(G^*_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i, \lambda-1))$$

$$= (\lambda-1)(\lambda-d-1)\, \psi(G^*, \lambda) + (\lambda-1)^2$$

$$\times \sum_{i=2}^{d} \sum_{\substack{1 \leqslant j < i \\ (x_j, x_i) \notin E(G)}} \psi(G^*_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i, \lambda). \quad \blacksquare$$

Proof of Lemma 2.3. Now let Q be any connected graph with $v(Q) = n$. We shall show that $\psi(Q, \lambda) \geqslant 0$ for $\lambda \geqslant n$. Let x be an end-vertex of some spanning tree of Q. Then $Q - x$ is connected. By the condition in Lemma 2.3, when $\lambda \geqslant n$,

$$\phi(Q, x, \lambda) \geqslant 0.$$

Let $Q^* = Q - x$, $d_G(x) = d$ and $N_G(x) = \{x_1, x_2, ..., x_d\}$. Since Q^* is connected and $v(Q^*) = n-1$, by the condition in Lemma 2.3, when $\lambda \geqslant n-1$,

$$\psi(Q^*, \lambda) \geqslant 0.$$

Observe that the graph $Q^*_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i$, where $1 \leqslant j < i \leqslant d$, is connected with order $n-2$. By the condition in Lemma 2.3, when $\lambda \geqslant n-2$,

$$\psi(Q^*_{x_1, ..., x_{j-1}, x_i} \cdot x_j x_i, \lambda) \geqslant 0.$$

Therefore, by Lemma 5.1, $\psi(Q, \lambda) \geqslant 0$ when $\lambda \geqslant n$. This completes the proof. $\blacksquare$

6. PROOF OF THEOREM 2.2

Theorem 2.1 follows directly from Lemmas 2.1, 2.2 and 2.3. We can now prove Theorem 2.2.

Proof of Theorem 2.2. Let G be any graph with order n, where $n \geqslant 1$. Let H be the graph obtained from G by adding a new vertex and adding edges joining this vertex to all vertices in G. Then by Lemma 4.1, we have

$$P(H, \lambda) = \lambda P(G, \lambda - 1). \tag{21}$$

It is clear that H is connected with order $n+1$. By Theorem 2.1(i), we have $\psi(H, \lambda) \geqslant 0$, i.e.,

$$(\lambda-2)^n P(H, \lambda) - \lambda(\lambda-1)^{n-1} P(H, \lambda-1) \geqslant 0, \tag{22}$$

for $\lambda \geqslant n+1$. By (21), we have

$$(\lambda-2)^n \lambda P(G, \lambda-1) - \lambda(\lambda-1)^n P(G, \lambda-2) \geqslant 0, \tag{23}$$

for $\lambda \geqslant n+1$. Replacing $\lambda-1$ by λ, we have

$$(\lambda-1)^n P(G, \lambda) - \lambda^n P(G, \lambda-1) \geqslant 0, \tag{24}$$

for $\lambda \geqslant n$. This proves Theorem 2.2. ▮

Before the end of this paper, we have the following remarks.

Remarks. (a) Theorem 2.2 can be proved directly by the method used in the proof of Theorem 2.1.

(b) It is obvious that $(\lambda-1)^n/\lambda^n \geqslant (\lambda-2)^{n-1}/(\lambda(\lambda-1)^{n-2})$ for $n \geqslant 1$ and $\lambda \geqslant 2$. Thus the result in Theorem 2.1(i) is stronger than that in Theorem 2.2.

(c) Observe that $\psi(C_4, 3) = \phi(C_4, x, 3) = -6$, where x is any vertex in C_4. Thus the condition "$\lambda \geqslant v(G)$" in Theorem 2.1 can't be replaced by "$\lambda \geqslant v(G) - 1$."

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