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Bounds for mean colour numbers of graphs

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Abstract

Let $\mu(G)$ denote the mean colour number of a graph G. Mosca discovered some counterexamples which disproved a conjecture proposed by Bartels and Welsh that if H is a subgraph of G, then $\mu(H) \leqslant \mu(G)$. In this paper, we show that this conjecture holds under the condition that either G is a chordal graph or H is a graph which can be obtained from a tree by replacing a vertex by a clique. This result gives a method to find upper bounds and lower bounds for the mean colour number of any graph. We also prove that $\mu(G) < \mu(G \cup K_1)$ for an arbitrary graph G.

Keywords: Graph; Chromatic polynomial; Mean colour number

1. Introduction

In this paper, we just consider simple graphs. For a graph G, let $V(G)$, $E(G)$, $v(G)$ and $P(G, \lambda)$ denote the vertex set, edge set, order and chromatic polynomial of G, respectively. For any positive integer k, let $\alpha(G, k)$ denote the number of partitions of $V(G)$ into exactly k non-empty independent sets. Then

$$P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)(\lambda)_k, \tag{1}$$

where $(\lambda)_k = \lambda(\lambda - 1)\dots(\lambda - k + 1)$ and $n = v(G)$.

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Let G be a graph with $v(G) = n$. It is clear that there exist n-colourings of G. For any n-colouring Γ of G, let $l(\Gamma)$ denote the actual number of colours used. The *mean colour number* $\mu(G)$ of G, defined by Bartels and Welsh [1], is the average of $l(\Gamma)$'s over all n-colourings Γ. The number of n-colourings Γ of G with $l(\Gamma) = k$ is

$$\alpha(G,k)(n)_k. \tag{2}$$

Hence by the definition of $\mu(G)$, we have

$$\mu(G) = \frac{\sum_{k=1}^{n} k(n)_k \alpha(G,k)}{\sum_{k=1}^{n} (n)_k \alpha(G,k)}. \tag{3}$$

An expression of $\mu(G)$ in terms of the chromatic polynomials was obtained by Bartels and Welsh [1].

Theorem 1.1 (Bartels and Welsh [1]). *If $v(G) = n$, then*

$$\mu(G) = n\left(1 - \frac{P(G,n-1)}{P(G,n)}\right). \tag{4}$$

Theorem 1.1 shows that $\mu(G) \leqslant n$ where equality holds iff G is complete. For the empty graph O_n on n vertices, we have

$$\mu(O_n) = n\left(1 - \left(1 - \frac{1}{n}\right)^n\right).$$

Bartels and Welsh conjectured that $\mu(O_n)$ is a lower bound of $\mu(G)$ for any graph G on n vertices, and their conjecture was proved by Dong [3]. They also proposed a more general conjecture that if H is a spanning subgraph of G, then $\mu(G) \geqslant \mu(H)$. This conjecture is equivalent to another conjecture that $\mu(G) \geqslant \mu(G - xy)$ for any graph G and any edge xy in G, where $G - xy$ is the graph obtained from G by deleting an edge xy. But counterexamples have been found by Mosca [6].

For distinct vertices x and y in G, if $xy \notin E(G)$, let $G + xy$ be the graph obtained from G by adding a new edge xy; otherwise, let $G + xy = G$.

Theorem 1.2 (Mosca [6]). *Let G be the graph obtained from the complete graph K_{2m} by adding two new vertices u and v and adding new edges joining u to any m vertices in K_{2m} and joining v to the other m vertices in K_{2m}. If $m \geqslant 2$, then $\mu(G) > \mu(G + uv)$.*

Hence in general the following inequality is not true:

$$\mu(G) \geqslant \mu(H), \tag{5}$$

where H is a subgraph of G. But it is true for some special cases. It is obvious that (5) holds if G is complete. The results in [3] also showed that (5) holds if H is a spanning subgraph of G and H is either a tree or an empty graph. In this paper, some more general results regarding inequality (5) are obtained.

A cycle C in a graph G is called a *chordless cycle* if C is of length at least 4 and there are no edges joining two non-consecutive vertices on C. G is called a *chordal graph* if G contains no chordless cycles. A special family Ω of chordal graphs is defined below:

(a) $K_n \in \Omega$ for all n; and
(b) $H \in \Omega$ if there is a vertex w of degree 1 in H such that $H - w \in \Omega$, where $H - w$ is the graph obtained from H by deleting the vertex w.

In Sections 2 and 3 we prove that $\mu(H) \leqslant \mu(G)$ if G is a chordal graph and H is a subgraph of G, and the equality holds iff $H \cong G$. By this result, an upper bound for $\mu(H)$ in terms of $v(H)$ and its tree width $tw(H)$ is given. In Section 4, we prove that $\mu(H) \leqslant \mu(G)$ if H is a subgraph of G and $H \in \Omega$, where the equality holds iff $H \cong G$. Motivated by the results in this paper, we propose some problems on mean colour numbers in Section 5.

For two disjoint graphs G and H, let $G \cup H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup V(H)$. For any graph H and positive integer m, let $H \cup mK_1$ denote the graph obtained from H by adding m new vertices and no new edges.

2. Upper bounds

From Theorem 1.1, we observe that for graphs G and H with n vertices, $\mu(G) \geqslant \mu(H)$ iff

$$P(G,n)/P(G,n-1) \geqslant P(H,n)/P(H,n-1) \tag{6}$$

or equivalently

$$P(G,n)P(H,n-1) \geqslant P(G,n-1)P(H,n). \tag{7}$$

For graphs G, H and real number λ, define

$$\tau(G,H,\lambda) = P(G,\lambda)P(H,\lambda-1) - P(G,\lambda-1)P(H,\lambda). \tag{8}$$

Lemma 2.1. *For any graphs G and H with $v(G) = v(H) = n$, the inequality $\mu(G) \geqslant \mu(H)$ is equivalent to $\tau(G,H,n) \geqslant 0$.*

In this section, we first prove that $\tau(G,H,\lambda) \geqslant 0$ for $\lambda \geqslant v(G)$ if H is any subgraph of a chordal graph G. Then by Lemma 2.1, it follows that $\mu(H) \leqslant \mu(G)$ if G is a chordal graph and H is a spanning subgraph of G.

We need to introduce some other results on chordal graphs. For any edge $e = xy$ in G, let $G \cdot e$ (or $G \cdot xy$) denote the graph obtained from G by contracting x and y and replacing multi-edges by single ones. The next result follows directly from the definition of a chordal graph.

Lemma 2.2. *If G is a chordal graph and e is any edge in G, then $G \cdot e$ is also a chordal graph.*

For any vertex x in G, let $N_G(x)$ (or simply $N(x)$) denote the set of vertices in G which are adjacent to x, and let $d_G(x)$ (or simply $d(x)$) denote the degree of x in G, i.e., $d_G(x) = |N_G(x)|$. The vertex x is called a *simplicial vertex* of G if either $d(x) = 0$ or $G[N(x)]$ is a clique, where $G[S]$ denotes the subgraph of G induced by S for any subset S of $V(G)$. If x is a simplicial vertex of G, then

$$P(G,\lambda) = (\lambda - d(x))P(G - x, \lambda). \tag{9}$$

Note that any chordal graph contains a simplicial vertex (in fact, any non-complete chordal graph contains two non-adjacent simplicial vertices) and any subgraph of a chordal graph obtained by deleting a vertex is still a chordal graph. Hence the chromatic polynomial of a chordal graph contains only non-negative integral roots which are less than the order of the graph.

Lemma 2.3. *Let G be a non-empty chordal graph and e be any edge in G. Assume that*

$$P(G \cdot e, \lambda) = \prod_{i=1}^{n-1} (\lambda - b_i), \tag{10}$$

where $n = v(G)$ and b_i is a non-negative integer. Then the chromatic polynomial of G can be expressed as

$$P(G,\lambda) = \lambda \prod_{i=1}^{n-1} (\lambda - a_i), \tag{11}$$

where $a_i \in \{b_i, b_i + 1\}$ for $i = 1, 2, \ldots, n-1$.

Proof. The result is obvious if G is a complete graph. Thus it holds for $n = 2$. Now assume that $n \geqslant 3$ and that the result is true for all chordal graphs with order less than n.

Let G be a non-empty chordal graph of order n and e be any edge in G. Just consider the case that G is not complete. Then G contains two non-adjacent simplicial vertices. So G has a simplicial vertex, denoted by u, such that $u \notin \{x, y\}$, where x and y are the end-vertices of e.

It is clear that u is a simplicial vertex of $G \cdot e$ too. Then $d_{G\cdot e}(u) = d_G(u) - 1$ if $x, y \in N_G(u)$ and $d_{G\cdot e}(u) = d_G(u)$ otherwise. By (9),

$$P(G,\lambda) = (\lambda - d_G(u))P(G - u, \lambda)$$

and

$$P(G \cdot e, \lambda) = (\lambda - d_{G\cdot e}(u))P(G \cdot e - u, \lambda).$$

Observe that $G \cdot e - u = (G - u) \cdot e$ and e is an edge in the chordal graph $G - u$. By inductive assumption, the lemma holds for $G - u$. Hence the lemma holds for G. $\square$

Lemma 2.4. *Let G be a non-empty chordal graph and e be any edge in G. Then $\tau(G, G\cdot e, \lambda)>0$ for $\lambda \geqslant n$, where $n = v(G)$.*

Proof. We always assume that $\lambda \geqslant n \geqslant 2$. Let

$$P(G\cdot e, \lambda) = \prod_{i=1}^{n-1} (\lambda - b_i),$$

where b_i is an integer with $0 \leqslant b_i \leqslant n-2$. By Lemma 2.2,

$$\tau(G, G\cdot e, \lambda) = \lambda \prod_{i=1}^{n-1} (\lambda - a_i)(\lambda - b_i - 1) - (\lambda - 1)\prod_{i=1}^{n-1} (\lambda - a_i - 1)(\lambda - b_i),$$

where $a_i \in \{b_i, b_i+1\}$ for $i = 1, 2, \dots, n-1$. Since $b_i \leqslant n-2$ and $b_i \leqslant a_i \leqslant b_i + 1$, we have

$$(\lambda - a_i)(\lambda - b_i - 1) \geqslant (\lambda - a_i - 1)(\lambda - b_i) \geqslant 0,$$

implying that $\tau(G, G\cdot e, \lambda) \geqslant 0$. Since

$$\sum_{1\leqslant i \leqslant n-1} (a_i - b_i) = |E(G)| - |E(H)| > 0,$$

we have $a_i > b_i$ for some i, implying that

$$(\lambda - a_i)(\lambda - b_i - 1) - (\lambda - a_i - 1)(\lambda - b_i) = a_i - b_i > 0.$$

Further we notice that $(\lambda - a_i)(\lambda - b_i - 1) > 0$ for $i = 1, 2, \dots, n-1$. Hence $\tau(G, G\cdot e, \lambda) > 0$. □

Lemma 2.5 (Read and Tutte [7]). *For any graph G, if $\lambda \geqslant v(G) - 1$, then $P(G, \lambda) \geqslant 0$ where equality holds iff G is complete and $\lambda = v(G) - 1$.*

Lemma 2.5 also follows from Lemma 4.4.

Lemma 2.6. *Let G_i be any graph for $i = 1, 2, 3$ and $\lambda \geqslant \max_{1\leqslant i\leqslant 3} v(G_i)$. If $\tau(G_1, G_2, \lambda) \geqslant 0$ and $\tau(G_2, G_3, \lambda) \geqslant 0$, then*

(i) *$\tau(G_1, G_3, \lambda) \geqslant 0$, and*
(ii) *$\tau(G_1, G_3, \lambda) = 0$ implies that $\tau(G_i, G_{i+1}, \lambda) = 0$ for $i = 1, 2$.*

Proof. We always assume that $\lambda \geqslant \max_{1\leqslant i\leqslant 3} v(G_i)$ and $\tau(G_i, G_{i+1}, \lambda) \geqslant 0$ for $i = 1, 2$. By Lemma 2.5, $P(G_i, \lambda) > 0$ and $P(G_i, \lambda - 1) \geqslant 0$ for $i = 1, 2, 3$. By definition, for

$1 \leqslant i < j \leqslant 3$,

$$\tau(G_i, G_j, \lambda) = P(G_i, \lambda)P(G_j, \lambda - 1) - P(G_i, \lambda - 1)P(G_j, \lambda). \tag{12}$$

Claim. *For $i = 1, 2$, $P(G_{i+1}, \lambda - 1) = 0$ implies that $P(G_i, \lambda - 1) = 0$.*
If $P(G_{i+1}, \lambda - 1) = 0$, then by (12),

$$0 \leqslant \tau(G_i, G_{i+1}, \lambda) = -P(G_i, \lambda - 1)P(G_{i+1}, \lambda) \leqslant 0.$$

Since $P(G_{i+1}, \lambda) > 0$ and $P(G_i, \lambda - 1) \geqslant 0$, we have $P(G_i, \lambda - 1) = 0$. The claim holds.

By (12), it is obtained that

$$\begin{aligned} P(G_2, \lambda - 1)\tau(G_1, G_3, \lambda) = {} & P(G_3, \lambda - 1)\tau(G_1, G_2, \lambda) \\ & + P(G_1, \lambda - 1)\tau(G_2, G_3, \lambda). \end{aligned} \tag{13}$$

(i) If $P(G_2, \lambda - 1) > 0$, then $\tau(G_1, G_3, \lambda) \geqslant 0$ by (13). If $P(G_2, \lambda - 1) = 0$, then $P(G_1, \lambda - 1) = 0$ by the claim, implying that

$$\tau(G_1, G_3, \lambda) = P(G_1, \lambda)P(G_3, \lambda - 1) \geqslant 0.$$

(ii) Assume that $\tau(G_1, G_3, \lambda) = 0$.

If $P(G_3, \lambda - 1) = 0$, then $P(G_i, \lambda - 1) = 0$ for $i = 1, 2$ by the claim, implying that $\tau(G_i, G_j, \lambda) = 0$ for $1 \leqslant i < j \leqslant 3$ by (12). If $P(G_1, \lambda - 1) = 0$, then

$$0 = \tau(G_1, G_3, \lambda) = P(G_1, \lambda)P(G_3, \lambda - 1) \geqslant 0,$$

implies that $P(G_3, \lambda - 1) = 0$. By the previous result, $\tau(G_i, G_j, \lambda) = 0$ for $1 \leqslant i < j \leqslant 3$.
Now consider the remaining case that $P(G_i, \lambda - 1) > 0$ for $i = 1, 3$. Since $\tau(G_1, G_3, \lambda) = 0$, by (13),

$$P(G_3, \lambda - 1)\tau(G_1, G_2, \lambda) + P(G_1, \lambda - 1)\tau(G_2, G_3, \lambda) = 0, \tag{14}$$

implies that $\tau(G_i, G_{i+1}, \lambda) = 0$ for $i = 1, 2$. $\square$

Now we can prove the main result of this section.

Theorem 2.1. *Let G be any chordal graph and H be any subgraph of G. Then for $\lambda \geqslant v(G)$, $\tau(G, H, \lambda) \geqslant 0$ where equality holds iff $H \cong G$.*

Proof. Let G be any chordal graph with order n and H be any subgraph of G. We always assume that $\lambda \geqslant n$.
Case 1: *H is a spanning subgraph of G*. If either G is empty or $n = 2$, it is trivial. Assume that G has m edges where $m \geqslant 1$. Suppose that the theorem holds if $n < k$, where $k \geqslant 3$. Now let $n = k$.

If $H = G$, it is clear that $\tau(G, H, \lambda) = 0$. Suppose that the theorem holds if H is subgraph of G with more that q edges, where $q < m$. Now let H be any subgraph of G with q edges.

Let e be an edge of G which is not in H. It is clear that $H + e$ is a subgraph of G and $H + e$ has $q + 1$ edges. By inductive assumption, we have

$$\tau(G, H + e, \lambda) \geqslant 0.$$

Note that $H \cdot e$ is a subgraph of $G \cdot e$. Since $G \cdot e$ is a chordal graph by Lemma 2.2, we have

$$\tau(G \cdot e, H \cdot e, \lambda) \geqslant 0$$

by inductive assumption. By Lemma 2.4, we have $\tau(G, G \cdot e, \lambda) > 0$. Thus by Lemma 2.6,

$$\tau(G, H \cdot e, \lambda) > 0.$$

Since

$$P(H, \lambda) = P(H + e, \lambda) + P(H \cdot e, \lambda)$$

by the definition of $\tau(G, H, \lambda)$, we have

$$\tau(G, H, \lambda) = \tau(G, H + e, \lambda) + \tau(G, H \cdot e, \lambda) > 0.$$

Case 2: *H is a subgraph of G, but not spanning*. Let $t = v(G) - v(H) > 0$ and $H' = H \cup tK_1$. Then H' is a spanning subgraph of G. By the result in case 1, we have $\tau(G, H', \lambda) \geqslant 0$. Observe that

$$\begin{aligned}\tau(H', H, \lambda) &= \lambda^t P(H, \lambda) P(H, \lambda - 1) - (\lambda - 1)^t P(H, \lambda - 1) P(H, \lambda) \\ &= (\lambda^t - (\lambda - 1)^t) P(H, \lambda) P(H, \lambda - 1) \\ &> 0,\end{aligned}$$

where the last inequality follows from Lemma 2.5. Hence $\tau(G, H, \lambda) > 0$ by Lemma 2.6. $\square$

By Theorem 2.1 and Lemma 2.1, we have the following result on mean colour numbers.

Theorem 2.2. *Let G be any chordal graph and H be any spanning subgraph of G. Then $\mu(G) \geqslant \mu(H)$ where equality holds iff $H \cong G$.*

Using Theorem 2.2, we are able to find a upper bound of $\mu(H)$ for any graph H by adding edges on H so that we get a chordal graph. In the following, we show a upper bound for $\mu(H)$ given by the tree width of H, denoted by $tw(H)$, and the order of H. (See the definition of *tree width* in [8].) Let $\omega(G)$ denote the clique number of a graph G. There is a relation between the two invariants.

Lemma 2.7 (Diestel [2]). *For any graph H,*

$$tw(H) = \min\{\omega(G) - 1, \text{ where } G \text{ is a chordal graph such that } H \subseteq G\}.$$

Theorem 2.3. *For any graph H with order n and tree width k, we have*

$$\mu(H) \leqslant n - (n-k-1)^{n-k-1}/(n-k)^{n-k-2}.$$

Proof. By Lemma 2.7, H is a spanning subgraph of a chordal graph G whose clique number is $k+1$. By Theorem 2.2, $\mu(H) \leqslant \mu(G)$. Since G is a chordal graph, we have

$$P(G,\lambda) = (\lambda)_{k+1} \prod_{i=1}^{n-k-1} (\lambda - a_i),$$

where a_i is an integer with $0 \leqslant a_i \leqslant k$. Thus by Theorem 1.1,

$$\begin{aligned}\mu(G) &= n - (n-k) \prod_{i=1}^{n-k-1} \frac{n - a_i - 1}{n - a_i} \leqslant n - (n-k)\left(\frac{n-k-1}{n-k}\right)^{n-k-1} \\ &= n - (n-k-1)^{n-k-1}/(n-k)^{n-k-2}. \qquad \square\end{aligned}$$

3. Further results on the upper bounds

Assume that G is a chordal graph and H is a subgraph of G but not spanning. In this section, we shall further prove the result in Theorem 2.2 under this condition.

Let G_1 be the subgraph of G induced by $V(H)$. Then H is a spanning subgraph of G_1. Further let $G_2 = G_1 \cup mK_1$, where $m = v(G) - v(G_1)$. Then G_2 is a chordal graph, which is a spanning subgraph of G. By Theorem 2.2, we have

$$\mu(H) \leqslant \mu(G_1) \quad \text{and} \quad \mu(G_2) \leqslant \mu(G).$$

Thus, we need only to prove $\mu(G_1) < \mu(G_2) = \mu(G_1 \cup mK_1)$.

Lemma 3.1. *For any graph G, $\mu(G \cup K_1) > \mu(G)$ iff*

$$\lambda(P(G,\lambda))^2 < (\lambda P(G,\lambda-1) + P(G,\lambda))P(G,\lambda+1)$$

for $\lambda = v(G)$.

Proof. Let $n = v(G)$. By Theorem 1.1,

$$\mu(G) = n - nP(G,n-1)/P(G,n).$$

Since $P(G \cup K_1, \lambda) = \lambda P(G,\lambda)$, we have

$$\mu(G \cup K_1) = n + 1 - nP(G,n)/P(G,n+1).$$

Thus,

$$\mu(G \cup K_1) - \mu(G) = 1 - n\frac{(P(G,n))^2 - P(G,n-1)P(G,n+1)}{P(G,n)P(G,n+1)}.$$

Observe that $\mu(G \cup K_1) - \mu(G) > 0$ iff

$$n(P(G,n))^2 < (nP(G,n-1) + P(G,n))P(G,n+1). \qquad \square$$

Lemma 3.2. *For any graphs G and H, if $\lambda > \max\{v(G)-2, v(H)-2\}$, then*

$$2(\lambda+1)P(G,\lambda)P(H,\lambda) \leqslant (\lambda P(G,\lambda-1) + P(G,\lambda))P(H,\lambda+1)$$
$$+ (\lambda P(H,\lambda-1) + P(H,\lambda))P(G,\lambda+1), \tag{15}$$

where the equality holds iff $G \cong H \cong K_k$ for some k.

Proof. Let $v(G) = n$ and $v(H) = m$. By (1), we have

$$P(G,\lambda) = \sum_{k=1}^{n} b_k(\lambda)_k,$$

where b_k is a non-negative integer for $k = 1, 2, \ldots, n$, and

$$P(H,\lambda) = \sum_{k=1}^{m} c_k(\lambda)_k,$$

where c_k is a non-negative integer for $k = 1, 2, \ldots, m$. Thus

$$P(G,\lambda)P(H,\lambda) = \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} b_i c_j \cdot (\lambda)_i(\lambda)_j$$

and

$$(\lambda P(G,\lambda-1) + P(G,\lambda))P(H,\lambda+1)$$
$$+ (\lambda P(H,\lambda-1) + P(H,\lambda))P(G,\lambda+1)$$
$$= \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} b_i c_j((\lambda(\lambda-1)_i + (\lambda)_i)(\lambda+1)_j + (\lambda(\lambda-1)_j + (\lambda)_j)(\lambda+1)_i).$$

Observe that

$$
\begin{aligned}
&(\lambda(\lambda-1)_i+(\lambda)_i)(\lambda+1)_j+(\lambda(\lambda-1)_j+(\lambda)_j)(\lambda+1)_i\\
&\quad=(\lambda-i+1)(\lambda)_i(\lambda+1)_j+(\lambda-j+1)(\lambda)_j(\lambda+1)_i\\
&\quad=(\lambda+1)(\lambda)_{i-1}(\lambda)_{j-1}((\lambda-i+1)^2+(\lambda-j+1)^2)\\
&\quad=2(\lambda+1)(\lambda)_i(\lambda)_j+2(i-j)^2(\lambda+1)(\lambda)_{i-1}(\lambda)_{j-1}\\
&\quad\geqslant 2(\lambda+1)(\lambda)_i(\lambda)_j
\end{aligned}
$$

for $\lambda > \max\{n-2, m-2\}$, where the equality holds iff $i=j$. Thus the lemma holds. $\square$

Corollary 3.1. *For any graph G, if $\lambda > v(G)-1$, then*

$$\lambda(P(G,\lambda))^2 < (\lambda P(G,\lambda-1)+P(G,\lambda))P(G,\lambda+1). \tag{16}$$

Proof. If follows directly from Lemma 3.2 by letting $H \cong G$. $\square$

The next result follows directly from Lemma 3.1 and Corollary 3.1.

Theorem 3.1. *For any graph G, we have $\mu(G) < \mu(G \cup K_1)$.*

By Theorem 3.1, it is easy to get the main result in this section.

Theorem 3.2. *If G is a chordal graph and H is a non-spanning subgraph of G, then $\mu(H) < \mu(G)$.*

Proof. Let G_1 and G_2 be the graphs constructed in the beginning of this section. Then by Theorem 2.2, we have

$$\mu(H) \leqslant \mu(G_1) \quad \text{and} \quad \mu(G_2) \leqslant \mu(G).$$

Since $G_2 = G_1 \cup mK_1$, where $m = v(G) - v(H) > 0$, by Theorem 3.1, we have $\mu(G_1) < \mu(G_2)$. Therefore $\mu(H) < \mu(G)$. $\square$

4. Lower bounds

Our main result in this section is to prove that $\mu(H) \leqslant \mu(G)$ if H is a subgraph of G and $H \in \Omega$, where the equality holds iff $H \cong G$. Recall that Ω is a set of chordal graphs defined in Section 1. For any integer $p \geqslant 2$, let

$$\Omega_p = \{G | G \in \Omega \text{ and the clique number of } G \text{ is } p\}.$$

Observe that Ω_2 is the set of trees with order at least 2, and when $p \geqslant 3$, any graph in Ω_p can be obtained from a tree by replacing a vertex by K_p. The chromatic polynomial of any graph in Ω_p can be expressed in terms of p and its order.

For any positive integer n, let $\mathscr{G}_n$ denote the set of graphs with order n and $\Omega_{n,p} = \mathscr{G}_n \cap \Omega_p$.

Lemma 4.1. *$P(H, \lambda) = (\lambda)_p (\lambda - 1)^{n-p}$ for any $H \in \Omega_{n,p}$.*

Lemma 4.2. *A graph G contains a subgraph in $\Omega_{n,p}$ iff G has a component whose order and clique number are at least n and p, respectively.*

For integers n and p with $n \geqslant p \geqslant 2$ and any graph G, define

$$\Psi(G, n, p, \lambda) = \frac{P(G, \lambda)}{(\lambda)_p (\lambda - 1)^{n-p}}. \tag{17}$$

So $\Psi(G, n, p, \lambda) = P(G, \lambda)/P(H, \lambda)$ by Lemma 4.1, where H is any graph in $\Omega_{n,p}$. It is clear that $\tau(G, H, \lambda) \geqslant 0$ iff $\Psi(G, n, p, \lambda) \geqslant \Psi(G, n, p, \lambda - 1)$. Observe that

$$\frac{d(\Psi(G, n, p, \lambda))}{d(\lambda)} = \frac{P'(G, \lambda) - P(G, \lambda)(\frac{1}{\lambda} + \frac{1}{\lambda - 1} + \cdots + \frac{1}{\lambda - p + 1} + \frac{n-p}{\lambda - 1})}{(\lambda)_p (\lambda - 1)^{n-p}}. \tag{18}$$

Define

$$\varepsilon(G, p, \lambda) = P'(G, \lambda) - P(G, \lambda)\left(\frac{n-p}{\lambda - 1} + \sum_{i=0}^{p-1} \frac{1}{\lambda - i}\right), \tag{19}$$

where $n = v(G)$.

Lemma 4.3. *Let H be any graph in $\Omega_{n,p}$, where $n \geqslant p > 0$. If $\varepsilon(G, p, \lambda) \geqslant 0$ for $\lambda \geqslant v(G) - 1$, then $\tau(G, H, \lambda) \geqslant 0$ for $\lambda \geqslant v(G)$. Further, if $\varepsilon(G, p, \lambda) > 0$ for $\lambda > v(G) - 1$, then $\tau(G, H, \lambda) > 0$ for $\lambda \geqslant v(G)$.*

We shall prove that if $G \in \mathscr{G}_n$ and G contains a subgraph in $\Omega_{n,p}$, then $\varepsilon(G, p, \lambda) \geqslant 0$ for $\lambda \geqslant n - 1$. We need to introduce some supporting results.

Lemma 4.4 (Dong [3]). *For any graph G with order $n \geqslant 3$ and $w \in V(G)$, there is a set $\mathscr{B}(G, w)$ (or simply $\mathscr{B}$) of graphs with order $n - 2$ such that*

$$P(G, \lambda) = (\lambda - d)P(G^*, \lambda) + \sum_{R \in \mathscr{B}} P(R, \lambda), \tag{20}$$

where $G^* = G - w$ and $d = d_G(w)$. The set $\mathscr{B}(G,w)$ is empty iff w is a simplicial vertex of G, and every graph of $\mathscr{B}(G,w)$ can be obtained from $G - w$ by adding edges joining vertices in $N_G(w)$ and contracting a pair of nonadjacent vertices in $N_G(w)$.

In the following, we give a recursive expression for $\varepsilon(G,p,\lambda)$.

Lemma 4.5. *Let p be a positive integer and G be a graph with order $n \geqslant p$. Let w be a vertex in G with $d_G(w) = d$ and $G^* = G - w$. Then*

$$\varepsilon(G,p,\lambda) = (\lambda - d)\varepsilon(G^*,p,\lambda) + \sum_{R \in \mathscr{B}} \varepsilon(R,p,\lambda) + \frac{1}{\lambda - 1}((2\lambda - d - 1)P(G^*,\lambda) - 2P(G,\lambda)). \tag{21}$$

Proof. By the definition of $\varepsilon(G,p,\lambda)$ and Lemma 4.4,

$$\begin{aligned}
\varepsilon(G,p,\lambda) &= P(G^*,\lambda) + (\lambda - d)P'(G^*,\lambda) + \sum_{R \in \mathscr{B}} P'(R,\lambda) \\
&\quad - (\lambda - d)P(G^*,\lambda)\left(\frac{n-p}{\lambda - 1} + \sum_{i=0}^{p-1} \frac{1}{\lambda - i}\right) \\
&\quad - \sum_{R \in \mathscr{B}} P(R,\lambda)\left(\frac{n-p}{\lambda - 1} + \sum_{i=0}^{p-1} \frac{1}{\lambda - i}\right) \\
&= P(G^*,\lambda) + (\lambda - d)\varepsilon(G^*,p,\lambda) - \frac{\lambda - d}{\lambda - 1}P(G^*,\lambda) \\
&\quad + \sum_{R \in \mathscr{B}} \varepsilon(R,p,\lambda) - \frac{2}{\lambda - 1}\sum_{R \in \mathscr{B}} P(R,\lambda).
\end{aligned}$$

By Lemma 4.4 again,

$$\sum_{R \in \mathscr{B}} P(R,\lambda) = P(G,\lambda) - (\lambda - d)P(G^*,\lambda).$$

The result then follows. $\square$

Lemma 4.6. *Let G be any connected graph with order n and clique number at least p. Then $|E(G)| \geqslant \binom{p}{2} + (n - p)$, where equality holds iff $G \in \Omega_p$.*

Proof. Since G is connected, there exists an ordering $x_1, x_2, \ldots, x_n$ of vertices in G such that $G_p \cong K_p$ and $d_{G_i}(x_i) \geqslant 1$ for $i = p+1, \ldots, n$, where G_i is the subgraph of G

induced by $\{x_1, x_2, \ldots, x_i\}$. Thus

$$|E(G)| = |E(G_p)| + \sum_{i=p+1}^{n} d_{G_i}(x_i) \geqslant \binom{p}{2} + (n-p).$$

If the above equality holds, then $d_{G_i}(x_i) = 1$ for $i = p+1, \ldots, n$, implies that $G \in \Omega_p$. If $G \in \Omega_p$, then it is clear that $|E(G)| = \binom{p}{2} + (n-p)$. $\square$

Now we introduce another result from [4]. It also follows directly from a result in [5] if $d(w) \geqslant 4$.

Theorem 4.1 (Dong [4]). *For any graph G and $w \in V(G)$ with $d(w) \geqslant 1$, we have*

$$(2\lambda - d(w) - 1)P(G - w, \lambda) - 2P(G, \lambda) \geqslant 0 \tag{22}$$

for $\lambda \geqslant v(G) - 1$, where the inequality is strict if $d(w) \geqslant 2$ and $\lambda > v(G) - 1$.

Now we can prove one of the main results in this section.

Theorem 4.2. *Let G be any connected graph with order n and clique number at least p. Then $\varepsilon(G, p, \lambda) \geqslant 0$ for $\lambda \geqslant n-1$, where the inequality is strict if $G \notin \Omega_p$ and $\lambda > n-1$.*

Proof. If $n = p$, then $G = K_p$ and it is clear that $\varepsilon(G, p, \lambda) = 0$.

Now assume that $n \geqslant p+1$. First, consider a special case that G has a simplicial vertex w such that the clique number of $G - w$ is at least p. It is obvious that $G - w$ is connected. We have

$$P(G, \lambda) = (\lambda - d(w))P(G - w, \lambda).$$

Note that $1 \leqslant d(w) \leqslant n-1$. Then by Lemmas 4.4 and 4.5,

$$\varepsilon(G, p, \lambda) = (\lambda - d(w))\varepsilon(G - w, p, \lambda) + \frac{(d(w) - 1)P(G - w, \lambda)}{\lambda - 1} \geqslant 0$$

for $\lambda \geqslant n-1$, where the inequality is strict if $d(w) \geqslant 2$ and $\lambda \geqslant n-1$ by inductive assumption and the fact that $P(G - w, \lambda) > 0$ for $\lambda > n-2$. If $d(w) = 1$, then $G \in \Omega_p$ iff $G - w \in \Omega_p$, and

$$\varepsilon(G, p, \lambda) = (\lambda - 1)\varepsilon(G - w, p, \lambda) \geqslant 0,$$

where the inequality is strict if $G - w \notin \Omega_p$ and $\lambda > n-2$ by inductive assumption.

If $n = p+1$, then G contains a vertex w such that $G - w \cong K_p$. The vertex w must be a simplicial vertex. In this case the theorem follows from the above result.

Now suppose that $n \geqslant p+2$. We apply Lemma 4.5. There is a vertex w in G such that $G - w$ is connected and the clique number of $G - w$ is at least p. We may assume that w is not a simplicial vertex of G and so $d(w) \geqslant 2$. Thus

$$|E(G)| = d(w) + |E(G-w)| \geqslant 2 + \binom{p}{2} + n - p - 1 > \binom{p}{2} + n - p,$$

implies that $G \notin \Omega_p$ by Lemma 4.6. By inductive hypothesis, $\varepsilon(G^*, p, \lambda) \geqslant 0$ for $\lambda \geqslant n-2$. Let $R \in \mathscr{B}(G, w)$. By Lemma 4.4, R is connected with clique number at least p. By inductive hypothesis, we have $\varepsilon(R, p, \lambda) \geqslant 0$ for $\lambda \geqslant n-3$. Finally by Theorem 4.1,

$$(2\lambda - d(w) - 1)P(G-w, \lambda) - 2P(G, \lambda) \geqslant 0$$

for $\lambda \geqslant n-1$, where the inequality is strict if $\lambda > n-1$. Hence by Lemma 4.5, $\varepsilon(G, p, \lambda) \geqslant 0$ for $\lambda \geqslant n-1$, where the inequality is strict if $\lambda > n-1$. $\square$

Theorem 4.3. *Let $H \in \Omega_{n,p}$ and $G \notin \Omega_p$ where $n \geqslant p \geqslant 2$. If H is a subgraph of G, then $\tau(G, H, \lambda) > 0$ for $\lambda \geqslant v(G)$.*

Proof. Assume that H is a subgraph of G. Then the order of G is at least n.

Case 1: $v(G) = n$. Since $H \in \Omega_{n,p}$ and H is a spanning subgraph of G, G must be connected and with clique number at least p. Since $G \notin \Omega_p$, by Theorem 4.2, we have $\varepsilon(G, p, \lambda) \geqslant 0$ for $\lambda \geqslant n-1$, where the inequality is strict if $\lambda > n-1$. Thus by Lemma 4.3, we have $\tau(G, H, \lambda) > 0$ for $\lambda \geqslant v(G) = n$.

Case 2: *G is connected with order $m > n$.*

So G contains a subgraph $H' \in \Omega_{m,p}$. Thus by the result in case 1, we have

$$\tau(G, H', \lambda) = P(G, \lambda)(\lambda-1)_p(\lambda-2)^{m-p} - P(G, \lambda-1)(\lambda)_p(\lambda-1)^{m-p} \geqslant 0$$

for $\lambda \geqslant m$. (It is possible that $G \in \Omega_{m,p}$. In this case, the equality holds.) Since $m > n$, the following inequality immediately follows:

$$\begin{aligned}\tau(G, H, \lambda) &= P(G, \lambda)(\lambda-1)_p(\lambda-2)^{n-p} - P(G, \lambda-1)(\lambda)_p(\lambda-1)^{n-p} \\ &> \frac{1}{(\lambda-1)^{m-n}}\tau(G, H', \lambda) \\ &\geqslant 0\end{aligned}$$

for $\lambda \geqslant v(G) = m$.

Case 3: *G is disconnected.* Then H is a subgraph of some component G_1 of G. By the result in cases 1 and 2, we have

$$\tau(G_1, H, \lambda) = P(G_1, \lambda)(\lambda-1)_p(\lambda-2)^{n-p} - P(G_1, \lambda-1)(\lambda)_p(\lambda-1)^{n-p} \geqslant 0$$

for $\lambda \geqslant v(G_1)$. Let G_2 be the subgraph of G induced by $V(G) \setminus V(G_1)$. Then $P(G,\lambda) = P(G_1,\lambda)P(G_2,\lambda)$. Since $P(G_2,\lambda) > P(G_2,\lambda-1)$ for $\lambda \geqslant v(G_2)$, we have

$$\tau(G,H,\lambda) = P(G,\lambda)(\lambda-1)_p(\lambda-2)^{n-p} - P(G,\lambda-1)(\lambda)_p(\lambda-1)^{n-p} > 0$$

for $\lambda \geqslant v(G)$. This completes the proof. $\square$

Any connected graph contains a spanning tree, i.e., a graph in Ω_2. Thus, the next result follows directly from Theorem 4.3 by letting $p = 2$. It is the main result in [3].

Corollary 4.1. *For any connected graph G, we have*

$$P(G,\lambda)(\lambda-2)^{n-1} - P(G,\lambda-1)\lambda(\lambda-1)^{n-2} \geqslant 0$$

for $\lambda \geqslant n$, where n is the order of G.

Lemma 4.7. *Let G be a graph with components $G_1, G_2, \ldots, G_k$ whose order and clique number are, respectively, n_i and p_i for $i = 1, 2, \ldots, k$. Then*

$$\mu(G) \geqslant n - \frac{(n-2)^{n-p'}}{n^{k-1}(n-1)^{n-p'}} \prod_{i=1}^{k} (n - p_i),$$

where

$$n = \sum_{i=1}^{k} n_i \text{ and } p' = \sum_{i=1}^{k} p_i,$$

and the inequality is strict iff $p_i \geqslant 2$ and $G_i \notin \Omega_{p_i}$ for some i.

Proof. First we have

$$P(G_i,\lambda)(\lambda-1)_{p_i}(\lambda-2)^{n_i-p_i} \geqslant P(G_i,\lambda-1)(\lambda)_{p_i}(\lambda-1)^{n_i-p_i},$$

for $\lambda \geqslant n_i$, since if $p_i = 1$, then $n_i = 1$ and thus the equality holds; if $p_i \geqslant 2$, then it follows from Theorem 4.3 and the inequality is strict iff $G_i \notin \Omega_{p_i}$. Since

$$P(G,\lambda) = \prod_{i=1}^{k} P(G_i,\lambda),$$

we have

$$P(G,\lambda)(\lambda-2)^{n-p'} \prod_{i=1}^{k} (\lambda - p_i) \geqslant P(G,\lambda-1)\lambda^k(\lambda-1)^{n-p'}$$

for $\lambda \geqslant \max_{1 \leqslant i \leqslant k} n_i$, where

$$n = \sum_{i=1}^{k} n_i \text{ and } p' = \sum_{i=1}^{k} p_i.$$

Then by Theorem 1.1, the result follows immediately. $\square$

Theorem 4.4. *Let G be a graph with components $G_1, G_2, \dots, G_s$ and H be a graph with components $H_1, H_2, \dots, H_t$, where $s \geqslant t$, such that $H_i \in \Omega$ and H_i is a subgraph of G_i for $i = 1, 2, \dots, t$. Then $\mu(G) \geqslant \mu(H)$, where equality holds iff $G \cong H$.*

Proof. Let the order and clique number of G_i be n_i and p_i, respectively, and let G_i' be a spanning subgraph of G_i such that $G_i' \in \Omega_{n_i, p_i}$ if $n_i > 1$ and $G_i' \cong K_1$ otherwise for $i = 1, 2, \dots, s$. Let G' be the graph with components $G_1', G_2', \dots, G_s'$.

By Lemma 4.7, we have

$$\mu(G') = n - \frac{(n-2)^{n-p'}}{n^{k-1}(n-1)^{n-p'}} \prod_{i=1}^{k} (n - p_i),$$

where

$$n = \sum_{i=1}^{k} n_i \text{ and } p' = \sum_{i=1}^{k} p_i.$$

By Lemma 4.7 again, we have $\mu(G) \geqslant \mu(G')$ where the inequality is strict iff $p_i \geqslant 2$ and $G_i' \notin \Omega$ for some i. This also shows that the theorem holds if $s = t$ and H_i and G_i has the same order and same clique number for $i = 1, 2, \dots, s$.

For $i = 1, 2, \dots, t$, since H_i is a subgraph of G_i and $H_i \in \Omega$, G_i' contains a subgraph $H_i' \in \Omega$ such that H' and H_i has the same order and same clique number. By Lemma 4.1, H_i' and H_i has the same chromatic polynomial.

Let H' be the graph with components $H_1', H_2', \dots, H_t'$. Then $P(H', \lambda) = P(H, \lambda)$ and H' is a subgraph of G'. Since G' is chordal, we have $\mu(H') \leqslant \mu(G')$ by Theorems 2.2 and 3.2, where the equality holds iff $H' \cong G'$.

Since $P(H', \lambda) = P(H, \lambda)$, we have

$$\mu(H) = \mu(H') \leqslant \mu(G') \leqslant \mu(G).$$

Now suppose that $\mu(H) = \mu(G)$. Then $\mu(H') = \mu(G') = \mu(G)$. Since G' is chordal, by Theorems 2.2 and 3.2, we have $s = t$ and $H' \cong G'$, implies that H_i and G_i has the same order and same clique number for $i = 1, 2, \dots s$. By the result in the second paragraph of the proof, we have $H \cong G$. $\square$

5. Unsolved problems

In this section, we propose some new problems on mean colour numbers.

Motivated by Theorem 4.4, we believe that deleting all but one of the edges which are incident with a fixed vertex does not increase the mean colour numbers of graphs.

Conjecture 1. *For any graph G and a vertex w in G with $d(w) \geqslant 1$, if H is a graph obtained from G by deleting all but one of the edges which are incident to w, then $\mu(G) \geqslant \mu(H)$.*

The following conjecture is weaker.

Conjecture 2. *For any graph G and a vertex w in G, $\mu(G) \geqslant \mu((G - w) \cup K_1)$.*

Let Ω' denote the set of graphs G in which each block is a complete graph. By the definition, a graph in Ω' is not necessarily connected, but is chordal. It is clear that $\Omega \subset \Omega'$. Also motivated by Theorem 4.4, another conjecture was proposed below.

Conjecture 3. *For any graphs G and H, if $H \in \Omega'$ and H is a subgraph of G, then $\mu(G) \geqslant \mu(H)$.*

Although Bartels and Welsh's conjecture that $\mu(G) \geqslant \mu(G - xy)$ holds for any graph G and any edge xy in G was disapproved by counter-examples constructed by Mosca, it is possible that $\mu(G) \geqslant \mu(G - xy)$ holds for some edge xy.

Conjecture 4. *For any non-empty graph G, there exists some edge xy in G such that $\mu(G) \geqslant \mu(G - xy)$.*

The following conjecture is more general.

Conjecture 5. *For any non-empty graph G,*

$$|E(G)| \cdot \mu(G) \geqslant \sum_{xy \in E(G)} \mu(G - xy).$$

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References

[1] J.E. Bartels, D.J.A. Welsh, The Markov chain of colourings, in: Proceedings of the Fourth Conference on Integer Programming and Combinatorial Optimization (IPCO IV), Lecture Notes in Computer Science, Vol. 920, Springer, New York/Berlin, 1995, pp. 373–387.

[2] R. Diestel, Graph Theory, Springer, Berlin, 1991.

[3] F.M. Dong, Proof of a chromatic polynomial conjecture, J. Combin. Theory Ser. B 78 (2000) 35–44.

[4] F.M. Dong, Further results on mean colour numbers, J. Graph Theory, submitted for publication.

[5] F.M. Dong, K.L. Teo, C.H.C. Little, M.D. Hendy, Some inequalities on chromatic polynomials, New Zealand J. Math. 30 (2001) 111–118.

[6] Michele Mosca, Removing edges can increase the average number of colours in the colourings of a graph, Combin. Probab. Comput. 7 (1998) 211–216.

[7] R.C. Read, W.T. Tutte, Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory III, Academic Press, New York, 1988, pp. 15–42.

[8] N. Robertson, P.D. Seymour, Graph minors. III. Planar tree-width, J. Combin. Theory Ser. B 36 (1984) 49–64.