Note

Divisibility of certain coefficients of the chromatic polynomials

F.M. Dong\textsuperscript{a,∗}, K.M. Koh\textsuperscript{b}, C.A. Soh\textsuperscript{b}

\textsuperscript{a}Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore 637616, Singapore
\textsuperscript{b}Department of Mathematics, National University of Singapore, Singapore 117543, Singapore

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Abstract

This note presents some results on the divisibility of certain coefficients of the chromatic polynomials.

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Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \), and of order \( n (=|V(G)|) \) and size \( m (=|E(G)|) \). Suppose \( \beta : E(G) \to \{1, 2, \ldots, m\} \) is a bijection. For any cycle \( C \) in \( G \), let \( e \in E(C) \) such that \( \beta(e) \geq \beta(x) \) for any \( x \in E(C) \). We call the path \( C - e \), obtained by deleting \( e \) from \( C \), a \emph{broken cycle} in \( G \) with respect to \( \beta \).

Given a positive integer \( \lambda \), let \( P(G, \lambda) \) denote the number of distinct \( \lambda \)-colourings of \( G \). Whitney [6] showed that

\[
P(G, \lambda) = \sum_{i=1}^{n} (-1)^{n-i} a_i(G) \lambda^i,
\]

where \( a_i(G) \) counts the number of spanning subgraphs of \( G \) that have exactly \( n - i \) edges and that contain no broken cycles. We call (1) the \emph{chromatic polynomial} of \( G \) (see [3–5]). The coefficient \( a_1(G) \) in (1) is interesting in its own right. Read [4]
observed that \( G \) is connected iff \( a_1(G) \geq 1 \). Eisenberg [1] noted that \( G \) is a tree iff \( a_1(G) = 1 \). Let \( \chi(G) \) denote the chromatic number of \( G \). Hong [2] showed that:

(i) if \( G \) is connected, then \( a_1(G) \) is divisible by \((\chi(G) - 1)!\) and

(ii) \( G \) is connected and bipartite iff \( a_1(G) \) is odd.

In this note, we shall present some results on the divisibility of certain \( a_i(G) \) which extend Hong’s results mentioned above.

Let \( \lfloor x \rfloor \) denote the largest integer not exceeding the real \( x \). For any integers \( k \) and \( i \) with \( k \geq 0 \) and \( i > 0 \), define

\[
\Phi(k, i) = \begin{cases} 
1 & \text{if } k < 2i, \\
\left\lfloor k/i \right\rfloor! \Phi(k - \left\lfloor k/i \right\rfloor, i) & \text{if } k \geq 2i.
\end{cases}
\]

Observe that \( \Phi(k, 1) = k! \). The following table shows the values of \( \Phi(10, i) \) for \( i = 1, 2, \ldots, 10 \):

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>( \geq 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi(10, i) )</td>
<td>10!</td>
<td>( 5! \times 2! )</td>
<td>( 3! \times 2! )</td>
<td>2!</td>
<td>2!</td>
<td>1</td>
</tr>
</tbody>
</table>

Our first main result is given below.

**Theorem 1.** For any graph \( G \), \( a_i(G) \) is divisible by \( \Phi(\chi(G) - 1, i) \) for each \( i = 1, 2, \ldots, \chi(G) - 1 \).

To establish Theorem 1, we first prove the following two lemmas.

**Lemma 1.** Let \( k \) and \( i \) be positive integers, and \( x_1, x_2, \ldots, x_i \) be any \( i \) non-negative integers. If \( \sum_{j=1}^{i} x_j > k - i \), then \( \prod_{j=1}^{i} (x_j!) \) is divisible by \( \Phi(k, i) \).

**Proof.** If \( i = 1 \), then \( x_1 > k - 1 \) and \( \Phi(k, 1) = k! \), and so the result holds. Assume that \( i \geq 2 \). We shall prove the result by induction on \( k \). If \( k \leq 2i - 1 \), then \( \Phi(k, i) = 1 \) and thus the result holds.

Suppose that the result holds when \( k < m \), where \( m \geq 2i \). Now let \( k = m \).

We first note that \( x_j \geq \left\lfloor k/i \right\rfloor \) for some \( j \); otherwise, we have

\[
x_1 + x_2 + \cdots + x_i \leq i(\left\lfloor k/i \right\rfloor - 1) \leq i(k/i - 1) = k - i,
\]

a contradiction. We may assume that \( x_1 \geq \left\lfloor k/i \right\rfloor \). Observe that

\[
x_1! = \left\lfloor k/i \right\rfloor!(x_1 - \left\lfloor k/i \right\rfloor)! \binom{x_1}{\left\lfloor k/i \right\rfloor}.
\]
Let $x'_1 = x_1 - \lfloor k/i \rfloor$. Then
\[
\prod_{j=1}^{i} (x_j!) = \lfloor k/i \rfloor ! \left( \frac{x_1}{\lfloor k/i \rfloor} \right) x'_1! x_2! \cdots x_i!.
\] (3)

Since
\[
x'_1 + x_2 + \cdots + x_i = x_1 + x_2 + \cdots + x_i - \lfloor k/i \rfloor > (k - \lfloor k/i \rfloor) - i
\]
and $\lfloor k/i \rfloor \geq 2$, by the induction hypothesis, $x'_1! x_2! \cdots x_i!$ is divisible by $\Phi(k - \lfloor k/i \rfloor, i)$. Thus, by (3), $x'_1! x_2! \cdots x_i!$ is divisible by $\Phi(k,i)$, as required.

**Lemma 2.** Let $n$ and $k$ be any integers with $n > k \geq 1$. Then $a_i(K_n)$ is divisible by $\Phi(k,i)$ for each $i = 1, 2, \ldots, k$.

**Proof.** We have
\[
P(K_n, \lambda) = \prod_{t=0}^{n-1} (\lambda - t).
\]
Let $N = \{1, 2, \ldots, n-1\}$. Then
\[
a_i(K_n) = \sum_{|R| = n-i} \prod_{r \in R} r.
\] (4)

Observe that for $R \subseteq N$ with $|R| = n-i$, there exist $b_1, b_2, \ldots, b_{i-1} \in N$ with $b_1 < b_2 < \cdots < b_{i-1}$ such that
\[
\prod_{r \in R} r = \frac{(n-1)!}{b_1 b_2 \cdots b_{i-1}} = (b_1 - 1)! \prod_{j=2}^{i} (b_j - b_{j-1} - 1)! \left( \frac{b_j - 1}{b_j - b_{j-1} - 1} \right),
\]
where $b_i = n$. Let $x_1 = b_1 - 1$ and $x_j = b_j - b_{j-1} - 1$ for $j = 2, \ldots, i$. So $\prod_{r \in R} r$ is divisible by $\prod_{j=1}^{i} (x_j!)$. Observe that
\[
x_1 + x_2 + \cdots + x_i = n - i > k - i.
\]
By Lemma 1, $\prod_{j=1}^{i} (x_j!)$ is divisible by $\Phi(k,i)$. Hence $\prod_{r \in R} r$ is divisible by $\Phi(k,i)$. The result thus follows from (4).

Let $G$ be any graph, and $u, v$ be any two vertices in $G$. If $uv \notin E(G)$, let $G + uv$ be the graph obtained from $G$ by adding an edge joining $u$ and $v$; otherwise $G + uv = G$. If $uv \notin E(G)$, let $G_{uv}$ denote the graph obtained from $G$ by identifying $u$ and $v$ and replacing all multi-edges by single ones. The following fundamental reduction formula is well known (see [4]):
\[
P(G, \lambda) = P(G + uv, \lambda) + P(G_{uv}, \lambda).
\] (5)
where \( u \) and \( v \) are non-adjacent vertices of \( G \). It follows from (5) that
\[
a_i(G) = a_i(G + uv) - a_i(Guv),
\]
for any integer \( i \geq 1 \).

The chromatic polynomial \( P(G, \lambda) \) can also be expressed in factorial form (see, for instance, [3,4]) as shown below:
\[
P(G, \lambda) = \sum_{j=\chi(G)}^{n} b_j(\lambda)_j,
\]
(6)
where \( n \) is the order of \( G \),
\[
(\lambda)_j = \lambda(\lambda - 1) \cdots (\lambda - j + 1) = P(K_j, \lambda)
\]
and \( b_j \) is the number of ways of partitioning \( V(G) \) into \( j \) non-empty colour classes (or independent sets).

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Let \( n \) be the order of \( G \). By (6), we have
\[
P(G, \lambda) = \sum_{j=\chi(G)}^{n} b_j(\lambda)_j.
\]
Thus,
\[
a_i(G) = \sum_{j=\chi(G)}^{n} (-1)^{n-j} b_j a_i(K_j).
\]

By Lemma 2, \( a_i(K_j) \) is divisible by \( \Phi(\chi(G) - 1, i) \) for all \( j \geq \chi(G) \) and \( i = 1, 2, \ldots, \chi(G) - 1 \). Therefore, \( a_i(G) \) is divisible by \( \Phi(\chi(G) - 1, i) \) for \( i = 1, 2, \ldots, \chi(G) - 1 \).

Theorem 1 states that \( \Phi(\chi(G) - 1, i) | a_i(G) \) for each \( i = 1, 2, \ldots, \chi(G) - 1 \). As no explicit expression for \( \Phi(k,i) \) has been found, it is useful to find a factor of \( \Phi(k,i) \) which is quite close to it and has an explicit expression. We shall show in what follows that
\[
\left( \prod_{s \geq 1} \left[ \frac{(i-1)^{s-1}k}{i^s} \right] ! \right) | \Phi(k,i)
\]
for any non-negative integer \( k \), and from which we have \( (\chi(G) - 1)! | a_i(G) \) (which gives Hong’s observation (i)) and
\[
\left( \prod_{s \geq 1} \left[ \frac{(i-1)^{s-1}k}{i^s} \right] ! \right) | a_i(G)
\]
for \( i = 2, 3, \ldots, \chi(G) - 1 \).

This result will be found useful later.
Corollary 1. For any graph $G$, $a_i(G)$ is divisible by $(\chi(G) - 1)!$ and $a_i(G)$ is divisible by
\[ \prod_{s \geq 1} \left( \frac{(i - 1)^{s-1}(\chi(G) - 1)}{i^s} \right)! \]
for $i = 2, 3, \ldots, \chi(G) - 1$.

Proof. By Theorem 1, $a_i(G)$ is divisible by $\Phi(\chi(G) - 1, i)$ for $i = 1, 2, \ldots, \chi(G) - 1$. Since $\Phi(\chi(G) - 1, 1) = (\chi(G) - 1)!$, the result holds when $i = 1$. Assume that $i \geq 2$.

To prove this result, we shall prove that $\Phi(k, i)$ is divisible by
\[ \prod_{t \geq 1} \left( \frac{(i - 1)^{t-1}k}{i^t} \right)! \]
for any integer $k \geq 0$.

We shall prove the result by induction on $k$. If $k < 2i$, then $\Phi(k, i) = 1$ and
\[ \prod_{s \geq 1} \left( \frac{(i - 1)^{s-1}k}{i^s} \right)! = 1 \]
and so the result holds in this case. Suppose that the result holds when $k < m$, where $m \geq 2i$. Now let $k = m$.

By definition
\[ \Phi(k, i) = [k/i]! \Phi(k - [k/i], i). \]

Since $k - [k/i] < m$, by the induction hypothesis, $\Phi(k - [k/i], i)$ is divisible by
\[ \prod_{t \geq 1} \left( \frac{(i - 1)^{t-1}(k - [k/i])}{i^t} \right)! \]
Observe that
\[ \frac{(i - 1)^{t-1}(k - [k/i])}{i^t} \geq \frac{(i - 1)^{t-1}(k - k/i)}{i^t} = \frac{(i - 1)^{t}k}{i^{t+1}}. \]

Hence $\Phi(k, i)$ is divisible by $\prod_{s \geq 1} [(i - 1)^{s-1}k/i^s]!$. \hfill \Box

Our second corollary of Theorem 1 generalizes the ‘if’ part of Hong’s observation (ii).

Corollary 2. For any graph $G$ and any positive integers $i$ and $r$, if $a_i(G)$ is not divisible by $r!$, then $\chi(G) \leq ir$.

Proof. If $\chi(G) - 1 \geq ir$, then by Theorem 1, $a_i(G)$ is divisible by $[ir/i]!$, which is $r!$, a contradiction. \hfill \Box
A graph $G$ is said to be uniquely $k$-colourable if $\chi(G) = k$ and there exists a unique way of partitioning $V(G)$ into $k$ colour classes (i.e., $b_k = 1$ in (6)). It is clear that $G$ is connected and bipartite iff $G$ is uniquely 2-colourable. Our second main result is stated below.

**Theorem 2.** Let $G$ be a uniquely $k$-colourable graph. If $k = ip$ for some prime $p$ and positive integer $i$, then $a_i(G)$ is not divisible by $p$.

We first prove the following result.

**Lemma 3.** For any prime number $p$ and positive integer $i$, $a_i(K_{ip})$ is not divisible by $p$.

**Proof.** Let $N = \{1, 2, \ldots, ip - 1\}$. Since

$$P(K_{ip}, \lambda) = \prod_{j=0}^{ip-1} (\lambda - j),$$

we have

$$a_i(K_{ip}) = \sum_{N' \subseteq N, |N'| = ip - i} \prod_{t \in N'} t.$$

Let $Q = \{p, 2p, \ldots, (i-1)p\}$. Observe that $|N - Q| = ip - i$. Since $p$ is prime, $\prod_{t \in N'} t$ is divisible by $p$ iff $N' \neq N - Q$. Therefore, $a_i(K_{ip})$ is not divisible by $p$. \hfill $\square$

**Proof of Theorem 2.** Again, we have

$$a_i(G) = \sum_{j=ip}^{n} (-1)^{n-j} b_j a_i(K_j),$$

where $n$ is the order of $G$. By Lemma 3, $a_i(K_{ip})$ is not divisible by $p$. But by Corollary 1 to Theorem 1, $a_i(K_j)$ is divisible by $\lfloor (j - 1)/i \rfloor!$. So $a_i(K_j)$ is divisible by $p!$ if $j > ip$. Since $G$ is uniquely $ip$-colourable, we have $b_{ip} = 1$. Therefore $a_i(G)$ is not divisible by $p$. \hfill $\square$

While the ‘if’ part of Hong’s result (ii) follows from Corollary 2 to Theorem 1, the ‘only if’ part follows immediately from Theorem 2.

Finally, we have:

**Theorem 3.** For any uniquely $k$-colourable graph $G$, where $k \geq 2$, $a_1(G)$ is not divisible by $k!$.

**Proof.** We have

$$a_1(G) = \sum_{j=k}^{n} (-1)^{n-j} b_j a_1(K_j).$$
Observe that \( a_1(K_j) = (j - 1)! \). Hence \( a_1(K_j) \) is divisible by \( k! \) iff \( j > k \). Since \( G \) is uniquely \( k \)-colourable, we have \( b_k = 1 \). Therefore \( a_1(G) \) is not divisible by \( k! \).

**Corollary 1.** For any graph \( G \), if \( a_1(G) \) is divisible by \((k - 1)!\) but not by \( k! \), then \( \chi(G) \leq k \) and either:

(i) \( \chi(G) = k \); or

(ii) \( G \) is not uniquely \( r \)-colourable for any \( r \).

**Proof.** Since \( a_1(G) \) is not divisible by \( k! \), by Corollary 2 to Theorem 1, \( \chi(G) \leq k \).

Assume that \( \chi(G) \leq k - 1 \). If \( G \) is uniquely \( r \)-colourable for some \( r \leq k - 1 \), then by Theorem 3, \( a_1(G) \) is not divisible by \( r! \), contradicting the assumption that \( a_1(G) \) is divisible by \((k - 1)!\).

It is easy to construct a graph \( G \) with \( \chi(G) = k \) such that \( a_1(G) \) is divisible by \( k! \). For example, for the following graph \( G \), we have \( \chi(G) = 3 \) and \( a_1(G) = 6 \).

\[
(P(G, \lambda) = \lambda^6 - 7\lambda^5 + 20\lambda^4 - 29\lambda^3 + 21\lambda^2 - 6\lambda.)
\]

This, however, is not the case if \( G \) is uniquely \( k \)-colourable as stated below.

**Corollary 2.** If \( G \) is a uniquely \( k \)-colourable graph, where \( k \geq 2 \), then \( a_1(G) \) is divisible by \((k - 1)!\) but not by \( k! \).

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**References**


