Discrete Mathematics 258 (2002) 303–321

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Chromaticity of some families of dense graphs

F.M. Dong[1], K.L. Teo, C.H.C. Little*, M.D. Hendy

Institute of Fundamental Sciences (Mathematics), Massey University, Private Bag 11222, Palmerston North, New Zealand

Received 1 March 1999; received in revised form 2 March 2001; accepted 20 August 2001

Abstract

For a graph G, let $P(G,\lambda)$ be its chromatic polynomial and let $[G]$ be the set of graphs having $P(G,\lambda)$ as their chromatic polynomial. We call $[G]$ the chromatic equivalence class of G. If $[G]=\{G\}$, then G is said to be chromatically unique. In this paper, we first determine $[G]$ for each graph G whose complement $\bar{G}$ is of the form $aK_1 \cup bK_3 \cup \bigcup_{1 \leqslant i \leqslant s} P_{l_i}$, where a, b are any nonnegative integers and l_i is even. By this result, we find that such a graph G is chromatically unique iff $ab=0$ and $l_i \neq 4$ for all i. This settles the conjecture that the complement of P_n is chromatically unique for each even n with $n \neq 4$. We also determine $[H]$ for each graph H whose complement $\bar{H}$ is of the form $aK_3 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \cup \bigcup_{1 \leqslant j \leqslant t} C_{v_j}$, where $u_i \geqslant 3$ and $u_i \not\equiv 4 \pmod 5$ for all i. We prove that such a graph H is chromatically unique if $u_i + 1 \neq v_j$ for all i, j and u_i is even when $u_i \geqslant 6$.

Keywords: Graph; Chromatic polynomial; Chromatic uniqueness; Adjoint polynomial

1. Introduction

In this paper, all graphs considered are simple graphs. For a graph G, let $\bar{G}$, $V(G)$, $E(G)$, $v(G)$, $e(G)$, $t(G)$, $c(G)$, $\chi(G)$, $\delta(G)$ and $P(G,\lambda)$, respectively, be the complement, vertex set, edge set, order, size, number of triangles, number of components, chromatic number, minimum degree and chromatic polynomial of G. We will denote by P_n the path, C_n the cycle, and K_n the complete graph with n vertices.

A partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$, where k is a positive integer, is called a *k-independent partition* of a graph G if each A_i is a nonempty independent set of

* Corresponding author.
E-mail address: c.Little@messey.ac.nz (C.H.C. Little).

[1] Present address: Mathematics & Mathematics Education Group, National Institute of Education, Nanyang Technological University, Singapore.

0012-365X/02/$ - see front matter
PII: S0012-365X(02)00355-2

G. Let $\alpha(G,k)$ denote the number of k-independent partitions of G. Then

$$P(G,\lambda)=\sum_{k=1}^{v(G)}\alpha(G,k)(\lambda)_k, \tag{1}$$

where $(\lambda)_k=\lambda(\lambda-1)\cdots(\lambda-k+1)$. (See [17].)

Two graphs G and H are said to be *chromatically equivalent* (or simply *χ-equivalent*), symbolically $G\sim H$, if $P(G,\lambda)=P(H,\lambda)$. It is clear that the relation '$\sim$' is an equivalence relation on the family of graphs. The chromatic equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is said to be *chromatically unique* (or simply *χ-unique*) if $H\cong G$ whenever $H\sim G$. Observe that G is χ-unique iff $[G]=\{G\}$. By (1), we have

Lemma 1.1. *For any two graphs G and H, $G\sim H$ iff $v(G)=v(H)$ and $\alpha(G,k)=\alpha(H,k)$ for all k with $1\leqslant k\leqslant v(G)$.*

It is an interesting problem to determine $[G]$ for a given graph G. In this paper, we shall study this problem for some dense graphs G such that the components of $\bar{G}$ are isolated vertices, paths or cycles. We shall use the adjoint polynomial of a graph as a tool for this study.

Let G be a graph with order n. If H is a spanning subgraph of G and each component of H is complete, then H is called a *clique cover* (or an *ideal subgraph*) of G [4]. Two clique covers are considered to be different if they have different edge sets. For $k\geqslant 1$, let $N(G,k)$ be the number of clique covers H in G with $c(H)=k$. It is clear that $N(G,n)=1$ and $N(G,k)=0$ for $k>n$. Define

$$h(G,\mu)=\begin{cases}\sum_{k=1}^{n}N(G,k)\mu^k & \text{if } n\geqslant 1,\\ 1 & \text{if } n=0.\end{cases} \tag{2}$$

The polynomial $h(G,\mu)$ is called the *adjoint polynomial* of G. Observe that $h(G,\mu)=h(G',\mu)$ if $G\cong G'$. Hence, $h(G,\mu)$ is a well-defined graph-function. The notion of the adjoint polynomial of a graph was introduced by Liu [9]. Note that the adjoint polynomial is a special case of an F-polynomial [4].

Two graphs G and H are said to be *adjointly equivalent*, symbolically $G\sim_h H$, if $h(G,\mu)=h(H,\mu)$. Let $[G]_h=\{H|H\sim_h G\}$. A graph G is said to be *adjointly unique* if $[G]_h=\{G\}$. By (2), we have

Lemma 1.2. *For any two graphs G and H, $G\sim_h H$ iff $v(G)=v(H)$ and $N(G,k)=N(H,k)$ for all k with $1\leqslant k\leqslant v(G)$.*

Let $\bar{G}$ denote the complement of G, i.e., the graph with vertex set $V(G)$ and edge set $\{xy|xy\notin E(G),x,y\in V(G)\}$. Note that

$$\alpha(G,k)=N(\bar{G},k),\quad k=1,2,\ldots,n. \tag{3}$$

It follows from Lemmas 1.1 and 1.2 together with (3) that

Theorem 1.1. (i) $G \sim H$ *iff* $\bar{G} \sim_h \bar{H}$;

(ii) $[G] = \{H | \bar{H} \in [\bar{G}]_h\}$;
(iii) G *is chromatically unique iff* $\bar{G}$ *is adjointly unique.*

Hence the goal of determining $[G]$ for a given graph G can be realized by determining $[\bar{G}]_h$. Thus, as has been observed in [13,16], if $e(G)$ is very large, it may be easier to study $[\bar{G}]_h$ rather than $[G]$. Another polynomial used to study the chromaticity of dense graphs is the σ-polynomial, which was introduced by Korfhage [6]. The *σ-polynomial* of G is defined by

$$\sigma(G,\mu) = h(\bar{G},\mu)/\mu^{\chi(G)}.$$

Although some researchers, such as Du [3] and Li and Whitehead [7], have used σ-polynomials to study the chromaticity of some dense graphs, one disadvantage is that $\sigma(G,\mu)$ does not determine the order of G. This can be seen from the fact that $\sigma(G,\mu) = \sigma(G \cup mK_1,\mu)$ for any integer $m \geqslant 1$, where $G \cup mK_1$ is the graph obtained from G by adding m isolated vertices. The adjoint polynomial does not have this fault, and it contains all the information that the σ-polynomial has. Hence in this paper we shall use adjoint polynomials rather than σ-polynomials.

It is clear that $N(G,k)$ is an invariant for adjointly equivalent graphs for each positive integer k. Thus any expression in terms of $N(G,1), N(G,2), \ldots, N(G,n)$ is an invariant for adjointly equivalent graphs. However we prefer invariants which have some useful properties such as having constant upper bounds or lower bounds. One invariant that has been used by several researchers [5,11–16], to determine adjoint equivalence classes of graphs is $R_1(G)$. Given a polynomial $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$, let

$$R_1(f) = \begin{cases} -\binom{b_1}{2} + b_1 & \text{if } n = 1, \\ b_2 - \binom{b_1}{2} + b_1 & \text{if } n \geqslant 2. \end{cases} \quad (4)$$

For any graph G, we define

$$R_1(G) = R_1(h(G,\mu)). \quad (5)$$

It is clear that $R_1(G)$ is in fact an invariant for adjointly equivalent graphs. It is known that this invariant is additive over the components of a graph. Specifically we have the following lemma.

Lemma 1.3 (Liu [9]). *For any graph* G *with components* $G_1, G_2, \ldots, G_k$

$$R_1(G) = \sum_{i=1}^{k} R_1(G_i).$$

Liu [16] showed that $R_1(G) \leqslant 1$ for any connected graph G, and characterised the connected graphs G with $R_1(G) \geqslant 0$. For positive integers k,s and t, let $T_{k,s,t}$ be

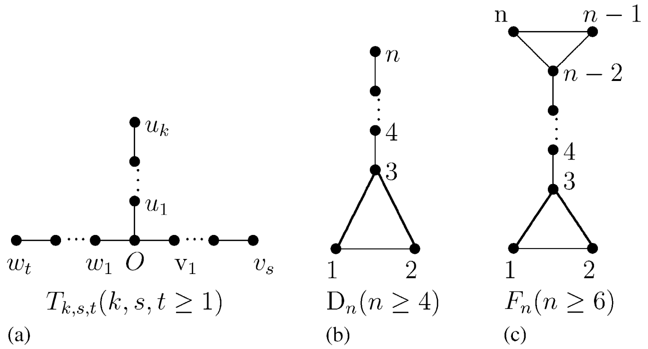

Fig. 1.

the graph in Fig. 1(a). Let

$$\mathscr{T}' = \{T_{k,s,t} \mid k \geqslant s \geqslant t \geqslant 1\}.$$

Let D_n and F_n be the graphs shown in Fig. 1(b) and (c).

Theorem 1.2 (Liu and Zhao [16]). *Let G be a connected graph. Then $R_1(G) \leqslant 1$ and*

(i) $R_1(G) = 1$ *iff* $G \in \{K_3\} \cup \{P_n \mid n \geqslant 2\}$,
(ii) $R_1(G) = 0$ *iff* $G \in \{K_1\} \cup \mathscr{T}' \cup \{C_n, D_n \mid n \geqslant 4\}$, *and*
(iii) $R_1(G) = -1$ *with* $e(G) \geqslant v(G) + 1$ *iff* $G \in \{K_4 - e\} \cup \{F_n \mid n \geqslant 6\}$.

A further result from [1] that is needed for the present paper is the following.

Theorem 1.3. *For any connected graph G with $G \notin \{K_3, K_4\}$,*

(i) *if* $-1 \leqslant R_1(G) \leqslant 1$, *then* $R_1(G) \leqslant v(G) - e(G)$ *with equality if and only if*

$$G \in \{K_4 - e\} \cup \{P_n, C_{n+1}, D_{n+2}, F_{n+4} \mid n \geqslant 2\};$$

(ii) *if* $R_1(G) \leqslant -2$, *then* $R_1(G) \leqslant v(G) - e(G) - 1$.

Note that $e(K_3) + R_1(K_3) = v(K_3) + 1$ and $e(K_4) + R_1(K_4) = v(K_4)$, since $R_1(K_4) = -2$ [1].

In [1] another invariant $R_2(G)$ is introduced. For any polynomial $f(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n$, let

$$R_2(f) = b_3 - \binom{b_1}{3} - (b_1 - 2)\left(b_2 - \binom{b_1}{2}\right) - b_1,$$

where $b_k = 0$ for $k > n$. For any graph G, define

$$R_2(G) = R_2(h(G, \mu)).$$

Since $R_2(G)$ is determined by $h(G,\mu)$, $R_2(G)$ is indeed an invariant for adjointly equivalent graphs. This invariant is also additive over the components of a graph (see [1]). We shall use it in combination with $R_1(G)$ to determine the adjoint equivalence classes of certain graphs and confirm a conjecture of Liu [13] that P_n is adjointly unique for each even $n \neq 4$.

We shall need the following theorem from [1]. Here Y_n denotes the graph $T_{n-3,1,1}$ where $n \geqslant 4$.

Theorem 1.4. (i) $R_2(P_1)=0$, $R_2(P_2)=-1$ *and* $R_2(P_n)=-2$ *for* $n \geqslant 3$;
(ii) $R_2(K_3)=-2$ *and* $R_2(C_n)=0$ *for* $n \geqslant 4$;
(iii) $R_2(Y_4)=-1$ *and* $R_2(Y_n)=0$ *for* $n \geqslant 5$;
(iv) $R_2(D_4)=0$ *and* $R_2(D_n)=1$ *for* $n \geqslant 5$;
(v) $R_2(F_6)=5$ *and* $R_2(F_n)=4$ *for* $n \geqslant 7$;
(vi) $R_2(K_4-e)=3$ *and* $R_2(K_4)=7$.

We also need the following results from [2] on the zeros of the adjoint polynomials of paths and cycles.

Theorem 1.5. *For any positive integer* n, *the zeros of* $h(P_n,\mu)$ *are*:

$$\underbrace{0,0,\ldots,0}_{\lceil n/2 \rceil}, \quad -2-2\cos\frac{2s\pi}{n+1}, \quad s=1,2,\ldots,\lfloor n/2 \rfloor.$$

Theorem 1.6. *For any integer* $n \geqslant 4$, $h(C_n,\mu)$ *has the following zeros*:

$$\underbrace{0,0,\ldots,0}_{\lceil n/2 \rceil}, \quad -2-2\cos\frac{(2s-1)\pi}{n}, \quad s=1,2,\ldots,\lfloor n/2 \rfloor.$$

For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. For any two graphs G and H, let $G \cup H$ denote the graph whose vertex set can be partitioned into sets V_1 and V_2 such that $(G \cup H)[V_1] \cong G$ and $(G \cup H)[V_2] \cong H$, and whose edge set is $E(G) \cup E(H)$. We call $G \cup H$ the union of G and H. For any positive integer k, let kG denote the union of k copies of G. Our main results determine $[G]$ for any graph G with $\bar{G}=aK_1 \cup bK_3 \cup \bigcup_{1 \leqslant i \leqslant s} P_{l_i}$, where a,b are any nonnegative integers and l_i is even, or $\bar{G}=aK_3 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \cup \bigcup_{1 \leqslant j \leqslant t} C_{v_j}$, where $u_i \geqslant 3$ and $u_i \not\equiv 4 \pmod 5$ for all i.

2. Chromatic equivalence classes of some dense graphs

Our aim is to determine the chromatic equivalence classes of some graphs G such that the components of $\bar{G}$ are members of the set

$$\{K_1, D_4\} \cup \{P_k, C_{k+1} \mid k \geqslant 2\},$$

but we realize this aim by determining the adjoint equivalence classes of graphs whose components belong to this set. We do not consider it necessary to translate the results on adjoint equivalence classes and adjoint uniqueness into those on chromatic equivalence classes and chromatic uniqueness.

We shall need the following reduction formula.

Theorem 2.1 (Liu [10]). *If xy is an edge not in any triangle of a graph G then*

$$h(G,\mu)=h(G-xy,\mu)+\mu h(G-\{x,y\},\mu).$$

For any graph G, let $\bar{\chi}(G)$ denote the minimum integer k such that $N(G,k)>0$. Observe that $\bar{\chi}(G)=\chi(\bar{G})$. Let $\tau(G)=N(G,\bar{\chi}(G))$. By the definition of $N(G,k)$, we have

Lemma 2.1. *Let $G_1,G_2,\ldots,G_k$ be the components of G. Then*

$$\bar{\chi}(G)=\sum_{i=1}^{k}\bar{\chi}(G_i), \tag{6}$$

$$\tau(G)=\prod_{i=1}^{k}\tau(G_i). \tag{7}$$

Since $v(G)$, $e(G)$, $R_1(G)$, $R_2(G)$, $\bar{\chi}(G)$ and $\tau(G)$ are determined by $h(G,\mu)$, we have

Lemma 2.2. *For any graph G, the parameters $v(G)$, $e(G)$, $R_1(G)$, $R_2(G)$, $\bar{\chi}(G)$ and $\tau(G)$ are invariants for graphs in $[G]_h$.*

But the following lemma shows that the number of components of a graph is not an invariant for adjointly equivalent graphs.

Lemma 2.3. (i) $h(P_4,\mu)=h(K_1\cup K_3,\mu)$;

(ii) $h(P_{2n+1},\mu)=h(P_n\cup C_{n+1},\mu)$ *for any* $n\geqslant 3$;
(iii) $h(Y_n,\mu)=h(K_1\cup C_{n-1},\mu)$ *for any* $n\geqslant 5$;
(iv) $h(C_4,\mu)=h(D_4,\mu)$.

Proof. (i) Observe that $h(P_4,\mu)=\mu^4+3\mu^3+\mu^2=\mu h(C_3,\mu)$.

(ii) By Theorem 2.1, for $n\geqslant 3$,

$$h(P_{2n+1},\mu)=h(P_n,\mu)h(P_{n+1},\mu)+\mu h(P_n,\mu)h(P_{n-1},\mu)$$

and

$$h(C_{n+1},\mu)=h(P_{n+1},\mu)+\mu h(P_{n-1},\mu).$$

The result then follows.

(iii) By Theorem 2.1, for $n\geqslant 5$,

$$h(Y_n,\mu)=\mu h(P_{n-1},\mu)+\mu^2 h(P_{n-3},\mu),$$

and

$$h(C_{n-1},\mu)=h(P_{n-1},\mu)+\mu h(P_{n-3},\mu).$$

The result follows.

(iv) Observe that $h(C_4,\mu)=h(D_4,\mu)=\mu^4+4\mu^3+2\mu^2$. □

By Lemma 2.3, we have

Theorem 2.2. *A graph is not adjointly unique if either one of its components or the union of two of its components is included in the set*

$$\{P_4,C_4,D_4\}\cup\{P_{2n+1},Y_{n+2},P_n\cup C_{n+1},K_1\cup C_n\mid n\geqslant 3\}.$$

In the following subsections, we shall focus on determining the adjoint equivalence class of each graph in the sets $\mathscr{S}_1$ and $\mathscr{S}_2$, where

$$\mathscr{S}_1=\left\{r_0K_1\cup r_1K_3\cup\bigcup_{1\leqslant i\leqslant s}P_{2l_i}\mid r_0,r_1\geqslant 0,\ s\geqslant 0,\ l_i\geqslant 1\right\},\quad\text{and}$$

$$\mathscr{S}_2=\left\{aK_3\cup bD_4\cup\bigcup_{1\leqslant i\leqslant s}P_{u_i}\cup\bigcup_{1\leqslant j\leqslant t}C_{v_j}\mid a,b\geqslant 0,\ u_i\geqslant 3,\ u_i\not\equiv 4\,(\mathrm{mod}\,5),\ v_j\geqslant 4\right\}.$$

2.1. The family $\mathscr{S}_1$

There are some known results on the adjoint uniqueness of graphs in $\mathscr{S}_1$, but the problem of determining all adjointly unique graphs in $\mathscr{S}_1$ has not hitherto been settled.

Theorem 2.3 (Du [3]). *The graph* $\bigcup_{i=1}^{s}P_{n_i}\cup lC_3$, *where each* n_i *is even and* $n_i\not\equiv 4$ (mod 10), *is adjointly unique.*

By Theorem 2.2, P_n is not adjointly unique for $n=4$ or odd $n\geqslant 7$. Is P_n adjointly unique for any even n with $n\neq 4$? Liu [13] proposed the following conjecture.

Conjecture. For each even number $n\neq 4$, P_n is adjointly unique.

Du's result partially proved the above conjecture. In this section, we shall determine $[G]_h$ for every $G\in\mathscr{S}_1$, and hence all adjointly unique graphs in $\mathscr{S}_1$. Consequently, we prove the above conjecture.

We shall prove that $h(G,\mu)\neq h(H,\mu)$ for any graphs $G\in\mathscr{S}_1$ and $H\notin\mathscr{S}_1$. For this purpose, we introduce the concept of being adjointly closed. A set $\mathscr{S}$ of graphs is said to be *adjointly closed* if $[G]_h \subseteq \mathscr{S}$ for any graph $G \in \mathscr{S}$. Thus, if $\mathscr{S}$ is adjointly closed, then $\mathscr{S} = \bigcup_{G\in\mathscr{S}} [G]_h$. In the following, we first prove the set $\mathscr{S}_1$ is adjointly closed, then determine the adjoint equivalence class for each graph in $\mathscr{S}_1$, and finally find all adjointly unique graphs in $\mathscr{S}_1$.

Lemma 2.4. *For any connected graph G with $v(G) \geqslant 2$, if $\tau(G)=1$, then $\bar{\chi}(G) \leqslant v(G)/2$.*

Proof. Let $\bar{\chi}(G)=k$. If $k=1$, the result holds. Let $k\geqslant 2$. Since $\tau(G)=N(G,k)=1$, G has only one clique cover H with k components, say $H_1, H_2, \ldots, H_k$. We shall show that $v(H_i)\geqslant 2$ for all $i=1,2,\ldots,k$. Suppose that $V(H_1)=\{x\}$. Let $y\in N_G(x)$, say $y \in V(H_2)$, and let $H_0 = G[\{x,y\}]$. If $V(H_2)=\{y\}$, then G has a clique cover H' with $k-1$ components H_0, $H_3, H_4, \ldots, H_k$, which contradicts the fact that $\bar{\chi}(G)=k$. Otherwise, we have another clique cover with k components $H_0, H_2-y, H_3, H_4, \ldots, H_k$, which contradicts the fact that $\tau(G)=1$. Hence $v(H_i)\geqslant 2$ for all $i=1,2,\ldots,k$, which implies that $k \leqslant v(G)/2$. □

Lemma 2.5. *For any connected graph G, if $\tau(G)=1$, then*

$$e(G)+R_1(G)+2\bar{\chi}(G) \leqslant 2v(G), \tag{8}$$

where equality holds iff $G \in \{K_1, K_3\} \cup \{P_{2i} \mid i \geqslant 1\}$.

Proof. Observe that the equality of (8) holds for any graph G in the set $\{K_1, K_3\} \cup \{P_{2i} \mid i \geqslant 1\}$. Now let G be a connected graph and $G \notin \{K_1, K_3\} \cup \{P_{2i} \mid i \geqslant 1\}$. It remains to show that the strict inequality holds for G.

If $G=K_4$, then

$$e(G)+R_1(G)+2\bar{\chi}(G) = 6-2+2 = 6 < 2v(G).$$

Now let $G \notin \{K_1, K_3, K_4\} \cup \{P_{2i} \mid i \geqslant 1\}$. By Theorem 1.3, we have

$$e(G)+R_1(G) \leqslant v(G).$$

Since $\tau(G)=1$, by Lemma 2.4, $\bar{\chi}(G) \leqslant v(G)/2$. Thus

$$e(G)+R_1(G)+2\bar{\chi}(G) \leqslant 2v(G).$$

Now suppose that the above equality holds. This occurs iff

$$e(G)+R_1(G) = v(G),$$
$$\bar{\chi}(G) = v(G)/2.$$

Since $e(G)+R_1(G)=v(G)$, again by Theorem 1.3, $R_1(G) \geqslant -1$. By Theorem 1.2,

$$G \in \{P_n, C_n, D_n, F_n\} \cup \{K_4 - e\},$$

where $n=v(G)$. But $\tau(P_n)>1$ when $n\geqslant 3$ and n is odd, $\tau(C_n)>1$ when $n\geqslant 4$, either $\tau(D_n)>1$ (when n is even) or $\bar{\chi}(D_n)<n/2$ (when n is odd), $\bar{\chi}(F_n)<n/2$ and $\tau(K_4-e)>1$, a contradiction. □

By Lemma 2.5, we find a necessary and sufficient condition for a graph to be in $\mathscr{S}_1$.

Theorem 2.4. *For any graph G, $G\in\mathscr{S}_1$ iff $\tau(G)=1$ and*

$$e(G)+R_1(G)+2\bar{\chi}(G)=2v(G).$$

Proof. It is straightforward to check the necessity. Now let $\tau(G)=1$ and $e(G)+R_1(G)+2\bar{\chi}(G)=2v(G)$. Let $G_1,G_2,\ldots,G_k$ be the components of G. By Lemma 2.1, $\tau(G_j)=1$ for all j as $\tau(G)=1$, and $\bar{\chi}(G)=\sum_{j=1}^{k}\bar{\chi}(G_j)$. Thus by Lemma 2.5,

$$e(G)+R_1(G)+2\bar{\chi}(G)=\sum_{j=1}^{k}(e(G_j)+R_1(G_j)+2\bar{\chi}(G_j))\leqslant\sum_{j=1}^{k}2v(G_j)=2v(G).$$

Since $e(G)+R_1(G)+2\bar{\chi}(G)=2v(G)$, we have $e(G_j)+R_1(G_j)+2\bar{\chi}(G_j)=2v(G_j)$ for each j. By Lemma 2.5 again, we have $G_j\in\{K_1,K_3\}\cup\{P_{2i}\,|\,i\geqslant 1\}$ for all j. Hence $G\in\mathscr{S}_1$. □

By Lemma 2.2, $\tau(G)$, $v(G)$, $e(G)$, $R_1(G)$ and $\bar{\chi}(G)$ are invariants for graphs in $[G]_h$. Thus by Theorem 2.4, we have $[G]_h\subseteq\mathscr{S}_1$ if $G\in\mathscr{S}_1$.

Theorem 2.5. *The set $\mathscr{S}_1$ is adjointly closed.*

Since $\mathscr{S}_1$ is adjointly closed, we need only search graphs within $\mathscr{S}_1$ for determining $[G]_h$ if $G\in\mathscr{S}_1$.

Theorem 2.6. *For any nonnegative integers s, r_0, r_1 and a_i, $i=1,\ldots,s$, if $G=r_0K_1\cup r_1K_3\cup\bigcup_{i=1}^{s}a_iP_{2i}$, then*

$$[G]_h=\left\{(r_0-a)K_1\cup(r_1-a)K_3\cup(a_2+a)P_4\cup\bigcup_{\substack{1\leqslant i\leqslant s\\ i\neq 2}}a_iP_{2i}\,\middle|\,-a_2\leqslant a\leqslant\min\{r_0,r_1\}\right\}.$$

Proof. Let $\mathscr{G}$ denote the set of graphs in the right-hand side of the theorem. Since

$$h(P_4,\mu)=\mu h(K_3,\mu)=h(K_1,\mu)h(K_3,\mu),$$

we have $\mathscr{G}\subseteq[G]_h$.

Observe that

$$h(G,\mu)=\mu^{r_0-r_1}(h(P_4,\mu))^{r_1+a_2}\prod_{\substack{1\leqslant i\leqslant s\\ i\neq 2}}(h(P_{2i},\mu))^{a_i}.$$

By Theorem 1.5, $h(G,\mu)$ has exactly a_i repeated zeros $-2-2\cos(2\pi/(2i+1))$ for $1\leqslant i\leqslant s$, $i\neq 2$, and has r_1+a_2 repeated zeros $-2-2\cos 2\pi/5$.

Now let H be a graph in $[G]_h$. By Theorem 2.5, $H\in\mathscr{S}_1$. Thus

$$H\cong r_0'K_1\cup r_1'K_3\cup\bigcup_{i=1}^{\infty}a_i'P_{2i},$$

for some nonnegative integers r_0', r_1' and a_i', $i=1,2,\ldots$, where $\sum_{i\geqslant 1}a_i'$ is finite.

By comparing the zeros of $h(H,\mu)$ and $h(G,\mu)$, we see that

$$a_1'=a_1,\ a_2'+r_1'=a_2+r_1,\quad a_i'=a_i \text{ for } i=3,4,\ldots,s, \text{ and } a_i'=0 \text{ for } i\geqslant s+1.$$

Let $a=r_1-r_1'$. Then $a_2'=a_2+a$ and $r_1'=r_1-a$. By considering the order of H, we have

$$r_0+3r_1+\sum_{i=1}^{s}2ia_i=r_0'+3r_1'+\sum_{i=1}^{s}2ia_i',$$

which implies that $r_0'=r_0-a$. Since a_2',r_1' and r_0' are nonnegative, we have $-a_2\leqslant a\leqslant \min\{r_0,r_1\}$. Thus $H\in\mathscr{G}$ and therefore $[G]_h\subseteq\mathscr{G}$. □

Theorem 2.7. *For nonnegative integers $r_0,r_1,s,a_1,a_2,\ldots,a_s$, the graph*

$$r_0K_1\cup r_1K_3\cup\bigcup_{i=1}^{s}a_iP_{2i}$$

is adjointly unique iff $r_0r_1+a_2=0$.

Proof. Let $G=r_0K_1\cup r_1K_3\cup\bigcup_{i=1}^{s}a_iP_{2i}$. Recall that G is adjointly unique iff $[G]_h=\{G\}$. By Theorem 2.6, $[G]_h=\{G\}$ iff $a_2=0$ and $r_0r_1=0$.

As a consequence of Theorem 2.7, we now know which paths are adjointly unique. In fact, a more general result is obtained.

Theorem 2.8. *For any integers n and r with $n\geqslant 1$ and $r\geqslant 0$, $rK_1\cup P_n$ is adjointly unique iff $n\in\{1,2,3,5\}\cup\{l\geqslant 6\mid l \text{ is even}\}$ and $r=0$ when $n=5$.*

Proof. It is easy to verify that $rK_1\cup P_n$ is adjointly unique when $1\leqslant n\leqslant 3$. By Theorem 2.2, $rK_1\cup P_n$ is not adjointly unique for $n=4$ or odd $n\geqslant 7$. By Theorem 2.7, $rK_1\cup P_n$ is adjointly unique for even $n\geqslant 6$. It remains to show that $rK_1\cup P_5$ is adjointly unique iff $r=0$.

If $r\geqslant 1$, we have

$$h(rK_1\cup P_5)=h((r-1)K_1\cup T_{1,1,1}\cup P_2).$$

Thus $rK_1\cup P_5$ is not adjointly unique if $r\geqslant 1$.

Let G be a graph with $h(G,\mu)=h(P_5,\mu)$. Then $R_1(G)=1$ by Lemma 2.2 and Theorem 1.2. By Lemma 1.3 and Theorem 1.2, G has a component, say G_0, such that

$R_1(G_0)=1$. Again by Theorem 1.2, either $G_0 \cong K_3$ or $G_0 \cong P_t$ for some $2 \leqslant t \leqslant 5$. Since $h(G_0,\mu)$ is a factor of $h(G,\mu)=h(P_5,\mu)$, by Theorem 1.5, either $G_0 \cong P_2$ or $G_0 \cong P_5$. If $G_0 \cong P_5$, then $G \cong P_5$. Now suppose that $G_0 \cong P_2$. Observe that

$$h(P_5,\mu)=\mu^5+4\mu^4+3\mu^3=(\mu^3+3\mu^2)(\mu^2+\mu)=(\mu^3+3\mu^2)h(P_2,\mu).$$

Let G_1 be the graph $G[V(G)-V(G_0)]$. We have

$$h(G,\mu)=h(G_0,\mu)h(G_1,\mu).$$

Thus

$$h(G_1,\mu)=\mu^3+3\mu^2.$$

Therefore the graph G_1 is of order 3 and size 3, which implies that $G_1 \cong K_3$. But

$$h(K_3,\mu)=\mu^3+3\mu^2+\mu,$$

a contradiction. Hence $G \cong P_5$ and P_5 is adjointly unique. □

Thus, the graph $rK_1 \cup P_5$ is not adjointly unique when $r \geqslant 1$. The following result gives its adjoint equivalence class.

Theorem 2.9. *For any positive integer r, the adjoint equivalence class of $rK_1 \cup P_5$ is*

$$\{rK_1 \cup P_5, (r-1)K_1 \cup T_{1,1,1} \cup P_2\}.$$

Proof. We have verify that $(r-1)K_1 \cup T_{1,1,1} \cup P_2 \sim_h rK_1 \cup P_5$. Let G be a graph with $G \sim_h rK_1 \cup P_5$. It suffices to show that $G \in \{rK_1 \cup P_5, (r-1)K_1 \cup T_{1,1,1} \cup P_2\}$.

Observe that $R_1(G)=R_1(rK_1 \cup P_5)=1$ by Lemmas 2.2 and 1.3. Thus by Lemma 1.3 and Theorem 1.2, G has a component, say G_0, such that $R_1(G_0)=1$. Again by Theorem 1.2, either $G_0 \cong K_3$ or $G_0 \cong P_t$ for some $2 \leqslant t \leqslant 5$. Since $h(G_0,\mu)$ is a factor of $h(G,\mu)=h(rK_1 \cup P_5,\mu)$, by Theorem 1.5, either $G_0 \cong P_2$ or $G_0 \cong P_5$. If $G_0 \cong P_5$, then we have $G \cong rK_1 \cup P_5$. Now assume that $G_0 \cong P_2$. Let $G_1 = G[V(G)-V(G_0)]$. Then G_1 is a graph of size 3. Thus

$$G_1 = r_0K_1 \cup r_1P_2 \cup r_2P_3 \cup r_3P_4 \cup r_4K_3 \cup r_5 T_{1,1,1}$$

for some non-negative integers $r_0, r_1, r_2, r_3, r_4, r_5$. Since $R_1(G_1)=R_1(G)-R_1(G_0)=0$, we have

$$0=R_1(G_1)=r_1+r_2+r_3+r_4,$$

implying that $r_i=0$ for $i=1,2,3,4$. Hence $G_1=r_0K_1 \cup r_5T_{1,1,1}$. Since G_1 is of size 3 and order $r+3$, we have $r_5=1$ and $r_0=r-1$. Therefore $G \cong (r-1)K_1 \cup T_{1,1,1} \cup P_2$. □

2.2. The family $\mathscr{S}_2$

We need two results before we can prove that $\mathscr{S}_2$ is adjointly closed. For any graph G, let $c_3(G)$ be the number of components of G which are isomorphic to K_3.

Lemma 2.6. *For any graph G,*

$$e(G)+R_1(G)\leqslant v(G)+c_3(G),$$

where equality holds iff G_i in $\{K_4-e,K_4\}\cup\{P_n,C_{n+1},D_{n+2},F_{n+4}\mid n\geqslant 2\}$ for each component G_i of G.

Proof. Let H be a connected graph. By Theorems 1.2 and 1.3, we have $e(H)+R_1(H)\leqslant v(H)+1$, and further

(i) $e(H)+R_1(H)=v(H)+1$ iff $H\cong K_3$;
(ii) $e(H)+R_1(H)=v(H)$ iff

$$H\in\{K_4-e,K_4\}\cup\{P_n,C_{n+2},D_{n+2},F_{n+4}\mid n\geqslant 2\}.$$

Let $G_1,G_2,\ldots,G_k$ be the components of G. Since $e(G)=\sum_{i=1}^k e(G_i)$, $R_1(G)=\sum_{i=1}^k R_1(G_i)$ and $v(G)=\sum_{i=1}^k v(G_i)$, the result then follows. □

Lemma 2.7. *If $e(G)+R_1(G)=v(G)+c_3(G)$, then*

$$2R_1(G)+R_2(G)\geqslant 0,$$

where equality holds iff $G_i\in\{D_4\}\cup\{P_n,C_n\mid n\geqslant 3\}$ for each component G_i of G.

Proof. Let G_i be any component of G. Since $e(G)+R_1(G)=v(G)+c_3(G)$, by Lemma 2.6,

$$G_i\in\{K_4-e,K_4\}\cup\{P_n,C_{n+1},D_{n+2},F_{n+4}\mid n\geqslant 2\}.$$

By Theorem 1.2 and Lemma 1.4 and the fact that $R_1(K_4)=-2$,

$$2R_1(G_i)+R_2(G_i)\begin{cases}=0 & \text{if } G_i\in\{D_4\}\cup\{P_n,C_n\mid n\geqslant 3\},\\ =1 & \text{if } G_i\in\{P_2,K_4-e\}\cup\{D_n\mid n\geqslant 5\},\\ >1 & \text{if } G_i\in\{K_4\}\cup\{F_n\mid n\geqslant 6\}.\end{cases}$$

Thus, $2R_1(G)+R_2(G)\geqslant 0$, where equality implies that $G_i\in\{D_4\}\cup\{P_n,C_n\mid n\geqslant 3\}$ for each component G_i of G. □

Now we can prove that $\mathscr{S}_2$ is adjointly closed. In fact, the set $\mathscr{S}_2$ can be partitioned into smaller adjointly closed sets. For any nonnegative integer a, let

$$\mathscr{S}_2(a)=\{G\in\mathscr{S}_2\mid c_3(G)=a\}.$$

Observe that

$$\mathscr{S}_2 = \bigcup_{a \geqslant 0} \mathscr{S}_2(a).$$

Theorem 2.10. *For any nonnegative integer* a, $\mathscr{S}_2(a)$ *is adjointly closed.*

Proof. Let $G \in \mathscr{S}_2(a)$, i.e.,

$$G = aK_3 \cup bD_4 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \cup \bigcup_{1 \leqslant j \leqslant t} C_{v_j},$$

for nonnegative integers a, b, u_i and v_j with $u_i \geqslant 3$, $u_i \not\equiv 4 \pmod 5$ and $v_j \geqslant 4$. Let H be any graph such that $h(H,\mu) = h(G,\mu)$. It suffices to show that $H \in \mathscr{S}_2(a)$. Since $h(K_3,\mu) = h(P_4,\mu)/\mu$ if $\mu \neq 0$, by Theorem 1.5, $-2 - 2\cos 4\pi/5$ is a zero of $h(K_3,\mu)$. But it is not a zero of $h(P_{u_i},\mu)$ when $u_i \not\equiv 4 \pmod 5$ by Theorem 1.5, and it is also not a zero of $h(C_{v_i},\mu)$ by Theorem 1.6. As $h(D_4,\mu) = h(C_4,\mu)$ by Lemma 2.3(iv), $-2 - 2\cos 4\pi/5$ is not a zero of $h(D_4,\mu)$. Hence the zero $-2 - 2\cos 4\pi/5$ of $h(G,\mu)$ is of multiplicity a, which implies that $c_3(H) \leqslant a$. Now let

$$H = a'K_3 \cup H',$$

where $a' \leqslant a$ and $c_3(H') = 0$. Then $h(H',\mu) = h(G',\mu)$, where

$$G' = (a - a')K_3 \cup bD_4 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \cup \bigcup_{1 \leqslant j \leqslant t} C_{v_j}.$$

Observe that $e(G') + R_1(G') = v(G') + a - a'$ by Lemma 2.6. Thus, we have

$$e(H') + R_1(H') = e(G') + R_1(G') = v(G') + a - a' = v(H') + a - a'.$$

Further,

$$\begin{aligned} e(H) + R_1(H) &= a'(e(K_3) + R_1(K_3)) + (e(H') + R_1(H')) \\ &= 4a' + v(H') + a - a' = v(H') + a + 3a' \\ &= v(H) + a \geqslant v(H) + a' = v(H) + c_3(H). \end{aligned}$$

By Lemma 2.6, we have $e(H) + R_1(H) \leqslant v(H) + c_3(H)$. Hence, $a = a'$ and $e(H) + R_1(H) = v(H) + c_3(H)$.

Observe that $2R_1(G) + R_2(G) = 0$ by Lemmas 2.6 and 2.7. Thus

$$2R_1(H) + R_2(H) = 2R_1(G) + R_2(G) = 0.$$

By Lemma 2.7, each component of H is a graph in the set

$$\{D_4\} \cup \{P_n, C_n \mid n \geqslant 3\}.$$

Assume that

$$H = aK_3 \cup b'D_4 \cup \bigcup_{1 \leqslant i \leqslant s'} P_{u'_i} \cup \bigcup_{1 \leqslant j \leqslant t'} C_{v'_j},$$

where $u_i' \geqslant 3$ and $v_j' \geqslant 4$. If $u_i' \equiv 4 \pmod 5$ for some i, then by Theorem 1.5, $-2-2\cos 4\pi/5$ is a zero of $h(P_{u_i'}, \mu)$, which implies that the zero $-2-2\cos 4\pi/5$ of $h(H,\mu)$ is of multiplicity more than a, a contradiction. Hence $u_i' \not\equiv 4 \pmod 5$ for $i=1,2,\ldots,s'$. Therefore, $H \in \mathscr{S}_2(a)$. $\square$

Corollary. *The set $\mathscr{S}_2$ is adjointly closed.*

Now we are ready to determine the adjoint equivalence class of each graph $G \in \mathscr{S}_2$, and all adjointly unique graphs in $\mathscr{S}_2$. For this purpose, we need the following two results. Let

$$\mathscr{S}_3 = \left\{ aK_3 \cup \bigcup_{1\leqslant i\leqslant s} P_{u_i} \cup \bigcup_{1\leqslant j\leqslant t} C_{v_j} \mid a \geqslant 0, v_j \geqslant 4, u_i \geqslant 2, u_i \text{ is even if } u_i \geqslant 6 \right\}.$$

Lemma 2.8. *Let $G, H \in \mathscr{S}_3$. If $h(G,\mu)=h(H,\mu)$, then $G \cong H$.*

Proof. Suppose that the result is not true. Then there exists a counterexample. Let $G, H \in \mathscr{S}_3$, where

$$G = aK_3 \cup \bigcup_{1\leqslant i\leqslant s} P_{u_i} \cup \bigcup_{1\leqslant j\leqslant t} C_{v_j},$$
$$H = a'K_3 \cup \bigcup_{1\leqslant i\leqslant s'} P_{u_i'} \cup \bigcup_{1\leqslant j\leqslant t'} C_{v_j'}.$$

Suppose that $h(G,\mu)=h(H,\mu)$, but $G \not\cong H$. We may also assume that the result holds for graphs $G', H' \in \mathscr{S}_3$ if $v(G') < v(G)$. Without loss of generality, let

$$u_1 \leqslant u_2 \leqslant \cdots \leqslant u_s,$$
$$u_1' \leqslant u_2' \leqslant \cdots \leqslant u_{s'}',$$
$$v_1 \leqslant v_2 \leqslant \cdots \leqslant v_t,$$
$$v_1' \leqslant v_2' \leqslant \cdots \leqslant v_{t'}'.$$

We shall show that $a=a'=s=s'=t=t'=0$.

Claim 1. $aa'=0$, $\{u_1,u_2,\ldots,u_s\} \cap \{u_1',u_2',\ldots,u_{s'}'\} = \emptyset$, *and* $\{v_1,v_2,\ldots,v_t\} \cap \{v_1',v_2',\ldots,v_{t'}'\} = \emptyset$.

Claim 1 is equivalent to saying that G and H do not contain any isomorphic components. For example, assume that both G and H have components isomorphic to K_3. Let G' be the graph obtained from G by deleting a component isomorphic to K_3, and let H' be the graph obtained from H by deleting a component isomorphic to K_3. Observe that $G', H' \in \mathscr{S}_3$ and $h(G',\mu)=h(H',\mu)$. Since $v(G') < v(G)$, we have $G' \cong H'$ by assumption, which implies that $G \cong H$, a contradiction.

Claim 2. $a=a'=0$.

By Lemma 2.6, $a = c_3(G) = e(G) + R_1(G) - v(G)$ and $a' = e(H) + R_1(H) - v(H)$. By Lemma 2.2, we have $a = a'$. Since $aa' = 0$, the claim holds.

Claim 3. $t = t' = 0$.

Suppose that the claim is not true. Then without loss of generality, we may assume that $t \geqslant 1$ and $v_t > v'_{t'}$ if $t' \geqslant 1$. By Theorem 1.6, $h(G,\mu)$ has a zero:

$$\alpha = -2 - 2\cos \pi / v_t.$$

Since $v'_i < v_t$ for $1 \leqslant i \leqslant t'$, by Theorem 1.6, α is not a zero of $h(C_{v'_i},\mu)$. Since u'_i is even when $u'_i \geqslant 6$, by Theorem 1.5, α is not a zero of $h(P_{u'_i},\mu)$. Hence α is not a zero of $h(H,\mu)$, a contradiction. Thus, the claim holds.

Claim 4. $s = s' = 0$.

By Claim 2, we have $s = R_1(G) = R_1(H) = s'$. Suppose that $s = s' > 0$ and $u_s > u'_s$. By Theorem 1.5, $h(G,\mu)$ contains the following zero:

$$\beta = -2 - 2\cos(2\pi/(u_s + 1)).$$

Since $u'_i < u_s$ for $i = 1, 2, \ldots, s'$, by Theorem 1.5, β is not a zero of $h(P_{u'_i},\mu)$. Thus β is not a zero of $h(H,\mu)$, a contradiction.

For any graph $G \in \mathscr{S}_2$, we construct a graph $\tilde{G}$ from G by the following operations until none of the components is isomorphic to P_{2n+1} ($n \geqslant 3$) or D_4:

(i) Replace each component P_{2n+1}, where $n \geqslant 3$, by two components P_n and C_{n+1};
(ii) Replace each component D_4 by C_4.

Lemma 2.9. *For any graph $G \in \mathscr{S}_2$, we have*

(i) $h(G,\mu) = h(\tilde{G},\mu)$; *and*
(ii) $\tilde{G} \in \mathscr{S}_2 \cap \mathscr{S}_3$.

Proof. Since $h(P_{2n+1},\mu) = h(P_n \cup C_{n+1},\mu)$ for $n \geqslant 3$ by Lemma 2.3(ii) and $h(D_4,\mu) = h(C_4,\mu)$, we have $h(G,\mu) = h(\tilde{G},\mu)$ by the definition of $\tilde{G}$.

It is clear that $\tilde{G} \in \mathscr{S}_3$. To show that $\tilde{G} \in \mathscr{S}_2$, it suffices to prove that $\tilde{G}$ does not contain any component P_n with $n = 2$ or $n \equiv 4 \pmod 5$. By definition, $\tilde{G}$ contains a component P_2 iff G does. Notice that if $n \equiv 4 \pmod 5$, then $2n + 1 \equiv 4 \pmod 5$. By the definition of $\tilde{G}$, $\tilde{G}$ does not contain any such component P_n since $G \in \mathscr{S}_2$. □

Now we give a necessary and sufficient condition for two graphs $G, H \in \mathscr{S}_2$ to be adjointly equivalent.

Theorem 2.11. *For any graphs $G, H \in \mathscr{S}_2$, $h(G,\mu) = h(H,\mu)$ iff $\tilde{G} \cong \tilde{H}$.*

Proof. By Lemma 2.9, the sufficiency holds. Now let $h(G,\mu)=h(H,\mu)$. By Lemma 2.9, $\tilde{G},\tilde{H}\in\mathscr{S}_3$ and $h(\tilde{G},\mu)=h(\tilde{H},\mu)$. Then by Lemma 2.8, $\tilde{G}\cong\tilde{H}$. □

Corollary. *For any graph* $G\in\mathscr{S}_2$,

$$[G]_h=\{H\in\mathscr{S}_2\mid\tilde{H}\cong\tilde{G}\}.$$

There is a numerical method to judge whether $G\sim_h H$ for two graphs G and H in $\mathscr{S}_2$. For a graph $G\in\mathscr{S}_2$, there are nonnegative integers a,b,n_i,m_i, with $n_i=0$ if $i\equiv 4\,(\mathrm{mod}\,5)$ and $\sum_{i\geqslant 3}(n_i+m_{i+1})$ finite such that

$$G=aK_3\cup bD_4\cup\bigcup_{i\geqslant 3}n_iP_i\cup\bigcup_{i\geqslant 4}m_iC_i.$$

Then by the definition of $\tilde{G}$, we have

Lemma 2.10. $\tilde{G}=a'K_3\cup\bigcup_{i\geqslant 3}n_i'P_i\cup\bigcup_{i\geqslant 4}m_i'C_i$, *where* $a'=a$ *and*

$$n_i'=0,\quad i\in\{4\}\cup\{7,9,11,\ldots\},$$

$$n_i'=\sum_{k\geqslant 0}n_{2^k(i+1)-1},\quad i\in\{3,5\}\cup\{6,8,10,\ldots\},$$

$$m_i'=m_i+\sum_{k\geqslant 1}n_{2^ki-1},\quad i\geqslant 5,$$

$$m_4'=b+m_4+\sum_{k\geqslant 1}n_{2^{k+2}-1}.$$

Given two graphs G and H in $\mathscr{S}_2$, it is easy to check by Lemma 2.10 whether $\tilde{G}\cong\tilde{H}$, and then answer immediately the question whether $G\sim_h H$. However it is not easy to list all graphs in $[G]_h$.

Now we are in a position to find the adjointly unique graphs in $\mathscr{S}_2$. Let $G\in\mathscr{S}_2$, where

$$G=aK_3\cup bD_4\cup\bigcup_{1\leqslant i\leqslant s}P_{u_i}\cup\bigcup_{1\leqslant j\leqslant t}C_{v_j}.$$

By Theorem 2.2, G is not adjointly unique if

(i) $b\geqslant 1$; or
(ii) $v_j=4$ for some j; or
(iii) $u_i\geqslant 7$ and u_i is odd for some i; or
(iv) $v_j=u_i+1$ for some i and j.

We shall show that G is adjointly unique otherwise.

Theorem 2.12. *Let* $G \in \mathscr{S}_2(a)$, *i.e.*,

$$G = aK_3 \cup bD_4 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \cup \bigcup_{1 \leqslant j \leqslant t} C_{v_j},$$

for nonnegative integers a, b, u_i *and* v_j *with* $u_i \geqslant 3$, $u_i \not\equiv 4 \,(\mathrm{mod}\, 5)$ *and* $v_j \geqslant 4$. *Then* G *is adjointly unique iff* $b = 0$, $v_j \geqslant 5$, u_i *is even when* $u_i \geqslant 6$ *and*

$$\{u_1 + 1, u_2 + 1, \ldots, u_s + 1\} \cap \{v_1, \ldots, v_t\} = \emptyset.$$

Proof. The necessity is clearly true. We prove just the sufficiency. Let H be a graph with $h(H,\mu) = h(G,\mu)$. By Theorem 2.10, $H \in \mathscr{S}_2(a)$. By Theorem 2.11, $\tilde{H} \cong \tilde{G}$. By definition, we have $G \cong \tilde{G}$, which implies that $\tilde{H} \cong G$.

If H contains a component C_4 or D_4, then $\tilde{H}$ contains a component C_4 by the definition of $\tilde{H}$, a contradiction. Assume that H contains a component P_{2n+1}, where $n \geqslant 3$. From the definition of $\tilde{H}$, we see that $\tilde{H}$ and hence G contains a pair of components P_m and C_{m+1}, contradicting the condition that G satisfies. Hence H does not contain any P_{2n+1} as a component, where $n \geqslant 3$. Therefore $H \cong \tilde{H}$, which implies that $H \cong G$. □

In particular, we have

Corollary 1. *Any graph in the following set is adjointly unique*:

$$\left\{ \bigcup_{1 \leqslant i \leqslant t} C_{v_i} \,|\, t \geqslant 1,\ v_i = 3 \text{ or } v_i \geqslant 5 \right\}.$$

Corollary 2. *Any graph in the following set is adjointly unique*:

$$\left\{ aK_3 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \,|\, s \geqslant 0,\ u_i \geqslant 3, u_i \not\equiv 4 \,(\mathrm{mod}\, 5) \text{ and } u_i \text{ is even when } u_i \geqslant 6 \right\}.$$

Du [3] also obtained the result of Corollary 1. Liu and Bao [14] showed that $\bigcup_{1 \leqslant i \leqslant t} C_{v_i}$ is adjointly unique for $v_i \geqslant 5$.

2.3. Open problems

Let

$$\mathscr{S}_4 = \left\{ aK_3 \cup bD_4 \cup \bigcup_{1 \leqslant i \leqslant s} P_{u_i} \cup \bigcup_{1 \leqslant j \leqslant t} C_{v_j} \,|\, a, b \geqslant 0,\ u_i \geqslant 3,\ v_j \geqslant 4 \right\}.$$

Observe that $\mathscr{S}_2$ is a proper subset of $\mathscr{S}_4$.. Is $\mathscr{S}_4$ adjointly closed? It is clear that while $P_4 \in \mathscr{S}_4$, $K_1 \cup K_3 \notin \mathscr{S}_4$, but $K_1 \cup K_3 \sim_h P_4$ by Lemma 2.3(i). Thus $\mathscr{S}_4$ is not adjointly closed. Then a problem arises.

For a set $\mathscr{S}$ of graphs, let

$$\min_h(\mathscr{S}) = \bigcup_{G\in\mathscr{S}} [G]_h.$$

Observe that $\min_h(\mathscr{S})$ is adjointly closed and $\mathscr{S} \subseteq \min_h(\mathscr{S})$. The set $\mathscr{S}$ is adjointly closed iff $\mathscr{S} = \min_h(\mathscr{S})$. Moreover, for any set $\mathscr{G}$ of graphs, if $\mathscr{S} \subseteq \mathscr{G}$ and $\mathscr{G}$ is adjointly closed, then $\min_h(\mathscr{S}) \subseteq \mathscr{G}$. Hence, $\min_h(\mathscr{S})$ is called the *adjoint closure* of $\mathscr{S}$.

Problem. *Determine* $\min_h(\mathscr{S}_4)$.

Since $K_1 \cup K_3 \sim_h P_4$ by Lemma 2.3(i), we have

$$\min_h(\mathscr{S}_4) \supseteq \left\{ rK_1 \cup aK_3 \cup bD_4 \cup \bigcup_{1\leqslant i\leqslant s} P_{u_i} \cup \bigcup_{1\leqslant j\leqslant t} C_{v_j} \right. \left| r,a,b,s,t \geqslant 0,\ r \leqslant a,\ u_i \geqslant 3,\ v_j \geqslant 4 \right\}.$$

Since $K_1 \cup C_{n-1} \sim_h Y_n$ for $n \geqslant 5$ by Lemma 2.3(iii), we have

$$\min_h(\mathscr{S}_4) \supseteq \left\{ rK_1 \cup aK_3 \cup bD_4 \cup \bigcup_{1\leqslant i\leqslant m} Y_{r_i} \cup \bigcup_{1\leqslant i\leqslant s} P_{u_i} \cup \bigcup_{1\leqslant j\leqslant t} C_{v_j} \right. \left| r,a,b,s,t,m \geqslant 0,\ m+r \leqslant a,\ r_i \geqslant 5,\ u_i \geqslant 3,\ v_j \geqslant 4 \right\}. \quad (9)$$

Conjecture 1. The set equality (9) holds.

Let

$$\mathscr{S}_5 = \left\{ rK_1 \cup \bigcup_{1\leqslant j\leqslant t} C_{v_j} \,\middle|\, r,t \geqslant 0,\ v_j \geqslant 4 \right\}.$$

Since $D_4 \sim_h C_4$ and $K_1 \cup C_{n-1} \sim_h Y_n$ for $n \geqslant 5$ by Lemma 2.3, we have

$$\min_h(\mathscr{S}_5) \supseteq \left\{ rK_1 \cup bD_4 \cup \bigcup_{1\leqslant i\leqslant m} Y_{r_i} \right. \left. \cup \bigcup_{1\leqslant j\leqslant t} C_{v_j} \,\right|\, r,b,m,t \geqslant 0,\ r_i \geqslant 5,\ v_j \geqslant 4 \Bigg\}. \quad (10)$$

Conjecture 2. The set equality (10) holds.

Acknowledgements

The authors thank the referees for their helpful suggestions.

References

[1] F.M. Dong, K.L. Teo, C.H.C. Little, M.D. Hendy, Two invariants for adjointly equivalent graphs, Australasian J. Combinat. 25 (2002) 133–143.
[2] F.M. Dong, K.L. Teo, C.H.C. Little, M.D. Hendy, Zeros of adjoint polynomials of paths and cycles, Australasian J. Combinat. 25 (2002) 167–174.
[3] Q.Y. Du, Chromaticity of the complements of paths and cycles, Discrete Math. 162 (1996) 109–125.
[4] E.J. Farrell, The impact of F-polynomials in graph theory, Ann. Discrete Math. 55 (1993) 173–178.
[5] Z.Y. Guo, Y.J. Li, Chromatic uniqueness of complement of the cycles union, Kexue Tongbao 33 (1988) 1676 (in Chinese, English summary).
[6] R.R. Korfhage, σ-polynomials and graph coloring, J. Combin. Theory Ser. B 24 (1978) 137–153.
[7] N.Z. Li, E.G. Whitehead, S.J. Xu, Classification of chromatically unique graphs having quadratic σ-polynomials, J. Graph Theory 11 (1987) 169–176.
[8] R.Y. Liu, On chromatic polynomials of two classes of graphs, Kexue Tongbao 32 (1987) 1147–1148 (In Chinese, English summary).
[9] R.Y. Liu, A new method to find chromatic polynomial of graphs and its applications, Kexue Tongbao 32 (1987) 1508–1509 (In Chinese, English summary).
[10] R.Y. Liu, Adjoint polynomials of graphs, J. Qinghai Normal Univ. 8 (1990) 1–9.
[11] R.Y. Liu, Chromatic uniqueness of $K_n - E(kP_s \cup rP_t)$, J. Systems Sci. Math. Sci. 12 (1992) 207–214 (In Chinese, English summary).
[12] R.Y. Liu, Chromatic uniqueness of complement of the irreducible cycles union, Math. Appl. 7 (1994) 200–205.
[13] R.Y. Liu, Adjoint polynomials and chromatically unique graphs, Discrete Math. 172 (1997) 85–92.
[14] R.Y. Liu, X.W. Bao, Chromatic uniqueness of the complements of 2-regular graphs, Pure Appl. Math. Suppl. 9 (1993) 69–71.
[15] R.Y. Liu, J.F. Wang, On chromatic uniqueness of complement of union of cycles and paths, Theoret. Comput. Sci. 1 (1992) 112–126.
[16] R.Y. Liu, L.C. Zhao, A new method for proving chromatic uniqueness of graphs, Discrete Math. 171 (1997) 169–177.
[17] R.C. Read, W.T. Tutte, Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory III, Academic Press, New York, 1988, pp. 15–42.