Chromatically unique bipartite graphs with low
3-independent partition numbers

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Abstract

For integers \(p,q,s\) with \(p > q > 2\) and \(s \geq 0\), let \(K_{-s}(p,q)\) denote the set of \(2\)-connected bipartite graphs which can be obtained from \(K_{p,q}\) by deleting a set of \(s\) edges. In this paper, we prove that for any graph \(G\) in \(K_{-s}(p,q)\) with \(p > q > 3\) and \(1 \leq s \leq q - 1\), if the number of 3-independent partitions of \(G\) is at most \(2p - 1 + 2q - 1 + s + 2\), then \(G\) is \(\chi\)-unique. It follows that any graph in \(K_{-s}(p,q)\) is \(\chi\)-unique if \(p > q > 3\) and \(1 \leq s \leq \min\{q - 1, 4\}\). © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Bipartite graph; Chromatic polynomial

1. Introduction

All graphs considered here are simple graphs. For a graph \(G\), let \(V(G)\), \(E(G)\), \(e(G)\), \(\delta(G)\), \(\Delta(G)\) and \(P(G, \lambda)\) be the vertex set, edge set, size, minimum degree, maximum degree and the chromatic polynomial of \(G\), respectively.

For integers \(p, q, s\) with \(p \geq q \geq 2\) and \(s \geq 0\), let \(H^{-s}(p, q)\) (resp. \(K_{-s}(p, q)\)) denote the set of connected (resp. 2-connected) bipartite graphs which can be obtained from \(K_{p,q}\) by deleting a set of \(s\) edges. The following result was obtained in [1].

Lemma 1.1. If \(p \geq q \geq 3\) and \(s \leq p + q - 4\), then for any \(G \in H^{-s}(p, q)\) with \(\delta(G) \geq 2\), \(G\) is \(2\)-connected.

For a bipartite graph \(G = (A,B;E)\) with bipartition \(A\) and \(B\) and edge set \(E\), let \(G' = (A',B';E')\) be the bipartite graph induced by the edge set \(E' = \{xy | xy \notin E, x \in A, y \in B\}\), where \(A' \subseteq A\) and \(B' \subseteq B\). We write \(G' = K_{p,q} - G\), where \(p = |A|\) and \(q = |B|\). Observe that \(\delta(G) = \min(q - \Delta(G'), p - \Delta(G'))\).

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PII: S0012-365X(00)00094-7
Corollary 1.1. For \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \), if \( G \in \mathcal{K}^{\rightarrow}(p,q) - \mathcal{K}^{\rightarrow}_{2}(p,q) \), then \( s = q - 1 \) and \( \Delta(G') = q - 1 \).

Two graphs \( G \) and \( H \) are said to be chromatically equivalent (or simply \( \chi \)-equivalent), symbolically \( G \sim H \), if \( P(G, \lambda) = P(H, \lambda) \). The equivalence class determined by \( G \) under \( \sim \) is denoted by \( [G] \). A graph \( G \) is chromatically unique (or simply \( \chi \)-unique) if \( H \cong G \) whenever \( H \sim G \). For a set \( \mathcal{G} \) of graphs, if \( [G] \subseteq \mathcal{G} \) for every \( G \in \mathcal{G} \), then \( \mathcal{G} \) is said to be \( \chi \)-closed. In [1], we established the following result.

Theorem 1.1. For integers \( p, q, s \) with \( p \geq q \geq 2 \) and \( 0 \leq s \leq q - 1 \), \( \mathcal{K}^{\rightarrow}_{2}(p,q) \) is \( \chi \)-closed.

The complete bipartite graph \( K_{p,q} \) is \( \chi \)-unique for any \( p \geq q \geq 2 \) (see [2,6]). In this paper, we shall search for \( \chi \)-unique graphs or \( \chi \)-equivalence classes from the set \( \mathcal{K}^{\rightarrow}_{2}(p,q) \), where \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \). Hence, in this paper, we fix the following conditions for \( p, q \) and \( s \):

\[
p \geq q \geq 3 \quad \text{and} \quad 0 \leq s \leq q - 1.
\]

For a graph \( G \) and a positive integer \( k \), a partition \( \{A_{1}, A_{2}, \ldots, A_{k}\} \) of \( V(G) \) is called a \( k \)-independent partition in \( G \) if each \( A_{i} \) is a non-empty independent set of \( G \). Let \( \alpha(G,k) \) denote the number of \( k \)-independent partitions in \( G \). For any bipartite graph \( G = (A, B, E) \), define

\[
\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2).
\]

In [1], we found the following sharp bounds for \( \alpha'(G,3) \):

Theorem 1.2. For \( G \in \mathcal{K}^{\rightarrow}(p,q) \) with \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \),

\[
s \leq \alpha'(G,3) \leq 2^s - 1,
\]

where \( \alpha'(G,3) = s \) iff \( \Delta(G') = 1 \) and \( \alpha'(G,3) = 2^s - 1 \) iff \( \Delta(G') = s \).

For \( t = 0, 1, 2, \ldots \), let \( \mathcal{B}(p,q,s,t) \) denote the set of graphs \( G \in \mathcal{K}^{\rightarrow}(p,q) \) with \( \alpha'(G,3) = s + t \). Thus, \( \mathcal{K}^{\rightarrow}(p,q) \) is partitioned into the following subsets:

\[
\mathcal{B}(p,q,s,0), \mathcal{B}(p,q,s,1), \ldots, \mathcal{B}(p,q,s,2^s - s - 1).
\]

Assume that \( \mathcal{B}(p,q,s,t) = \emptyset \) for \( t > 2^s - s - 1 \).

Lemma 1.2. For \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \), if \( 0 \leq t \leq 2^s - 1 - q - 1 \), then

\[
\mathcal{B}(p,q,s,t) \subseteq \mathcal{K}^{\rightarrow}_{2}(p,q).
\]

Proof. We consider the following two cases.

Case 1: \( s \leq q - 2 \). By the corollary to Lemma 1.1, \( \mathcal{K}^{\rightarrow}(p,q) = \mathcal{K}^{\rightarrow}_{2}(p,q) \) and thus \( \mathcal{B}(p,q,s,t) \subseteq \mathcal{K}^{\rightarrow}_{2}(p,q) \) for all \( t \).
Lemma 2.1. For any graph \( G \in \mathcal{B}(p,q,s,t) \), we have \( \Delta(G') \leq q - 2 \) and thus by the corollary to Lemma 1.1, \( G \) is 2-connected. Hence \( \mathcal{B}(p,q,s,t) \subseteq \mathcal{H}_{-t}^-(p,q) \) if \( 0 \leq t < 2^{q-1} - q - 1 \).

For any graph \( G \), we have \( P(G, \lambda) = \sum_{s \geq 1} x(G,k)\lambda(s-1) \cdots (\lambda-k+1) \) (see [5]). If \( G \sim H \), then \( x(G,k) = x(H,k) \) for \( k = 1,2,\ldots \). Thus, by Theorem 1.1, the following result is obtained.

**Theorem 1.3.** The set \( \mathcal{B}(p,q,s,t) \cap \mathcal{H}_{-t}^-(p,q) \) is \( \chi \)-closed for all \( t \geq 0 \).

**Corollary 1.2.** If \( 0 \leq t < 2^{q-1} - q - 1 \), then \( \mathcal{B}(p,q,s,t) \) is \( \chi \)-closed.

We have proved in [1] the following result.

**Theorem 1.4.** For any graph \( G \in \mathcal{B}(p,q,s,0) \cup \mathcal{B}(p,q,s,2^{q-1} - q - 1) \), if \( G \) is 2-connected, then \( G \) is \( \chi \)-unique.

In this paper, we shall show that every 2-connected graph in \( \mathcal{B}(p,q,s,t) \) is \( \chi \)-unique for \( 1 \leq t \leq 4 \). Further, we prove that every graph in \( \mathcal{H}_{-t}^-(p,q) \) is \( \chi \)-unique if \( 1 \leq s \leq \min(4,q-1) \).

2. \( \mathcal{B}(p,q,s,t) \) for \( t \leq 4 \)

In this section, we shall study the structure of graphs in \( \mathcal{B}(p,q,s,t) \) for \( t \leq 4 \).

**Lemma 2.1.** For \( G = (A,B;E) \in \mathcal{H}_{-t}^-(p,q) \) with \( |A| = p \) and \( |B| = q \), we have

\[
e(G') = \sum_{x \in A'} d_G(x) = \sum_{y \in B'} d_G(y) = s.
\]

For a graph \( G \) and \( x \in V(G) \), let \( N_G(x) \) or simply \( N(x) \) denote the set of vertices \( y \) such that \( xy \in E(G) \). Let \( G = (A,B;E) \) be a graph in \( \mathcal{H}_{-t}^-(p,q) \) with \( |A| = p \) and \( |B| = q \). Since \( s \leq q - 1 \leq p - 1 \), there exist vertices \( u \in A \) and \( v \in B \) such that \( N(u) = B \) and \( N(v) = A \). Thus, for any independent set \( Q \) in \( G \), if \( u \in Q \), then \( Q \subseteq A \); if \( v \in Q \), then \( Q \subseteq B \). Therefore for any 3-independent partition \( \{A_1,A_2,A_3\} \) in \( G \), there are at least two \( A_i \)'s, say \( A_2, A_3 \), such that \( A_2 \subseteq A \) and \( A_3 \subseteq B \). Hence \( G \) has only two types of 3-independent partitions \( \{A_1,A_2,A_3\} \):

- **Type 1:** either \( A_1 \cup A_2 = A, A_3 = B \) or \( A_1 \cup A_3 = A, A_2 = B \).
- **Type 2:** \( A_1 \cap A \neq \emptyset, A_1 \cap B \neq \emptyset, A_2 = A - A_1 \) and \( A_3 = B - A_1 \).

The number of 3-independent partitions of Type 1 is \( 2^{p-1} + 2^{q-1} - 2 \). Let \( \Psi(G) \) be the set of 3-independent partitions \( \{A_1,A_2,A_3\} \) of Type 2 in \( G \). Thus \( |\Psi(G)| = \chi'(G,3) \) by the definition of \( \chi'(G,3) \).

\[\Omega(G) = \{Q \mid Q \text{ is an independent set in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset\} \]
Since $s \leq q - 1 \leq p - 1$, $A - Q \neq \emptyset$ and $B - Q \neq \emptyset$ for any $Q \in \Omega(G)$. This implies that $Q \in \Omega(G)$ if and only if $\{Q, A - Q, B - Q\} \in \Psi(G)$. The following result is then obtained.

**Lemma 2.2.** $\chi'(G, 3) = |\Omega(G)|$ for any $G \in \mathcal{K}^{-1}(p, q)$.

We consider two special types of sets $Q \in \Omega(G)$: either $|Q \cap A| = 1$ or $|Q \cap B| = 1$. Let $\Omega_1(G) = \{Q \in \Omega(G) \mid |Q \cap A| = 1\}$ and $\Omega_2(G) = \{Q \in \Omega(G) \mid |Q \cap B| = 1\}$. Thus

$$|\Omega_1(G) \cap \Omega_2(G)| = s,$$

$$|\Omega_1(G)| = \sum_{x \in A'} (2^{d_{G'}(x)} - 1) \geq s,$$

$$|\Omega_2(G)| = \sum_{y \in B'} (2^{d_{G'}(y)} - 1) \geq s. \quad (1)$$

Let $\beta_i(G)$, or simply $\beta_i$, denote the number of vertices in $G$ with degree $i$, and let $n_i(G)$ denote the number of $i$-cycles in $G$.

**Lemma 2.3.** For $G = (A, B; E) \in \mathcal{K}^{-1}(p, q)$,

$$\chi'(G, 3) \geq s + \sum_{i \geq 2} \beta_i(G')(2^i - 1 - i) + n_4(G'), \quad (2)$$

where equality holds if and only if $|N_{G'}(x) \cap N_{G'}(y)| \leq 2$ for every $x, y \in A'$ or $x, y \in B'$.

**Proof.** The number of $Q \in \Omega(G)$ with $|Q \cap A| = 1$ or $|Q \cap B| = 1$ is

$$|\Omega_1(G) \cup \Omega_2(G)|$$

$$= -s + \sum_{x \in N_{G'}(x)} (2^{d_{G'}(x)} - 1)$$

$$= -s + \sum_{i \geq 1} \beta_i(G')(2^i - 1)$$

$$= -s + \sum_{i \geq 1} i\beta_i(G') + \sum_{i \geq 1} \beta_i(G')(2^i - 1 - i)$$

$$= -s + 2s + \sum_{i \geq 1} \beta_i(G')(2^i - 1 - i)$$

$$= s + \sum_{i \geq 2} \beta_i(G')(2^i - 1 - i).$$

Notice that the number of $Q$’s in $\Omega(G)$ such that $|Q \cap A| = 2$ and $|Q \cap B| = 2$ is exactly the number of 4-cycles in $G'$. Thus (2) is obtained by Lemma 2.2. The equality in (2) holds if and only if there is no $Q \in \Omega(G)$ such that either $|Q \cap A| \geq 3$ and $|Q \cap B| \geq 2$, or $|Q \cap A| \geq 2$ and $|Q \cap B| \geq 3$, i.e., $|N_{G'}(x) \cap N_{G'}(y)| \geq 3$ for $x, y \in A'$ or $x, y \in B'$. □
Corollary 2.1. For $G = (A, B; E) \in \mathcal{X}^{-1}(p, q)$,

(i) if $\Delta(G') \leq 2$, then $\chi'(G, 3) = s + \beta_2(G') + n_4(G')$;
(ii) if $\Delta(G') = 3$, then $\chi'(G, 3) \geq s + \beta_2(G') + 4\beta_3(G') + n_4(G')$, where equality holds
    iff $|N_{G'}(u) \cap N_{G'}(v)| \leq 2$ for all $u, v \in A'$ or $u, v \in B'$;
(iii) $\chi'(G, 3) \geq 2^{h(G')} + s - 1 - \Delta(G')$.

For two disjoint graphs $H_1$ and $H_2$, let $H_1 \cup H_2$ denote the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Let $kH = \underbrace{H \cup \cdots \cup H}_{k}$ for $k \geq 1$ and let $kH$ be null if $k = 0$.

Lemma 2.4. Let $G \in \mathcal{X}^{-1}(p, q)$. If $\chi'(G, 3) = s + t \leq s + 4$, then either

(i) each component of $G'$ is a path and $\beta_2(G') = t$, or
(ii) $G' \cong K_{1,3} \cup (s - 3)K_2$.

Proof. Since $\chi'(G, 3) \leq s + 4$, $\Delta(G') \leq 3$ by corollary (iii) to Lemma 2.3. If $\Delta(G') = 3$, then $\beta_3(G') = 0$ and $\beta_3(G') = 1$ by corollary (ii) to Lemma 2.3, and thus $G' \cong K_{1,3} \cup (s - 3)K_2$. If $\Delta(G') = 2$, then $\beta_2(G') + n_4(G') \leq 4$ by corollary (i) to Lemma 2.3, and thus $G'$ contains no cycles. Hence when $\Delta(G') = 2$, each component of $G'$ is a path, and $\beta_2(G') = t$ by corollary (i) to Lemma 2.3. □

Let $P_n$ denote the path with $n$ vertices. By Lemma 2.4, we establish the following result.

Theorem 2.1. Let $G \in \mathcal{X}^{-1}(p, q)$ and $\chi'(G, 3) = s + t$, where $0 \leq t \leq 4$. Then

$$G' \in \{ \begin{cases} \{sK_2\} & \text{if } t = 0, \\
\{P_3 \cup (s - 2)K_2\} & \text{if } t = 1, \\
\{P_4 \cup (s - 3)K_2, 2P_3 \cup (s - 4)K_2\} & \text{if } t = 2, \\
\{P_3 \cup (s - 4)K_2, P_4 \cup P_3 \cup (s - 5)K_2, 3P_3 \cup (s - 6)K_2\} & \text{if } t = 3, \\
\{P_6 \cup (s - 5)K_2, P_5 \cup P_3 \cup (s - 6)K_2, 2P_4 \cup (s - 6)K_2, \\
P_4 \cup 2P_3 \cup (s - 7)K_2, 4P_3 \cup (s - 8)K_2, K_{1,3} \cup (s - 3)K_2\} & \text{if } t = 4, \\
\end{cases} \}$$

where $H \cup (s - i)K_2$ does not exist if $s < i$.

3. Chromaticity of graphs in $\mathcal{B}(p, q, s, t)$, $t \leq 4$

In this section, we shall show that each graph in $\bigcup_{1 \leq i \leq 4} \mathcal{B}(p, q, s, t)$ is $\chi$-unique.

For a bipartite graph $G = (A, B; E)$, the number of 4-independent partitions $\{A_1, A_2, A_3, A_4\}$ with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is
\[ (2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \]
\[ = (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|}-1 + 3^{|B|}-1) - 2. \]  

(3)

Let \( x'(G, 4) = x(G, 4) - ((2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|}-1 + 3^{|B|}-1) - 2) \). Observe that for \( G, H \in \mathcal{K}(p, q) \), \( x(G, 4) = x(H, 4) \) iff \( x'(G, 4) = x'(H, 4) \).

**Lemma 3.1.** For \( G = (A, B; E) \in \mathcal{K}(p, q) \) with \( |A| = p \) and \( |B| = q \),
\[ x'(G, 4) = \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) \]
\[ + \vert\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset\}\vert. \]

**Proof.** As \( s \leq q - 1 \leq p - 1 \), there exist \( x \in A \) and \( y \in B \) such that \( N_G(x) = B \) and \( N_G(y) = A \). Thus, for any 4-independent partition \( \{A_1, A_2, A_3, A_4\} \), there are at least two \( A_i \)'s with \( A_i \subseteq A \) or \( A_i \subseteq B \). This means that \( G \) has only three types of 4-independent partitions \( \{A_1, A_2, A_3, A_4\} \): for \( k = 0, 1, 2 \), we call the partition type \( k \) if there are exactly \( k \) \( A_i \)'s with \( A_i \in \Omega(G) \). The number of 4-independent partitions of type 0 is given in (3). The number of 4-independent partitions of type 1 is
\[ \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) \]
and the number of 4-independent partitions of type 2 is
\[ \vert\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset\}\vert. \]
The lemma holds. \( \square \)

For a bipartite graph \( G = (A, B; E) \), let \( \beta_i(G, A) \) (resp. \( \beta_i(G, B) \)) be the number of vertices in \( A \) (resp. \( B \)) with degree \( i \).

**Remark.** For \( G \in \mathcal{B}(p, q; s, t) \), if each component of \( G' \) is a path, then \( x'(G, 3) = s + \beta_2(G') \) by Corollary 2.1(i) to Lemma 2.3. Thus \( \beta_2(G') = t \).

**Lemma 3.2.** For \( G \in \mathcal{B}(p, q; s, t) \), if each component of \( G' \) is a path, then
\[ \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) \]
\[ = s(2^{p-2} + 2^{q-2} - 2) + t(2^{p-3} + 2^{q-3} - 2) + (2^{p-3} - 2^{q-3})\beta_2(G', A'). \]

**Proof.** Since each component of \( G' \) is a path, \( |Q| \leq 3 \) for every \( Q \in \Omega(G) \). There are exactly \( s \) sets \( Q \) in \( \Omega(G) \) with \( |Q| = 2 \), there are exactly \( \beta_2(G', A') \) sets \( Q \) in \( \Omega(G) \)
with \(|Q \cap A| = 1\) and \(|Q \cap B| = 2\), and there are exactly \(\beta_2(G',B')\) sets \(Q\) in \(\Omega(G)\) with \(|Q \cap A| = 2\) and \(|Q \cap B| = 1\). Thus

\[
\sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2)
\]

\[= s(2^{p-2} + 2^{q-2} - 2) + \beta_2(G',A')(2^{p-2} + 2^{q-3} - 2)
+ \beta_2(G',B')(2^{p-3} + 2^{q-2} - 2)
\]

\[= s(2^{p-2} + 2^{q-2} - 2) + (\beta_2(G',A') + \beta_2(G',B'))(2^{p-3} + 2^{q-2} - 2)
+ (2^{p-3} - 2^{q-3})\beta_2(G',A')
\]

\[= s(2^{p-2} + 2^{q-2} - 2) + \beta_2(G')(2^{p-3} + 2^{q-2} - 2) + (2^{p-3} - 2^{q-3})\beta_2(G',A').
\]

Since \(\beta_2(G') = t\), the lemma is obtained. \(\square\)

Let \(p_t(G)\) denote the number of paths \(P_t\) in \(G\).

**Lemma 3.3.** For \(G \in \mathcal{B}(p,q,s,t)\), if each component of \(G'\) is a path, then

\[
|\{(Q_1,Q_2) | Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset\}|
\]

\[= \left(\frac{s + t}{2}\right) - 3t - 3p_s(G') - ps(G').
\]

**Proof.** Since each component of \(G'\) is a path, \(|Q| \leq 3\) for every \(Q \in \Omega(G)\). We also have \(\beta_2(G') = t\). There are exactly \(s\) (resp. \(t\)) sets \(Q\) in \(\Omega(G)\) with \(|Q| = 2\) (resp. \(|Q| = 3\)). Observe that

\[
|\{(Q_1,Q_2) | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 2, Q_1 \cap Q_2 = \emptyset\}|
\]

\[= |\{(Q_1,Q_2) | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 2, |Q_1 \cap Q_2| \leq 1\}|
\]

\[-|\{(Q_1,Q_2) | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 2, |Q_1 \cap Q_2| = 1\}|
\]

\[= \left(\frac{s}{2}\right) - t.
\]

(4)

There are exactly \(t\) \(Q_t\)’s with \(|Q_1| = 3\). For each \(Q_1 \in \Omega(G)\) with \(|Q_1| = 3\), there are exactly \(s - 2\) \(Q_2\)’s in \(\Omega(G)\) with \(|Q_2| = 2\) and \(|Q_1 \cap Q_2| \leq 1\). Observe that \(|Q_1 \cap Q_2| = 1\) iff \(Q_1 \cup Q_2\) induces a path \(P_3\) in \(G'\), and that for each path \(P_4\) in \(G'\), there are exactly two pairs \(Q_1,Q_2\) with \(|Q_1| = 3\), \(|Q_2| = 2\) and \(|Q_1 \cap Q_2| = 1\) such that \(Q_1 \cup Q_2\) induces this path \(P_4\). Thus the number of sets \(\{Q_1,Q_2\}\) with \(|Q_1| = 3\), \(|Q_2| = 2\) and \(|Q_1 \cap Q_2| = 1\) is \(2p_s(G')\). Hence

\[
|\{(Q_1,Q_2) | Q_1, Q_2 \in \Omega(G), |Q_1| = 3, |Q_2| = 2, Q_1 \cap Q_2 = \emptyset\}|
\]
Observe that for $j$ and $Q_1, Q_2 \in \mathcal{O}(G), |Q_1| = 3, |Q_2| = 2, |Q_1 \cap Q_2| \leq 1$]

$$= \binom{t}{2} - p_4(G') - p_5(G').$$

By (4)–(6), the result is obtained. 

For $G \in \mathcal{B}(p,q,s,t)$, define

$$z''(G,4) = z'(G,4) - (s(2^p-2 + 2^q-2 - 2) + t(2^p-3 + 2^q-2 - 2))$$

$$+ (s + t)(s + t - 1)/2 - 3t.$$  \hfill (7)

Observe that for $G, H \in \mathcal{B}(p,q,s,t)$, $z''(G,4) = z''(H,4)$ iff $z(G,4) = z(H,4)$.

**Lemma 3.4.** For $G \in \mathcal{B}(p,q,s,t)$, if each component of $G'$ is a path, then

$$z''(G,4) = (2^{p-3} - 2^{q-3})b_2(G',A') - 3p_4(G') - p_5(G').$$

**Proof.** It follows from Lemmas 3.1–3.3. 

For a graph $G$ with $uv \notin E(G)$, let $G + uv$ (resp. $G \cdot uv$) denote the graph obtained from $G$ by adding the edge $uv$ (resp. by identifying $u$ and $v$). For any vertex set $A \subseteq V(G)$, let $G - A$ denote the graph obtained from $G$ by deleting all vertices in $A$ and all edges incident to vertices in $A$.

For two disjoint graphs $G$ and $H$, let $G + H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy| x \in V(G), y \in V(H)\}$.
Lemma 3.5. For a bipartite graph $G=(A,B;E)$, if $uw$ is a path in $G'$ with $d_{G'}(u)=1$ and $d_{G'}(v)=2$, then for any $k \geq 2$,
\[ \alpha(G,k) = \alpha(G + uw,k) + \alpha(G - \{u,v\},k - 1) + \alpha(G - \{u,v,w\},k - 1). \]

Proof. Since $P(G,\lambda) = P(G + uw,\lambda) + P(G \cdot uv,\lambda)$, we have
\[ \alpha(G,k) = \alpha(G + uw,k) + \alpha(G \cdot uv,k). \]

Let $x$ be the vertex in $G \cdot uv$ produced by identifying $u$ and $v$. Notice that $x$ is adjacent to all vertices in $V(G \cdot uv) - \{x,w\}$. Thus $G \cdot uv + xw = K_1 + (G - \{u,v\})$ and $G \cdot uv \cdot xw = K_1 + (G - \{u,v,w\})$. We also observe that for any graph $H$, $\alpha(K_1 + H,k) = \alpha(H,k - 1)$, since
\[ P(K_1 + H,\lambda) = \lambda P(H,\lambda - 1). \]
Hence
\[ \alpha(G \cdot uv,k) = \alpha(G \cdot uv + xw,k) + \alpha(G \cdot uv \cdot xw,k) \]
\[ = \alpha(K_1 + (G - \{u,v\}),k) + \alpha(K_1 + (G - \{u,v,w\}),k) \]
\[ = \alpha(G - \{u,v\},k - 1) + \alpha(G - \{u,v,w\},k - 1). \]

The lemma is then obtained. \qed

Theorem 3.1. Let $p,q$ and $s$ be integers with $p \geq q \geq 3$ and $0 \leq s \leq q - 1$. For every $G \in \bigcup_{t=1}^{4} \mathcal{B}(p,q,s,t)$, if $G$ is 2-connected, then $G$ is $\chi$-unique.

Proof. By Theorem 1.3, $\mathcal{B}(p,q,s,t) \cap \mathcal{H}_2^{-\lambda}(p,q)$ is $\chi$-closed for each $t \geq 0$. To show that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is $\chi$-unique, it suffices to show that for every two graphs $G$ and $H$ in $\mathcal{B}(p,q,s,t)$, if $G \not\cong H$, then $\alpha(G,4) \neq \alpha(H,4)$ or $\alpha(G,5) \neq \alpha(H,5)$. Recall that for $G,H \in \mathcal{B}(p,q,s,t)$, $\chi''(G,4) \neq \chi''(H,4)$ if $\chi(G,4) \neq \chi(H,4)$.

For each $t = 1,2,3,4$, the graphs in $\mathcal{B}(p,q,s,t)$ are named as $G_{t,1}, G_{t,2}, \ldots$, and are shown in a table together with the values $\chi''(G_{t,1}), \chi''(G_{t,2}), \ldots$. For each graph $G_{t,i}$, if every component of $G_{t,i}'$, is a path, then $\chi''(G_{t,i},4)$ can be obtained by Lemma 3.4; otherwise, we must first find $\chi'(G_{t,i},4)$ by Lemma 3.1, and then find $\chi''(G_{t,i},4)$ by (7).

1) $\mathcal{B}(p,q,s,1)$: The set $\mathcal{B}(p,q,s,1)$ includes two graphs by Theorem 2.1, $G_{1,1}$ and $G_{1,2}$ (see Table 1). Notice that $\chi''(G_{1,1},4) \neq \chi''(G_{1,2},4)$ when $p \neq q$. But when $p = q$, $G_{1,1} \cong G_{1,2}$.

2) $\mathcal{B}(p,q,s,2)$: The set $\mathcal{B}(p,q,s,2)$ includes four graphs by Theorem 2.1, $G_{2,1}, G_{2,2}, G_{2,3}$ and $G_{2,4}$ (see Table 2). Notice that only $\chi''(G_{2,1},4)$ is odd. If $p > q$, the three values $\chi''(G_{2,2},4)$, $\chi''(G_{2,3},4)$ and $\chi''(G_{2,4},4)$ are distinct. If $p = q$, then $G_{2,2} \cong G_{2,3}$ and we shall show that $\alpha(G_{2,3},5) > \alpha(G_{2,4},5)$. When $p = q$, by Lemma 3.5
<table>
<thead>
<tr>
<th>Table 1</th>
<th>$\mathcal{A}(p, q, s, 1)$</th>
</tr>
</thead>
</table>
| name of graph | graphs $G_{1,i}$  
($G'_{1,i} = K_{p,q} - G_{1,i}$)  
($|A| = p, |B| = q$) | $\alpha'(G_{1,i}, 4)$ | conditions on $s$ |
| $G_{1,1}$ | ![Diagram](image1) | 0 | $2 \leq s \leq q - 1$ |
| $G_{1,2}$ | ![Diagram](image2) | $2^{p-3} - 2^{q-3}$ | $2 \leq s \leq q - 1$ |

<table>
<thead>
<tr>
<th>Table 2</th>
<th>$\mathcal{A}(p, q, s, 2)$</th>
</tr>
</thead>
</table>
| name of graph | graphs $G'_{2,i}$  
($G''_{2,i} = K_{p,q} - G_{2,i}$)  
($|A| = p, |B| = q$) | $\alpha''(G_{2,i}, 4)$ | conditions on $s$ |
| $G_{2,1}$ | ![Diagram](image3) | $2^{p-3} - 2^{q-3} - 3$ | $3 \leq s \leq q - 1$ |
| $G_{2,2}$ | ![Diagram](image4) | 0 | $4 \leq s \leq q - 1$ |
| $G_{2,3}$ | ![Diagram](image5) | $2(2^{p-3} - 2^{q-3})$ | $4 \leq s \leq q - 1$ |
| $G_{2,4}$ | ![Diagram](image6) | $2^{p-3} - 2^{q-3}$ | $4 \leq s \leq q - 1$ |
and Table 1, we have

\[ \alpha(G_{2,3}, 5) - \alpha(G_{2,4}, 5) \]

\[ = \alpha(G_{2,3} + u_1v_1, 5) + \alpha(G_{2,3} - \{u_1, v_1\}, 4) + \alpha(G_{2,3} - \{u_1, v_1, w_1\}, 4) \]

\[ - (\alpha(G_{2,4} + u_2v_2, 5) + \alpha(G_{2,4} - \{u_2, v_2\}, 4) + \alpha(G_{2,4} - \{u_2, v_2, w_2\}, 4)) \]

\[ = \alpha(G_{2,3} - \{u_1, v_1, w_1\}, 4) - \alpha(G_{2,4} - \{u_2, v_2, w_2\}, 4) \]

\[ = \alpha''(G_{2,3} - \{u_1, v_1\}, 4) - \alpha''(G_{2,4} - \{u_2, v_2\}, 4) \]

\[ = 2^{q-4} - 2^{q-5} \]

\[ > 0, \quad (8) \]

since \( G_{2,3} + u_1v_1 \cong G_{2,4} + u_2v_2 \), and both \( G_{2,3} - \{u_1, v_1, w_1\} \) and \( G_{2,4} - \{u_2, v_2, w_2\} \) belong to \( \mathcal{B}(q - 1, q - 2, s - 1) \).

(3) \( \mathcal{B}(p, q, s, 3) \): The set \( \mathcal{B}(p, q, s, 3) \) contains eight graphs by Theorem 2.1, \( G_{31}, G_{32}, \ldots, G_{38} \) (see Table 3). Notice that \( \alpha''(G_{3,1}, 4) \) is odd when \( 1 \leq i \leq 4 \) and even when \( i \geq 5 \). Thus \( \alpha''(G_{3,1}, 4) \neq \alpha''(G_{3,j}, 4) \) if \( 1 \leq i \leq 4 \) and \( 5 \leq j \leq 8 \). Observe that \( \alpha''(G_{3,1}, 4) + 7 \) contains a factor \( 2^{q-3} \) for \( i = 1, 2 \), but no factor 8 for \( i = 3, 4 \). Thus \( \alpha''(G_{3,i}, 4) \neq \alpha''(G_{3,j}, 4) \) for all \( i = 1, 2 \) and all \( j = 3, 4 \). When \( p > q \), \( \alpha''(G_{3,1}, 4) \neq \alpha''(G_{3,2}, 4) \) and \( \alpha''(G_{3,3}, 4) \neq \alpha''(G_{3,4}, 4) \). When \( p = q \), \( G_{3,1} \cong G_{3,2} \) and \( G_{3,3} \cong G_{3,4} \).

For \( i = 5, \ldots, 8 \), the \( \alpha''(G_{4,i}, 4) \)'s are distinct if \( p > q \). If \( p = q \), then \( G_{3,5} \cong G_{3,8} \) and \( G_{3,6} \cong G_{3,7} \), and by using the method in (8), we have

\[ \alpha(G_{3,7}, 5) - \alpha(G_{3,8}, 5) = \alpha''(G_{3,7} - \{u_1, v_1, w_1\}, 4) - \alpha''(G_{3,8} - \{u_2, v_2, w_2\}, 4) \]

\[ = -2^{q-4} < 0. \]

(4) \( \mathcal{B}(p, q, s, 4) \): The set \( \mathcal{B}(p, q, s, 4) \) has 16 graphs by Theorem 2.1, \( G_{4,1}, G_{4,2}, \ldots, G_{4,16} \) (see Table 4). Partition \( \mathcal{B}(p, q, s, 4) \) into subsets:

\[ \mathcal{I}_1 = \{G_{4,1}\}, \]

\[ \mathcal{I}_2 = \{G_{4,2}, G_{4,3}, G_{4,4}, G_{4,5}\}, \]

\[ \mathcal{I}_3 = \{G_{4,6}, G_{4,7}, G_{4,8}\}, \]

\[ \mathcal{I}_4 = \{G_{4,9}\}, \]

\[ \mathcal{I}_5 = \{G_{4,10}, G_{4,11}\}, \]

\[ \mathcal{I}_6 = \{G_{4,12}, G_{4,13}, G_{4,14}, G_{4,15}, G_{4,16}\}. \]

For non-empty sets \( W_1, \ldots, W_k \) of graphs, let \( \eta(W_1, \ldots, W_k) = 0 \) if \( \alpha(G_{1,4}) \neq \alpha(G_{2,4}) \) for every two graphs \( G_1 \in W_i \) and \( G_2 \in W_j \), where \( i \neq j \), and let \( \eta(W_1, \ldots, W_k) = 1 \) otherwise.
Table 3

<table>
<thead>
<tr>
<th>name of graph</th>
<th>graphs $G_{3,i}$ $(G_{3,i} = K_{p,q} - G_{3,i})$</th>
<th>$\chi''(G_{3,i}, 4)$</th>
<th>conditions on $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{3,1}$</td>
<td>$A$ $B$ $s = 4$</td>
<td>$(2^{p-3} - 2^{q-3}) - 7$</td>
<td>$4 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,2}$</td>
<td>$A$ $B$ $s = 4$</td>
<td>$(2^{p-3} - 2^{q-3}) - 7$</td>
<td>$4 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,3}$</td>
<td>$A$ $B$ $s = 5$</td>
<td>$2(2^{p-3} - 2^{q-3}) - 3$</td>
<td>$5 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,4}$</td>
<td>$A$ $B$ $s = 5$</td>
<td>$2(2^{p-3} - 2^{q-3}) - 3$</td>
<td>$5 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,5}$</td>
<td>$A$ $B$ $s = 6$</td>
<td>$0$</td>
<td>$6 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,6}$</td>
<td>$A$ $B$ $s = 6$</td>
<td>$(2^{p-3} - 2^{q-3})$</td>
<td>$6 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,7}$</td>
<td>$A$ $B$ $s = 6$</td>
<td>$(2^{p-3} - 2^{q-3})$</td>
<td>$6 \leq s \leq q - 1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{3,8}$</td>
<td>$A$ $B$ $s = 6$</td>
<td>$3(2^{p-3} - 2^{q-3})$</td>
<td>$6 \leq s \leq q - 1$</td>
</tr>
</tbody>
</table>

The values of $\chi''(G_{4,10,4})$ and $\chi''(G_{4,11,4})$ are not given by Lemma 3.4, but can be obtained by Lemma 3.1 and (7). We have

$$
\chi''(G_{4,10,4}) = s(2^{p-2} + 2^{q-2} - 2) + 3(2^{p-3} + 2^{q-2} - 2) + (2^{p-4} + 2^{q-2} - 2) + \left(\frac{s}{2}\right) - 3 + 4(s - 3) - s(2^{p-2} + 2^{q-2} - 2) - 4(2^{p-3} + 2^{q-2} - 2) - \left(\frac{s + 4}{2}\right) + 12
$$

$$
= -2^{p-4} - 9.
$$
Similarly, we find $x''(G_{4,11}, 4) = 2^{p-1} - 9 \cdot 2^{q-4} - 9$.

**Claim 1.** $\eta(\mathcal{S}_1, \ldots, \mathcal{S}_6) = 0$:  

(a) If $s \leq 4$, only $\mathcal{S}_5$ is non-empty.  

(b) For $s \geq 5$, $x''(G, 4)$ is odd if $G \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_5$ and even if $G \in \mathcal{S}_4 \cup \mathcal{S}_6$.  

Hence $\eta(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4, \mathcal{S}_5 \cup \mathcal{S}_6) = 0$.  

<table>
<thead>
<tr>
<th>name of graph</th>
<th>graphs $G'_{4,i}$</th>
<th>conditions on $s$</th>
<th>$\alpha''(G_{4,i}, 4)$</th>
<th>$\beta''(p,q,4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{4,1}$</td>
<td></td>
<td></td>
<td>$2(2^{p-3} - 2^{q-3})$</td>
<td>$-11$</td>
</tr>
<tr>
<td>$G_{4,2}$</td>
<td></td>
<td></td>
<td>$(2^{p-3} - 2^{q-3})$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$G_{4,3}$</td>
<td></td>
<td></td>
<td>$2(2^{p-3} - 2^{q-3})$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$G_{4,4}$</td>
<td></td>
<td></td>
<td>$2(2^{p-3} - 2^{q-3})$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$G_{4,5}$</td>
<td></td>
<td></td>
<td>$3(2^{p-3} - 2^{q-3})$</td>
<td>$-7$</td>
</tr>
<tr>
<td>$G_{4,6}$</td>
<td></td>
<td></td>
<td>$3(2^{p-3} - 2^{q-3})$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$G_{4,7}$</td>
<td></td>
<td></td>
<td>$2(2^{p-3} - 2^{q-3})$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$G_{4,8}$</td>
<td></td>
<td></td>
<td>$(2^{p-3} - 2^{q-3})$</td>
<td>$-3$</td>
</tr>
</tbody>
</table>

Table 4
$\beta''(p,q,s,4)$
Table 4. (continued)

<table>
<thead>
<tr>
<th>name of graph</th>
<th>graphs $G_{4,i}$ ((G'<em>{4,i} = K</em>{p,q} - G_{4,i}))</th>
<th>$\alpha^p(G_{4,i}, 4)$</th>
<th>conditions on $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{4,9}$</td>
<td>$s - 6 \quad A$ \quad ... \quad $B$</td>
<td>$2(2^{p-3} - 2^{q-3}) - 6$</td>
<td>$6 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,10}$</td>
<td>$s - 3 \quad A$ \quad ... \quad $B$</td>
<td>$-2^{p-4}$ \quad $-9$</td>
<td>$3 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,11}$</td>
<td>$s - 3 \quad A$ \quad ... \quad $B$</td>
<td>$(2^{p-1} - 9 \cdot 2^{q-4}) - 9$</td>
<td>$3 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,12}$</td>
<td>$s - 8 \quad A$ \quad ... \quad $B$</td>
<td>$4(2^{p-3} - 2^{q-3})$</td>
<td>$8 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,13}$</td>
<td>$s - 8 \quad A$ \quad ... \quad $B$</td>
<td>$3(2^{p-3} - 2^{q-3})$</td>
<td>$8 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,14}$</td>
<td>$b_0 \quad b_0 \quad b_0 \quad b_0$ \quad $A$ \quad ... \quad $B$</td>
<td>$2(2^{p-3} - 2^{q-3})$</td>
<td>$8 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,15}$</td>
<td>$c_0 \quad c_0 \quad c_0 \quad c_0 \quad A$ \quad ... \quad $B$</td>
<td>$(2^{p-3} - 2^{q-3})$</td>
<td>$8 \leq s \leq q - 1$</td>
</tr>
<tr>
<td>$G_{4,16}$</td>
<td>$d_0 \quad d_0 \quad d_0 \quad d_0 \quad A$ \quad ... \quad $B$</td>
<td>$0$</td>
<td>$8 \leq s \leq q - 1$</td>
</tr>
</tbody>
</table>

(c) For $s \geq 5$, we have $q \geq 6$ and $2^{q-4}$ is a factor of $\chi''(G,4) + 9$ for every $G \in \mathcal{S}_5$, but 4 is not a factor of $\chi''(G,4) + 9$ for every $G \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Hence $\eta(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3, \mathcal{S}_5) = 0$.

(d) For $s \geq 5$, we have $q \geq 6$, and $2^{q-2}$ is a factor of $\chi''(G,4) + 11$ for $G \in \mathcal{S}_1$, $2^3$ is not a factor of $\chi''(G,4) + 11$ for $G \in \mathcal{S}_2$, and $2^3$ is a factor of $\chi''(G,4) + 11$ but $2^4$ is not for $G \in \mathcal{S}_3$. Hence $\eta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = 0$. 
(e) For \( s \geq 6 \), we have \( q \geq 7 \), and \( 2^2 \) is a factor of \( \omega'(G, 4) \) for every \( G \in \mathcal{S}_6 \) but it is not for every \( G \in \mathcal{S}_4 \). Hence \( \eta(\mathcal{S}_4, \mathcal{S}_6) = 0 \).

By (b)–(e), Claim 1 holds.

The remaining work is to compare every two graphs in each \( \mathcal{S}_i \). Both \( \mathcal{S}_1 \) and \( \mathcal{S}_4 \) contain only one graph. For \( \mathcal{S}_5 \), when \( p=q \), \( G_{4,10} \cong G_{4,11} \); when \( p>q \), \( \omega''(G_{4,10}, 4) \neq \omega''(G_{4,11}, 4) \). In the following, we shall study the three sets \( \mathcal{S}_2 \), \( \mathcal{S}_3 \) and \( \mathcal{S}_6 \).

(4.1) \( \mathcal{S}_2 \): When \( p>q \), \( \omega''(G_{4,6}, 4) < \omega''(G_{4,7}, 4) < \omega''(G_{4,8}, 4) \). When \( p=q \), we have \( G_{4,6} \cong G_{4,8} \) and by the method used in (8),

\[
\omega(G_{4,7}, 5) - \omega(G_{4,8}, 5) = \omega(G_{4,7} - \{a_4, b_4, c_4\}, 4) - \omega(G_{4,8} - \{a_5, b_5, c_5\}, 4) = -2^{q-5} \neq 0.
\]

(4.2) \( \mathcal{S}_6 \): When \( p>q \),

\[
\omega''(G_{4,12}, 4) > \omega''(G_{4,13}, 4) > \omega''(G_{4,14}, 4) > \omega''(G_{4,15}, 4) > \omega''(G_{4,16}, 4).
\]

When \( p=q \), \( G_{4,12} \cong G_{4,16} \), \( G_{4,13} \cong G_{4,15} \) and by the method used in (8),

\[
\omega(G_{4,14}, 5) - \omega(G_{4,15}, 5) = -2^{q-5},
\]

\[
\omega(G_{4,15}, 5) - \omega(G_{4,16}, 5) = -3 \times 2^{q-5} < 0.
\]

(4.3) \( \mathcal{S}_3 \): Observe that \( \omega''(G_{4,3}, 4) = \omega''(G_{4,4}, 4) \). When \( p>q \),

\[
\omega''(G_{4,2}, 4) < \omega''(G_{4,3}, 4) < \omega''(G_{4,4}, 4).
\]

When \( p=q \), \( G_{4,2} \cong G_{4,5} \) and \( G_{4,3} \cong G_{4,4} \). In the following, we shall compare \( \omega(G_{4,4}, 5) \) with \( \omega(G_{4,5}, 5) \) for \( p=q \), and \( \omega(G_{4,3}, 5) \) with \( \omega(G_{4,4}, 5) \) for \( p>q \).

By Lemma 3.5, when \( p=q \), by the method used in (8),

\[
\omega(G_{4,4}, 5) - \omega(G_{4,5}, 5) = \omega(G_{4,4} - \{a_4, b_4, c_4\}, 4) - \omega(G_{4,5} - \{a_5, b_5, c_5\}, 5)
\]

\[
= -2^{q-5} < 0.
\]

For \( G_{4,3} \) and \( G_{4,4} \), we prove the following claim:

Claim 2. \( \omega(G_{4,4}, 5) - \omega(G_{4,4}, 5) = 3(2^{p-5} - 2^{q-5}) \).

By Lemma 3.5,

\[
\omega(G_{4,3}, 5) = \omega(G_{4,3} + a_1b_1, 5) + \omega(G_{4,3} - \{a_1, b_1\}, 4) + \omega(G_{4,3} - \{a_1, b_1, c_1\}, 4)
\]

\[
= \omega(G_{4,3} + a_1b_1 + b_1c_1, 5) + \omega(G_{4,3} - \{b_1, c_1\}, 4) + \omega(G_{4,3} - \{b_1, c_1, d_1\}, 4)
\]

\[
+ \omega(G_{4,3} - \{a_1, b_1\}, 4) + \omega(G_{4,3} - \{a_1, b_1, c_1\}, 4).
\]
and
\[
\alpha(G_{4,4}, 5) = \alpha(G_{4,4} + a_2 b_2, 5) + \alpha(G_{4,4} - \{a_2, b_2\}, 4) + \alpha(G_{4,4} - \{a_2, b_2, c_2\}, 4)
\]
\[
= \alpha(G_{4,4} + a_2 b_2 + b_2 c_2, 5) + \alpha(G_{4,4} - \{b_2, c_2\}, 4) + \alpha(G_{4,4} - \{b_2, c_2, d_2\}, 4)
\]
\[
+ \alpha(G_{4,4} - \{a_2, b_2\}, 4) + \alpha(G_{4,4} - \{a_2, b_2, c_2\}, 4).
\]
Observe that
\[
G_{4,3} + a_1 b_1 + b_1 c_1 \cong G_{4,4} + a_2 b_2 + b_2 c_2,
\]
\[
\alpha(G_{4,3} - \{b_1, c_1\}, 4) - \alpha(G_{4,4} - \{b_2, c_2\}, 4) = 2^{p-4} - 2^{q-4},
\]
\[
G_{4,3} - \{a_1, b_1\} \cong G_{4,4} - \{a_2, b_2\}.
\]  
(9)

Since
\[
G_{4,3} - \{a_1, b_1, c_1\} \in \mathcal{B}(p - 2, q - 1, s - 3, 1),
\]
\[
G_{4,4} - \{b_2, c_2, d_2\} \in \mathcal{B}(p - 2, q - 1, s - 4, 1),
\]
by Lemma 3.1, we have
\[
\alpha(G_{4,3} - \{a_1, b_1, c_1\}, 4) - \alpha(G_{4,4} - \{b_2, c_2, d_2\}, 4)
\]
\[
= \alpha'(G_{4,3} - \{a_1, b_1, c_1\}, 4) - \alpha'(G_{4,4} - \{b_2, c_2, d_2\}, 4)
\]
\[
= (s - 3)(2^{p-4} + 2^{q-3} - 2) + (2^{p-4} + 2^{q-4} - 2) + \left(\frac{s - 3}{2}\right) - 1 + (s - 5)
\]
\[
- ((s - 4)(2^{p-4} + 2^{q-3} - 2) + (2^{p-5} + 2^{q-3} - 2)
\]
\[
+ \left(\frac{s - 4}{2}\right) - 1 + (s - 6)
\]
\[
= 2^{p-4} + 2^{p-5} + 2^{q-4} + s - 5.
\]  
(10)
Similarly, since
\[
G_{4,3} - \{b_1, c_1, d_1\} \in \mathcal{B}(p - 1, q - 2, s - 4, 1),
\]
\[
G_{4,4} - \{a_2, b_2, c_2\} \in \mathcal{B}(p - 1, q - 2, s - 3, 1),
\]
by Lemma 3.1, we have
\[
\alpha(G_{4,3} - \{b_1, c_1, d_1\}, 4) - \alpha(G_{4,4} - \{a_2, b_2, c_2\}, 4)
\]
\[
= \alpha'(G_{4,3} - \{b_1, c_1, d_1\}, 4) - \alpha'(G_{4,4} - \{a_2, b_2, c_2\}, 4)
\]
\[
= (s - 4)(2^{p-3} + 2^{q-4} - 2) + (2^{p-3} + 2^{q-5} - 2) + \left(\frac{s - 4}{2}\right) - 1 + (s - 6)
\]
By \((9)-(11)\), Claim 2 is proved. \(\square\)

Finally, we conclude that for every two graphs \(G_1, G_2 \in \mathcal{B}(p, q, s, 4)\), if \(G_1 \neq G_2\), then either \(\chi''(G_1, 4) \neq \chi''(G_2, 4)\) or \(\chi(G_1, 5) \neq \chi(G_2, 5)\). This completes the proof of the result. \(\square\)

**Theorem 3.2.** For any \(G \in \mathcal{K}_{2}^{-}(p, q)\) with \(p \geq q \geq 3\) and \(0 \leq s \leq \min\{4, q - 1\}\), \(G\) is \(\chi\)-unique.

**Proof.** Let \(G \in \mathcal{K}_{2}^{-}(p, q)\). If \(s \leq 3\), then by Theorem 1.2, \(\chi'(G, 3) \leq 2^s - 1 \leq s + 4\). Thus by Theorem 3.1, \(G\) is \(\chi\)-unique if \(s \leq 3\). Now suppose that \(s = 4\). We have \(q \geq 5\). If \(\Delta(G') \in \{1, 4\}\), then \(\chi'(G, 3) = s\) or \(\chi'(G, 3) = 2^s - 1\) and thus \(G\) is \(\chi\)-unique by Theorem 1.4. If \(\Delta(G') = 2\) and \(G' \neq K_{2,2}\), then \(\chi'(G, 3) \leq s + 3\) by Corollary (i) to Lemma 2.3, and thus \(G\) is \(\chi\)-unique by Theorem 3.1. If \(G' = K_{1,1} \cup K_{2}\), then \(\chi'(G, 3) = 8 = s + 4\), and thus \(G\) is \(\chi\)-unique by Theorem 3.1. Otherwise, there are only two possible structures for \(G'\). They are shown in Table 5. For \(i=1, 2, 3\), \(\chi'(R_i, 4)\) is obtained by Lemma 3.1. From Table 5, observe that \(\chi'(R_i, 4)\) is even when \(i = 1\) and odd when

<table>
<thead>
<tr>
<th>name of graph</th>
<th>graphs (R_i^p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_1)</td>
<td>({A})</td>
</tr>
<tr>
<td>(R_2)</td>
<td>({B})</td>
</tr>
<tr>
<td>(R_3)</td>
<td>({A})</td>
</tr>
</tbody>
</table>

\[
\frac{-(s-3)(2^{p-3}+2^{q-4}-2)+(2^{p-4}+2^{q-4}-2)}{+\left(\frac{s-3}{2}\right)-1+(s-5)} = -2^{p-4} - 2^{q-4} - 2^{q-5} - s + 5.
\]
i = 2, 3. When \( p = q \), \( R_2 \cong R_3 \); when \( p > q \), 
\[
\Delta'(R_2, 4) - \Delta'(R_3, 4) = 7(2^{p-4} - 2^{q-4}) > 0.
\]

Hence \( G \) is \( \chi \)-unique when \( \Delta(G') \in \{2, 3\} \). This completes the proof. \( \square \)

4. For further reading

The following references are also of interest to the reader: [3,4].

References


