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Note

Non-chordal graphs having integral-root chromatic polynomials II

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Abstract

It is known that the chromatic polynomial of any chordal graph has only integer roots. However, there also exist non-chordal graphs whose chromatic polynomials have only integer roots. Dmitriev asked in 1980 if for any integer $p \geqslant 4$, there exists a graph with chordless cycles of length p whose chromatic polynomial has only integer roots. This question has been given positive answers by Dong and Koh for $p = 4$ and $p = 5$. In this paper, we construct a family of graphs in which all chordless cycles are of length p for any integer $p \geqslant 4$. It is shown that the chromatic polynomial of such a graph has only integer roots iff a polynomial of degree $p - 1$ has only integer roots. By this result, this paper extends Dong and Koh's result for $p = 5$ and answer the question affirmatively for $p = 6$ and 7. © 2002 Elsevier Science B.V. All rights reserved.

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In this paper, we consider only simple graphs. Let G be a graph. A cycle (say $v_1 v_2 \dots v_r v_1$) in G is called a *chordless cycle* (or *pure cycle*) of G if $r \geqslant 4$ and no pair of non-consecutive vertices on the cycle forms an edge of G. If G contains no chordless cycles, then G is called a *chordal graph*.

Let $\mathscr{C}$ (resp., $\bar{\mathscr{C}}$) denote the family of chordal (resp., non-chordal) graphs. For any $G \in \mathscr{C}$, there exist non-negative integers $d_1, d_2, \dots, d_n$ such that

$$P(G, \lambda) = \prod_{i=1}^{n} (\lambda - d_i), \tag{1}$$

where n is the order of G (see [8]). Thus all roots of $P(G, \lambda)$ are non-negative integers if $G \in \mathscr{C}$.

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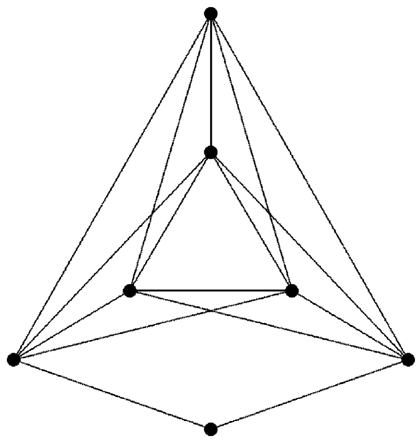

Fig. 1.

A polynomial is called an *integral-root* polynomial if all its roots are integers. Recently, Voloshin and Zhou [10] published a paper which discussed the integral-root chromatic polynomials for some hypergraphs. In this paper, we deal with integral-root chromatic polynomials for graphs. Let $\mathscr{I}$ denote the family of graphs G such that $P(G,\lambda)$ is an integral-root polynomial. (Since the chromatic polynomial of any graph has no negative real roots (see [8]), $P(G,\lambda)$ contains only non-negative integer roots iff $G\in\mathscr{I}$.) Thus $\mathscr{C}\subseteq\mathscr{I}$. It had been conjectured by Braun et al. [1] and Vaderlind [9] that $\mathscr{C}=\mathscr{I}$. This was disproved by Read [6], who discovered the graph in Fig. 1 whose chromatic polynomial is

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3)^3(\lambda-4). \tag{2}$$

Therefore $\mathscr{C}\subset\mathscr{I}$.

It is known that the graph in Fig. 1 is the graph in $\mathscr{I}\cap\bar{\mathscr{C}}$ with least order (see [2]). It means that any graph with order less than 7 belongs to $\mathscr{C}$ iff it belongs to $\mathscr{I}$.

Dmitriev [3] constructed a family of graphs in $\mathscr{I}\cap\bar{\mathscr{C}}$. All chordless cycles in the graphs of this family are of order 4. He asked the following:

Problem 1. *For any integer $p\geqslant 5$, does there exist a graph $G\in\mathscr{I}\cap\bar{\mathscr{C}}$ such that G has a chordless cycle of order p?*

We now construct a family of graphs. For any positive integers k_i, $i=1,2,\ldots,n$, where $n\geqslant 2$, let $H_{k_1,k_2,\ldots,k_n}$ denote the graph obtained from the disjoint union of n complete graphs $K_{k_1},K_{k_2},\ldots,K_{k_n}$ and a vertex w by adding edges joining each vertex in K_{k_i} to each vertex in $K_{k_{i+1}}$ for $i=1,2,\ldots,n-1$, and edges joining w to each vertex in K_{k_1} and K_{k_n}, as shown in Fig. 2. Clearly, when $n\geqslant 3$, $H_{k_1,k_2,\ldots,k_n}\in\bar{\mathscr{C}}$ and $H_{k_1,k_2,\ldots,k_n}$ contains chordless cycles of order $n+1$.

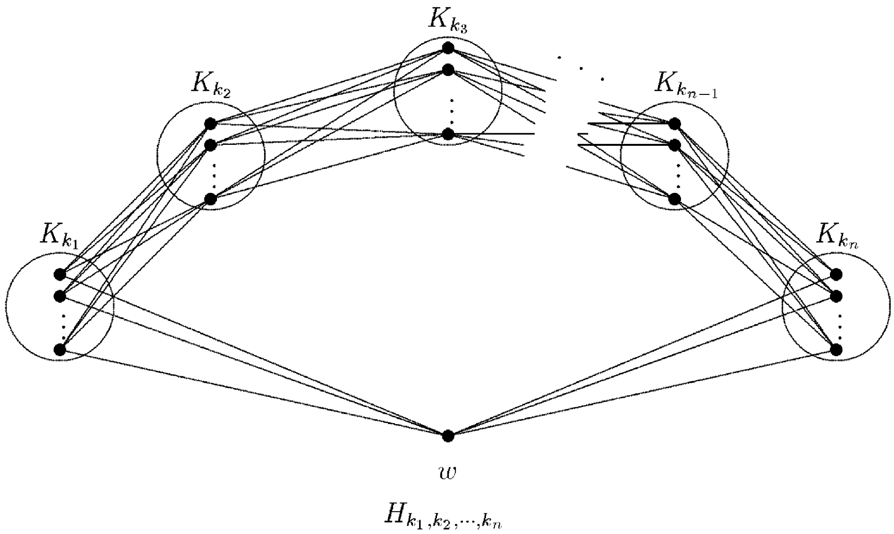

Fig. 2.

Problem 2. *For $n \geqslant 3$, are there positive integers $k_1, k_2, \ldots, k_n$ such that*

$$H_{k_1,k_2,\ldots,k_n} \in \mathscr{I}? \tag{3}$$

Read [7] obtained a formula for the chromatic polynomial of a graph he called a 'ring of cliques', which is like the graph in Fig. 2 but with the single vertex w replaced by another complete graph. Thus, the following result for the chromatic polynomial of $H_{k_1,k_2,\ldots,k_n}$ is a special case of Read's result. But we think it is worth to give a short proof of this result here. For a real number λ and a positive integer r, denote

$$(\lambda)_r = \prod_{i=0}^{r-1} (\lambda - i). \tag{4}$$

Lemma 1. *For any positive integers $n, k_1, k_2, \ldots, k_n$, where $n \geqslant 2$,*

$$P(H_{k_1,k_2,\ldots,k_n}, \lambda) = \frac{\prod_{i=1}^{n-1} (\lambda)_{k_i+k_{i+1}}}{\lambda \prod_{i=2}^{n-1} (\lambda)_{k_i+1}} \left(\prod_{i=1}^{n} (\lambda - k_i) - (-1)^n \prod_{i=1}^{n} k_i \right). \tag{5}$$

Proof. It is easy to verify the result for $n = 2, 3$. Now let $n \geqslant 4$. Consider the graph H' obtained from $H_{k_1,k_2,\ldots,k_n}$ by adding edges joining w to all vertices in the clique $K_{k_{n-1}}$ (see Fig. 2). The chromatic polynomial of H' is

$$P(H', \lambda) = \frac{(\lambda)_{k_{n-1}+k_n+1}}{(\lambda)_{k_{n-1}+1}} P(H_{k_1,k_2,\ldots,k_{n-1}}, \lambda). \tag{6}$$

Thus, by repeatedly using the Fundamental Reduction Theorem of chromatic polynomials (see [5] and [8]), we have

$$P(H_{k_1,k_2,\dots,k_n},\lambda)=\frac{(\lambda)_{k_{n-1}+k_n+1}}{(\lambda)_{k_{n-1}+1}}P(H_{k_1,k_2,\dots,k_{n-1}},\lambda)+\frac{k_{n-1}(\lambda)_{k_{n-1}+k_n}(\lambda)_{k_{n-2}+k_{n-1}}}{(\lambda)_{k_{n-1}}(\lambda)_{k_{n-2}+1}}P(H_{k_1,k_2,\dots,k_{n-2}},\lambda). \tag{7}$$

The result follows by induction from the above recursive expression. □

For positive integers $n,k_1,k_2,\dots,k_n$ and real number x, define

$$f(k_1,k_2,\dots,k_n,x)=\prod_{i=1}^{n}(x-k_i)-(-1)^n\prod_{i=1}^{n}k_i. \tag{8}$$

Lemma 1 reduces Problem 2 to a much simpler one.

Lemma 2. *Let $k_1,k_2,\dots,k_n$ be any positive integers. For any permutation t of $1,2,\dots,n$, $H_{k_{t(1)},k_{t(2)},\dots,k_{t(n)}}\in\mathscr{I}$ iff $f(k_1,k_2,\dots,k_n,x)$ is an integral-root polynomial of x.* □

For positive integers $k_1,k_2,\dots,k_n$, let $\mathscr{H}(k_1,k_2,\dots,k_n)$ denote the set of graphs $H_{k_{t(1)},k_{t(2)},\dots,k_{t(n)}}$ for all permutations t of $1,2,\dots,n$. By Lemma 2, we have

Lemma 3. *Let $k_1,k_2,\dots,k_n$ be any positive integers. If $f(k_1,k_2,\dots,k_n,x)$ is an integral-root polynomial of x, then $\mathscr{H}(k_1,k_2,\dots,k_n)\subseteq\mathscr{I}$.* □

Problem 2 is thus equivalent to the following.

Problem 3. *For $n\geqslant 3$, are there positive integers $k_1,k_2,\dots,k_n$ such that $f(k_1,k_2,\dots,k_n,x)$ is an integral-root polynomial?*

We shall see that Lemma 3 is very useful. Almost all graphs in $\bar{\mathscr{C}}\cap\mathscr{I}$ that have been found can be verified easily by using Lemma 3. The graph in Fig. 1 is $H_{1,4,1}$. Observe that

$$f(1,1,4,x)=(x-1)^2(x-4)+4=x(x-3)^2. \tag{9}$$

By Lemma 3, we have $\mathscr{H}(1,1,4)\subseteq\mathscr{I}$. Further we observe that

$$f(1,ab,(a+1)(b+1),x)=x(x-1-a-ab)(x-1-b-ab). \tag{10}$$

Thus by Lemma 3, $\mathscr{H}(1,ab,(a+1)(b+1))\subseteq\mathscr{I}$ for any positive integers a and b. This result was first found by Dmitriev [3].

Theorem 1 (Dmitriev [3]). *If $k_1=1,k_2=(k+1)(k+l+1)$ and $k_3=(k+2)(k+l+2)$ for any non-negative integers k and l, then $\mathscr{H}(k_1,k_2,k_3)\subseteq\mathscr{I}$.* □

Dmitriev's result was extended by Dong and Koh [4].

Theorem 2 (Dong and Koh [4]). *If $k_1=a_1b_1, k_2=a_2b_2$ and $k_3=(a_1+a_2)(b_1+b_2)$ for any positive integers a_1,a_2,b_1,b_2, then $\mathscr{H}(k_1,k_2,k_3)\subseteq\mathscr{I}$.*

This result can also be verified by using Lemma 3. Let $k_1=a_1b_1, k_2=a_2b_2$ and $k_3=(a_1+a_2)(b_1+b_2)$. Then

$$f(k_1,k_2,k_3,x)=x(x-a_1b_1-a_2b_2-a_1b_2)(x-a_1b_1-a_2b_2-a_2b_1). \tag{11}$$

By Lemma 3, $\mathscr{H}(k_1,k_2,k_3)\subseteq\mathscr{I}$.

Dong and Koh [4] also gave a positive answer to Problem 3 for $n=4$.

Theorem 3 (Dong and Koh [4]). *If $k_1=1, k_2=ab, k_3=(a+2)(b+1)$ and $k_4=(a+1)(2b+1)$ for any positive integers a and b, then $\mathscr{H}(k_1,k_2,k_3,k_4)\subseteq\mathscr{I}$.* □

To prove Theorem 3 by using Lemma 3, we observe that

$$\begin{aligned}&f(1,ab,(a+2)(b+1),(a+1)(2b+1),x)\\&\quad=x(x-1-a-ab)(x-1-2b-ab)(x-2-2b-a-2ab).\end{aligned} \tag{12}$$

We shall now apply Lemma 3 to establish some new results. The first one given below is an extension of Theorem 3.

Theorem 4. *If $k_1=a_1b_1, k_2=a_2b_2, k_3=(a_1+a_2)(2b_1+b_2)$ and $k_4=(a_1+2a_2)(b_1+b_2)$, where a_1,a_2,b_1 and b_2 are arbitrary positive integers, then $\mathscr{H}(k_1,k_2,k_3,k_4)\subseteq\mathscr{I}$.*

Proof. Given the k_i's, we have

$$\begin{aligned}f(k_1,k_2,k_3,k_4,x)&=x(x-a_2b_2-2a_2b_1-a_1b_1)(x-a_2b_2-a_1b_2-a_1b_1)\\&\quad\times(x-2a_2b_2-2a_2b_1-a_1b_2-2a_1b_1).\end{aligned} \tag{13}$$

The result then follows from Lemma 3. □

Using Lemma 3, we also obtain positive solutions to Problem 3 for $n=5,6$.

Theorem 5. *If $k_1=1$, $k_2=a(a-3)/2$, $k_3=a(a+3)/2$, $k_4=(a-1)(a+2)$ and $k_5=(a-2)(a+1)$, where a is an arbitrary integer with $a\geqslant 4$, then $\mathscr{H}(k_1,k_2,k_3,k_4,k_5)\subseteq\mathscr{I}$.*

Proof. Given the k_i's, we observe that

$$\begin{aligned}&f(k_1,k_2,k_3,k_4,k_5,x)\\&\quad=x(x-a^2)(x-a^2-1)\\&\qquad\times\left(x-\frac{a(a-1)}{2}+2\right)\left(x-\frac{a(a+1)}{2}+2\right).\end{aligned} \tag{14}$$

The result then follows from Lemma 3. □

We have also found the following multisets $\{k_1,k_2,k_3,k_4,k_5\}$ such that the function $f(k_1,k_2,k_3,k_4,k_5,x)$ is an integral-root polynomial in x, some of which are included in the family of Theorem 5:

$\{k_1,k_2,k_3,k_4,k_5\}$	zeros of $f(k_1,k_2,k_3,k_4,k_5,x)$	
$\{1,2,10,14,18\}$	$\{0,4,8,16,17\}$	
$\{1,3,12,14,20\}$	$\{0,6,8,17,19\}$	
$\{1,4,20,21,30\}$	$\{0,6,16,25,29\}$	
$\{1,5,18,20,28\}$	$\{0,8,13,25,26\}$	
$\{1,7,20,24,28\}$	$\{0,12,13,29,31\}$	
$\{1,9,27,28,40\}$	$\{0,13,19,36,37\}$	
$\{1,11,30,35,48\}$	$\{0,15,23,41,46\}$	(15)
$\{1,12,30,35,52\}$	$\{0,17,22,40,51\}$	
$\{1,13,32,35,54\}$	$\{0,19,22,41,53\}$	
$\{1,14,35,40,54\}$	$\{0,19,26,49,50\}$	
$\{1,19,42,48,65\}$	$\{0,27,29,58,61\}$	
$\{1,20,44,54,70\}$	$\{0,26,34,64,65\}$	
$\{1,21,48,52,75\}$	$\{0,27,36,61,73\}$	
$\{1,26,55,60,84\}$	$\{0,34,81,71,40\}$.	

Theorem 6. *If* $k_1=a_1b_1$, $k_2=a_2b_2$, $k_3=(a_1+a_2)(b_1+3b_2)$, $k_4=(3a_1+a_2)(b_1+b_2)$, $k_5=(2a_1+a_2)(2b_1+3b_2)$ *and* $k_6=(3a_1+2a_1)(b_1+2b_2)$, *where* a_1,a_2,b_1 *and* b_2 *are arbitrary positive integers, then* $\mathscr{H}(k_1,k_2,k_3,k_4,k_5,k_6)\subseteq\mathscr{I}$.

Proof. Given the k_i's, we observe that

$$\begin{aligned} &f(k_1,k_2,k_3,k_4,k_5,k_6,x)\\ &\quad = x(x-3a_1b_2-3a_1b_1-3a_2b_2-2a_2b_1)\\ &\qquad \times(x-3a_1b_2-a_1b_1-a_2b_2)(x-6a_1b_2-4a_1b_1-4a_2b_2-2a_2b_1)\\ &\qquad \times(x-a_1b_1-a_2b_2-a_2b_1)(x-3a_1b_1-3a_2b_2-a_2b_1-6a_1b_2). \end{aligned} \tag{16}$$

The result then follows from Lemma 3. □

Theorems 5 and 6 answer affirmatively Problem 3 and hence Problem 2 for $n=5$ and $n=6$. Thus, the problem asked by Dmitriev has positive answers for $p=6$ and $p=7$. For $n\geqslant 7$, Problem 3 remains open, and so does Dmitriev's problem for $p\geqslant 8$.

Acknowledgements

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