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Chromatically unique bipartite graphs with low 3-independent partition numbers

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Abstract

For integers p,q,s with $p\geqslant q\geqslant 2$ and $s\geqslant 0$, let $\mathscr{K}_2^{-s}(p,q)$ denote the set of 2-connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. In this paper, we prove that for any graph $G\in\mathscr{K}_2^{-s}(p,q)$ with $p\geqslant q\geqslant 3$ and $1\leqslant s\leqslant q-1$, if the number of 3-independent partitions of G is at most $2^{p-1}+2^{q-1}+s+2$, then G is χ-unique. It follows that any graph in $\mathscr{K}_2^{-s}(p,q)$ is χ-unique if $p\geqslant q\geqslant 3$ and $1\leqslant s\leqslant \min\{q-1,4\}$.

Keywords: Bipartite graph; Chromatic polynomial

1. Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $E(G)$, $e(G)$, $\delta(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, edge set, size, minimum degree, maximum degree and the chromatic polynomial of G, respectively.

For integers p,q,s with $p\geqslant q\geqslant 2$ and $s\geqslant 0$, let $\mathscr{K}^{-s}(p,q)$ (resp. $\mathscr{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2-connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. The following result was obtained in [1].

Lemma 1.1. *If* $p\geqslant q\geqslant 3$ *and* $s\leqslant p+q-4$, *then for any* $G\in\mathscr{K}^{-s}(p,q)$ *with* $\delta(G)\geqslant 2$, G *is* 2-*connected.*

For a bipartite graph $G=(A,B;E)$ with bipartition A and B and edge set E, let $G'=(A',B';E')$ be the bipartite graph induced by the edge set $E'=\{xy|xy\notin E, x\in A, y\in B\}$, where $A'\subseteq A$ and $B'\subseteq B$. We write $G'=K_{p,q}-G$, where $p=|A|$ and $q=|B|$. Observe that $\delta(G)=\min(q-\Delta(G'), p-\Delta(G'))$.

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Corollary 1.1. *For* $p \geqslant q \geqslant 3$ *and* $0 \leqslant s \leqslant q-1$, *if* $G \in K^{-s}(p,q) - K_2^{-s}(p,q)$, *then* $s = q-1$ *and* $\Delta(G') = q-1$.

Two graphs G and H are said to be *chromatically equivalent* (or simply χ-*equivalent*), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ-unique) if $H \cong G$ whenever $H \sim G$. For a set $\mathscr{G}$ of graphs, if $[G] \subseteq \mathscr{G}$ for every $G \in \mathscr{G}$, then $\mathscr{G}$ is said to be χ-*closed*. In [1], we established the following result.

Theorem 1.1. *For integers* p,q,s *with* $p \geqslant q \geqslant 2$ *and* $0 \leqslant s \leqslant q-1$, $\mathscr{K}_2^{-s}(p,q)$ *is* χ-*closed.*

The complete bipartite graph $K_{p,q}$ is χ-unique for any $p \geqslant q \geqslant 2$ (see [2,6]). In this paper, we shall search for χ-unique graphs or χ-equivalence classes from the set $\mathscr{K}_2^{-s}(p,q)$, where $p \geqslant q \geqslant 3$ and $0 \leqslant s \leqslant q-1$. Hence, in this paper, we fix the following conditions for p,q and s:

$$p \geqslant q \geqslant 3 \quad \text{and} \quad 0 \leqslant s \leqslant q-1.$$

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a k-*independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any bipartite graph $G = (A,B;E)$, define

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2).$$

In [1], we found the following sharp bounds for $\alpha'(G,3)$:

Theorem 1.2. *For* $G \in \mathscr{K}^{-s}(p,q)$ *with* $p \geqslant q \geqslant 3$ *and* $0 \leqslant s \leqslant q-1$,

$$s \leqslant \alpha'(G,3) \leqslant 2^s - 1,$$

where $\alpha'(G,3) = s$ *iff* $\Delta(G') = 1$ *and* $\alpha'(G,3) = 2^s - 1$ *iff* $\Delta(G') = s$.

For $t = 0,1,2,\ldots$, let $\mathscr{B}(p,q,s,t)$ denote the set of graphs $G \in \mathscr{K}^{-s}(p,q)$ with $\alpha'(G,3) = s+t$. Thus, $\mathscr{K}^{-s}(p,q)$ is partitioned into the following subsets:

$$\mathscr{B}(p,q,s,0),\ \mathscr{B}(p,q,s,1), \ldots, \mathscr{B}(p,q,s,2^s - s - 1).$$

Assume that $\mathscr{B}(p,q,s,t) = \emptyset$ for $t > 2^s - s - 1$.

Lemma 1.2. *For* $p \geqslant q \geqslant 3$ *and* $0 \leqslant s \leqslant q-1$, *if* $0 \leqslant t \leqslant 2^{q-1} - q - 1$, *then*

$$\mathscr{B}(p,q,s,t) \subseteq \mathscr{K}_2^{-s}(p,q).$$

Proof. We consider the following two cases.

Case 1: $s \leqslant q-2$. By the corollary to Lemma 1.1, $\mathscr{K}^{-s}(p,q) = \mathscr{K}_2^{-s}(p,q)$ and thus $\mathscr{B}(p,q,s,t) \subseteq \mathscr{K}_2^{-s}(p,q)$ for all t.

Case 2: $s=q-1$. If $0\leqslant t\leqslant 2^{q-1}-q-1$, by Theorem 1.2, for any $G\in\mathscr{B}(p,q,s,t)$, we have $\Delta(G')\leqslant q-2$ and thus by the corollary to Lemma 1.1, G is 2-connected. Hence $\mathscr{B}(p,q,s,t)\subseteq\mathscr{K}_2^{-s}(p,q)$ if $0\leqslant t\leqslant 2^{q-1}-q-1$. □

For any graph G, we have $P(G,\lambda)=\sum_{k\geqslant 1}\alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1)$ (see [5]). If $G\sim H$, then $\alpha(G,k)=\alpha(H,k)$ for $k=1,2,\ldots$. Thus, by Theorem 1.1, the following result is obtained.

Theorem 1.3. *The set* $\mathscr{B}(p,q,s,t)\cap\mathscr{K}_2^{-s}(p,q)$ *is* χ*-closed for all* $t\geqslant 0$.

Corollary 1.2. *If* $0\leqslant t\leqslant 2^{q-1}-q-1$, *then* $\mathscr{B}(p,q,s,t)$ *is* χ*-closed.*

We have proved in [1] the following result.

Theorem 1.4. *For any graph* $G\in\mathscr{B}(p,q,s,0)\cup\mathscr{B}(p,q,s,2^s-s-1)$, *if* G *is* 2*-connected, then* G *is* χ*-unique.*

In this paper, we shall show that every 2-connected graph in $\mathscr{B}(p,q,s,t)$ is χ-unique for $1\leqslant t\leqslant 4$. Further, we prove that every graph in $\mathscr{K}_2^{-s}(p,q)$ is χ-unique if $1\leqslant s\leqslant\min\{4,q-1\}$.

2. $\mathscr{B}(p,q,s,t)$ for $t\leqslant 4$

In this section, we shall study the structure of graphs in $\mathscr{B}(p,q,s,t)$ for $t\leqslant 4$.

Lemma 2.1. *For* $G=(A,B;E)\in\mathscr{K}^{-s}(p,q)$ *with* $|A|=p$ *and* $|B|=q$, *we have*

$$e(G')=\sum_{x\in A'}d_{G'}(x)=\sum_{y\in B'}d_{G'}(y)=s.$$

For a graph G and $x\in V(G)$, let $N_G(x)$ or simply $N(x)$ denote the set of vertices y such that $xy\in E(G)$. Let $G=(A,B;E)$ be a graph in $\mathscr{K}^{-s}(p,q)$ with $|A|=p$ and $|B|=q$. Since $s\leqslant q-1\leqslant p-1$, there exist vertices $u\in A$ and $v\in B$ such that $N(u)=B$ and $N(v)=A$. Thus, for any independent set Q in G, if $u\in Q$, then $Q\subseteq A$; if $v\in Q$, then $Q\subseteq B$. Therefore for any 3-independent partition $\{A_1,A_2,A_3\}$ in G, there are at least two A_i's, say A_2,A_3, such that $A_2\subseteq A$ and $A_3\subseteq B$. Hence G has only two types of 3-independent partitions $\{A_1,A_2,A_3\}$:

Type 1: either $A_1\cup A_2=A$, $A_3=B$ or $A_1\cup A_3=B$, $A_2=A$.

Type 2: $A_1\cap A\neq\emptyset$, $A_1\cap B\neq\emptyset$, $A_2=A-A_1$ and $A_3=B-A_1$.

The number of 3-independent partitions of Type 1 is $2^{p-1}+2^{q-1}-2$. Let $\Psi(G)$ be the set of 3-independent partitions $\{A_1,A_2,A_3\}$ of Type 2 in G. Thus $|\Psi(G)|=\alpha'(G,3)$ by the definition of $\alpha'(G,3)$. Let

$$\Omega(G)=\{Q|Q \text{ is an independent set in } G \text{ with } Q\cap A\neq\emptyset, Q\cap B\neq\emptyset\}.$$

Since $s \leqslant q-1 \leqslant p-1$, $A-Q \neq \emptyset$ and $B-Q \neq \emptyset$ for any $Q \in \Omega(G)$. This implies that $Q \in \Omega(G)$ iff $\{Q, A-Q, B-Q\} \in \Psi(G)$. The following result is then obtained.

Lemma 2.2. *$\alpha'(G,3) = |\Omega(G)|$ for any $G \in \mathscr{K}^{-s}(p,q)$.*

We consider two special types of sets $Q \in \Omega(G)$: either $|Q \cap A| = 1$ or $|Q \cap B| = 1$. Let $\Omega_1(G) = \{Q \in \Omega(G) \mid |Q \cap A| = 1\}$ and $\Omega_2(G) = \{Q \in \Omega(G) \mid |Q \cap B| = 1\}$. Thus

$$|\Omega_1(G) \cap \Omega_2(G)| = s,$$

$$|\Omega_1(G)| = \sum_{x \in A'} (2^{d_{G'}(x)} - 1) \geqslant s,$$

$$|\Omega_2(G)| = \sum_{y \in B'} (2^{d_{G'}(y)} - 1) \geqslant s. \qquad (1)$$

Let $\beta_i(G)$, or simply β_i, denote the number of vertices in G with degree i, and let $n_i(G)$ denote the number of i-cycles in G.

Lemma 2.3. *For $G = (A,B;E) \in \mathscr{K}^{-s}(p,q)$,*

$$\alpha'(G,3) \geqslant s + \sum_{i \geqslant 2} \beta_i(G')(2^i - 1 - i) + n_4(G'), \qquad (2)$$

where equality holds iff $|N_{G'}(x) \cap N_{G'}(y)| \leqslant 2$ for every $x, y \in A'$ or $x, y \in B'$.

Proof. The number of $Q \in \Omega(G)$ with $|Q \cap A| = 1$ or $|Q \cap B| = 1$ is

$$\begin{aligned} &|\Omega_1(G) \cup \Omega_2(G)| \\ &\quad = -s + \sum_{x \in V(G')} (2^{d_{G'}(x)} - 1) \\ &\quad = -s + \sum_{i \geqslant 1} \beta_i(G')(2^i - 1) \\ &\quad = -s + \sum_{i \geqslant 1} i\beta_i(G') + \sum_{i \geqslant 1} \beta_i(G')(2^i - 1 - i) \\ &\quad = -s + 2s + \sum_{i \geqslant 1} \beta_i(G')(2^i - 1 - i) \\ &\quad = s + \sum_{i \geqslant 2} \beta_i(G')(2^i - 1 - i). \end{aligned}$$

Notice that the number of Q's in $\Omega(G)$ such that $|Q \cap A| = 2$ and $|Q \cap B| = 2$ is exactly the number of 4-cycles in G'. Thus (2) is obtained by Lemma 2.2. The equality in (2) holds iff there is no $Q \in \Omega(G)$ such that either $|Q \cap A| \geqslant 3$ and $|Q \cap B| \geqslant 2$, or $|Q \cap A| \geqslant 2$ and $|Q \cap B| \geqslant 3$, i.e., $|N_{G'}(x) \cap N_{G'}(y)| \geqslant 3$ for $x, y \in A'$ or $x, y \in B'$. $\square$

Corollary 2.1. *For* $G=(A,B;E)\in \mathscr{K}^{-s}(p,q)$,

(i) *if* $\Delta(G')\leqslant 2$, *then* $\alpha'(G,3)=s+\beta_2(G')+n_4(G')$;
(ii) *if* $\Delta(G')=3$, *then* $\alpha'(G,3)\geqslant s+\beta_2(G')+4\beta_3(G')+n_4(G')$, *where equality holds iff* $|N_{G'}(u)\cap N_{G'}(v)|\leqslant 2$ *for all* $u,v\in A'$ *or* $u,v\in B'$;
(iii) $\alpha'(G,3)\geqslant 2^{\Delta(G')}+s-1-\Delta(G')$.

For two disjoint graphs H_1 and H_2, let $H_1\cup H_2$ denote the graph with vertex set $V(H_1)\cup V(H_2)$ and edge set $E(H_1)\cup E(H_2)$. Let $kH=\underbrace{H\cup\cdots\cup H}_{k}$ for $k\geqslant 1$ and let kH be null if $k=0$.

Lemma 2.4. *Let* $G\in\mathscr{K}^{-s}(p,q)$. *If* $\alpha'(G,3)=s+t\leqslant s+4$, *then either*

(i) *each component of* G' *is a path and* $\beta_2(G')=t$, *or*
(ii) $G'\cong K_{1,3}\cup(s-3)K_2$.

Proof. Since $\alpha'(G,3)\leqslant s+4$, $\Delta(G')\leqslant 3$ by corollary (iii) to Lemma 2.3. If $\Delta(G')=3$, then $\beta_2(G')=0$ and $\beta_3(G')=1$ by corollary (ii) to Lemma 2.3, and thus $G'\cong K_{1,3}\cup(s-3)K_2$. If $\Delta(G')=2$, then $\beta_2(G')+n_4(G')\leqslant 4$ by corollary (i) to Lemma 2.3, and thus G' contains no cycles. Hence when $\Delta(G')=2$, each component of G' is a path, and $\beta_2(G')=t$ by corollary (i) to Lemma 2.3. □

Let P_n denote the path with n vertices. By Lemma 2.4, we establish the following result.

Theorem 2.1. *Let* $G\in\mathscr{K}^{-s}(p,q)$ *and* $\alpha'(G,3)=s+t$, *where* $0\leqslant t\leqslant 4$. *Then*

$$G'\in\begin{cases}\{sK_2\} & \text{if } t=0,\\ \{P_3\cup(s-2)K_2\} & \text{if } t=1,\\ \{P_4\cup(s-3)K_2,2P_3\cup(s-4)K_2\} & \text{if } t=2,\\ \{P_5\cup(s-4)K_2,P_4\cup P_3\cup(s-5)K_2,3P_3\cup(s-6)K_2\} & \text{if } t=3,\\ \{P_6\cup(s-5)K_2,P_5\cup P_3\cup(s-6)K_2,2P_4\cup(s-6)K_2, & \\ \ P_4\cup 2P_3\cup(s-7)K_2,4P_3\cup(s-8)K_2,K_{1,3}\cup(s-3)K_2\} & \text{if } t=4,\end{cases}$$

where $H\cup(s-i)K_2$ *does not exist if* $s<i$.

3. Chromaticity of graphs in $\mathscr{B}(p,q,s,t)$, $t\leqslant 4$

In this section, we shall show that each graph in $\bigcup_{1\leqslant t\leqslant 4}\mathscr{B}(p,q,s,t)$ is χ-unique.

For a bipartite graph $G=(A,B;E)$, the number of 4-independent partitions $\{A_1,A_2,A_3,A_4\}$ with $A_i\subseteq A$ or $A_i\subseteq B$ for all $i=1,2,3,4$ is

$$(2^{|A|-1}-1)(2^{|B|-1}-1)+\frac{1}{3!}(3^{|A|}-3\cdot 2^{|A|}+3)+\frac{1}{3!}(3^{|B|}-3\cdot 2^{|B|}+3)$$

$$=(2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2. \qquad (3)$$

Let $\alpha'(G,4)=\alpha(G,4)-((2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2)$. Observe that for $G,H\in\mathscr{K}^{-s}(p,q)$, $\alpha(G,4)=\alpha(H,4)$ iff $\alpha'(G,4)=\alpha'(H,4)$.

Lemma 3.1. *For $G=(A,B;E)\in\mathscr{K}^{-s}(p,q)$ with $|A|=p$ and $|B|=q$,*

$$\alpha'(G,4)=\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)$$

$$+|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\emptyset\}|.$$

Proof. As $s\leqslant q-1\leqslant p-1$, there exist $x\in A$ and $y\in B$ such that $N_G(x)=B$ and $N_G(y)=A$. Thus, for any 4-independent partition $\{A_1,A_2,A_3,A_4\}$, there are at least two A_i's with $A_i\subseteq A$ or $A_i\subseteq B$. This means that G has only three types of 4-independent partitions $\{A_1,A_2,A_3,A_4\}$: for $k=0,1,2$, we call the partition type k if there are exactly k A_i's with $A_i\in\Omega(G)$. The number of 4-independent partitions of type 0 is given in (3). The number of 4-independent partitions of type 1 is

$$\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)$$

and the number of 4-independent partitions of type 2 is

$$|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\emptyset\}|.$$

The lemma holds. □

For a bipartite graph $G=(A,B;E)$, let $\beta_i(G,A)$ (resp. $\beta_i(G,B)$) be the number of vertices in A (resp. B) with degree i.

Remark. For $G\in\mathscr{B}(p,q,s,t)$, if each component of G' is a path, then $\alpha'(G,3)=s+\beta_2(G')$ by Corollary 2.1(i) to Lemma 2.3. Thus $\beta_2(G')=t$.

Lemma 3.2. *For $G\in\mathscr{B}(p,q,s,t)$, if each component of G' is a path, then*

$$\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)$$

$$=s(2^{p-2}+2^{q-2}-2)+t(2^{p-3}+2^{q-2}-2)+(2^{p-3}-2^{q-3})\beta_2(G',A').$$

Proof. Since each component of G' is a path, $|Q|\leqslant 3$ for every $Q\in\Omega(G)$. There are exactly s sets Q in $\Omega(G)$ with $|Q|=2$, there are exactly $\beta_2(G',A')$ sets Q in $\Omega(G)$

with $|Q\cap A|=1$ and $|Q\cap B|=2$, and there are exactly $\beta_2(G',B')$ sets Q in $\Omega(G)$ with $|Q\cap A|=2$ and $|Q\cap B|=1$. Thus

$$\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)$$

$$=s(2^{p-2}+2^{q-2}-2)+\beta_2(G',A')(2^{p-2}+2^{q-3}-2)$$

$$+\beta_2(G',B')(2^{p-3}+2^{q-2}-2)$$

$$=s(2^{p-2}+2^{q-2}-2)+(\beta_2(G',A')+\beta_2(G',B'))(2^{p-3}+2^{q-2}-2)$$

$$+(2^{p-3}-2^{q-3})\beta_2(G',A')$$

$$=s(2^{p-2}+2^{q-2}-2)+\beta_2(G')(2^{p-3}+2^{q-2}-2)+(2^{p-3}-2^{q-3})\beta_2(G',A').$$

Since $\beta_2(G')=t$, the lemma is obtained. □

Let $p_i(G)$ denote the number of paths P_i in G.

Lemma 3.3. *For $G\in\mathscr{B}(p,q,s,t)$, if each component of G' is a path, then*

$$|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\emptyset\}|$$

$$=\binom{s+t}{2}-3t-3p_4(G')-p_5(G').$$

Proof. Since each component of G' is a path, $|Q|\leqslant 3$ for every $Q\in\Omega(G)$. We also have $\beta_2(G')=t$. There are exactly s (resp. t) sets Q in $\Omega(G)$ with $|Q|=2$ (resp. $|Q|=3$). Observe that

$$|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),|Q_1|=|Q_2|=2,Q_1\cap Q_2=\emptyset\}|$$

$$=|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),|Q_1|=|Q_2|=2,|Q_1\cap Q_2|\leqslant 1\}|$$

$$-|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),|Q_1|=|Q_2|=2,|Q_1\cap Q_2|=1\}|$$

$$=\binom{s}{2}-t. \tag{4}$$

There are exactly t Q_1's with $|Q_1|=3$. For each $Q_1\in\Omega(G)$ with $|Q_1|=3$, there are exactly $s-2$ Q_2's in $\Omega(G)$ with $|Q_2|=2$ and $|Q_1\cap Q_2|\leqslant 1$. Observe that $|Q_1\cap Q_2|=1$ iff $Q_1\cup Q_2$ induces a path P_4 in G', and that for each path P_4 in G', there are exactly two pairs Q_1,Q_2 with $|Q_1|=3$, $|Q_2|=2$ and $|Q_1\cap Q_2|=1$ such that $Q_1\cup Q_2$ induces this path P_4. Thus the number of sets $\{Q_1,Q_2\}$ with $|Q_1|=3$, $|Q_2|=2$ and $|Q_1\cap Q_2|=1$ is $2p_4(G')$. Hence

$$|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),|Q_1|=3,|Q_2|=2,Q_1\cap Q_2=\emptyset\}|$$

$$= |\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), |Q_1| = 3, |Q_2| = 2, |Q_1 \cap Q_2| \leqslant 1\}|$$

$$- |\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), |Q_1| = 3, |Q_2| = 2, |Q_1 \cap Q_2| = 1\}|$$

$$= t(s-2) - 2p_4(G'). \tag{5}$$

There are exactly $\binom{t}{2}$ sets $\{Q_1, Q_2\}$, where $Q_1, Q_2 \in \Omega(G)$, with $|Q_1| = |Q_2| = 3$ and $Q_1 \neq Q_2$. Thus $|Q_1 \cap Q_2| \leqslant 2$ for such Q_1, Q_2. The number of sets $\{Q_1, Q_2\}$ with $|Q_1| = |Q_2| = 3$ and $|Q_1 \cap Q_2| = 2$ is $p_4(G')$, because

(i) $|Q_1 \cap Q_2| = 2$ iff $Q_1 \cup Q_2$ induces a path P_4 in G' and
(ii) for each path P_4 in G', there is only one set $\{Q_1, Q_2\}$ with $|Q_1| = |Q_2| = 3$ such that $Q_1 \cup Q_2$ induces this path P_4.

Similarly, the number of sets $\{Q_1, Q_2\}$ with $|Q_1| = |Q_2| = 3$ and $|Q_1 \cap Q_2| = 1$ is $p_5(G')$. Hence

$$|\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 3, Q_1 \cap Q_2 = \emptyset\}|$$

$$= |\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 3, |Q_1 \cap Q_2| \leqslant 2\}|$$

$$- |\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 3, |Q_1 \cap Q_2| = 2\}|$$

$$- |\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), |Q_1| = |Q_2| = 3, |Q_1 \cap Q_2| = 1\}|$$

$$= \binom{t}{2} - p_4(G') - p_5(G'). \tag{6}$$

By (4)–(6), the result is obtained. □

For $G \in \mathscr{B}(p, q, s, t)$, define

$$\alpha''(G, 4) = \alpha'(G, 4) - (s(2^{p-2} + 2^{q-2} - 2) + t(2^{p-3} + 2^{q-2} - 2) + (s+t)(s+t-1)/2 - 3t). \tag{7}$$

Observe that for $G, H \in \mathscr{B}(p, q, s, t)$, $\alpha''(G, 4) = \alpha''(H, 4)$ iff $\alpha(G, 4) = \alpha(H, 4)$.

Lemma 3.4. *For $G \in \mathscr{B}(p, q, s, t)$, if each component of G' is a path, then*

$$\alpha''(G, 4) = (2^{p-3} - 2^{q-3})\beta_2(G', A') - 3p_4(G') - p_5(G').$$

Proof. It follows from Lemmas 3.1–3.3. □

For a graph G with $uv \notin E(G)$, let $G + uv$ (resp. $G \cdot uv$) denote the graph obtained from G by adding the edge uv (resp. by identifying u and v). For any vertex set $A \subseteq V(G)$, let $G - A$ denote the graph obtained from G by deleting all vertices in A and all edges incident to vertices in A.

For two disjoint graphs G and H, let $G + H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$.

Lemma 3.5. *For a bipartite graph $G=(A,B;E)$, if uvw is a path in G' with $d_{G'}(u)=1$ and $d_{G'}(v)=2$, then for any $k\geqslant 2$,*

$$\alpha(G,k)=\alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1).$$

Proof. Since $P(G,\lambda)=P(G+uv,\lambda)+P(G\cdot uv,\lambda)$, we have

$$\alpha(G,k)=\alpha(G+uv,k)+\alpha(G\cdot uv,k).$$

Let x be the vertex in $G\cdot uv$ produced by identifying u and v. Notice that x is adjacent to all vertices in $V(G\cdot uv)-\{x,w\}$. Thus $G\cdot uv+xw=K_1+(G-\{u,v\})$ and $G\cdot uv\cdot xw=K_1+(G-\{u,v,w\})$. We also observe that for any graph H, $\alpha(K_1+H,k)=\alpha(H,k-1)$, since

$$P(K_1+H,\lambda)=\lambda P(H,\lambda-1).$$

Hence

$$\begin{aligned}\alpha(G\cdot uv,k)&=\alpha(G\cdot uv+xw,k)+\alpha(G\cdot uv\cdot xw,k)\\&=\alpha(K_1+(G-\{u,v\}),k)+\alpha(K_1+(G-\{u,v,w\}),k)\\&=\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1).\end{aligned}$$

The lemma is then obtained. □

Theorem 3.1. *Let p,q and s be integers with $p\geqslant q\geqslant 3$ and $0\leqslant s\leqslant q-1$. For every $G\in\bigcup_{t=1}^{4}\mathscr{B}(p,q,s,t)$, if G is 2-connected, then G is χ-unique.*

Proof. By Theorem 1.3, $\mathscr{B}(p,q,s,t)\cap\mathscr{K}_2^{-s}(p,q)$ is χ-closed for each $t\geqslant 0$. To show that every 2-connected graph in $\mathscr{B}(p,q,s,t)$ is χ-unique, it suffices to show that for every two graphs G and H in $\mathscr{B}(p,q,s,t)$, if $G\not\cong H$, then $\alpha(G,4)\neq\alpha(H,4)$ or $\alpha(G,5)\neq\alpha(H,5)$. Recall that for $G,H\in\mathscr{B}(p,q,s,t)$, $\alpha''(G,4)\neq\alpha''(H,4)$ iff $\alpha(G,4)\neq\alpha(H,4)$.

For each $t=1,2,3,4$, the graphs in $\mathscr{B}(p,q,s,t)$ are named as $G_{t,1},G_{t,2},\ldots$, and are shown in a table together with the values $\alpha''(G_{t,1}),\alpha''(G_{t,2}),\ldots$. For each graph $G_{t,i}$, if every component of $G'_{t,i}$ is a path, then $\alpha''(G_{t,i},4)$ can be obtained by Lemma 3.4; otherwise, we must first find $\alpha'(G_{t,i},4)$ by Lemma 3.1, and then find $\alpha''(G_{t,i},4)$ by (7).

(1) $\mathscr{B}(p,q,s,1)$: The set $\mathscr{B}(p,q,s,1)$ includes two graphs by Theorem 2.1, $G_{1,1}$ and $G_{1,2}$ (see Table 1). Notice that $\alpha''(G_{1,1},4)\neq\alpha''(G_{1,2},4)$ when $p\neq q$. But when $p=q$, $G_{1,1}\cong G_{1,2}$.

(2) $\mathscr{B}(p,q,s,2)$: The set $\mathscr{B}(p,q,s,2)$ includes four graphs by Theorem 2.1, $G_{2,1}$, $G_{2,2},G_{2,3}$ and $G_{2,4}$ (see Table 2). Notice that only $\alpha''(G_{2,1},4)$ is odd. If $p>q$, the three values $\alpha''(G_{2,2},4)$, $\alpha''(G_{2,3},4)$ and $\alpha''(G_{2,4},4)$ are distinct. If $p=q$, then $G_{2,2}\cong G_{2,3}$ and we shall show that $\alpha(G_{2,3},5)>\alpha(G_{2,4},5)$. When $p=q$, by Lemma 3.5

Table 1
$\mathscr{B}(p,q,s,1)$

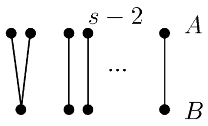
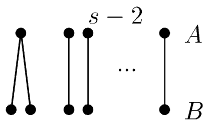
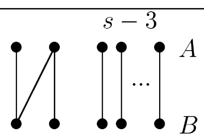
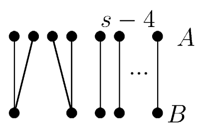
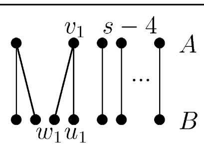
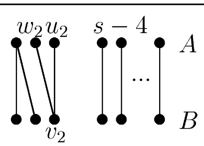
name of graph	graphs $G'_{1,i}$ ($G'_{1,i}=K_{p,q}-G_{1,i}$) ($\|A\|=p$, $\|B\|=q$)	$\alpha''(G_{1,i},4)$	conditions on s
$G_{1,1}$		0	$2\le s\le q-1$
$G_{1,2}$		$2^{p-3}-2^{q-3}$	$2\le s\le q-1$

Table 2
$\mathscr{B}(p,q,s,2)$

name of graph	graphs $G'_{2,i}$ ($G'_{2,i}=K_{p,q}-G_{2,i}$) ($\|A\|=p,\|B\|=q$)	$\alpha''(G_{2,i},4)$	conditions on s
$G_{2,1}$		$(2^{p-3}-2^{q-3})-3$	$3\le s\le q-1$
$G_{2,2}$		0	$4\le s\le q-1$
$G_{2,3}$		$2(2^{p-3}-2^{q-3})$	$4\le s\le q-1$
$G_{2.4}$		$(2^{p-3}-2^{q-3})$	$4\le s\le q-1$

and Table 1, we have

$$\begin{aligned}
&\alpha(G_{2,3},5)-\alpha(G_{2,4},5)\\
&\quad=\alpha(G_{2,3}+u_1v_1,5)+\alpha(G_{2,3}-\{u_1,v_1\},4)+\alpha(G_{2,3}-\{u_1,v_1,w_1\},4)\\
&\qquad-(\alpha(G_{2,4}+u_2v_2,5)+\alpha(G_{2,4}-\{u_2,v_2\},4)+\alpha(G_{2,4}-\{u_2,v_2,w_2\},4))\\
&\quad=\alpha(G_{2,3}-\{u_1,v_1,w_1\},4)-\alpha(G_{2,4}-\{u_2,v_2,w_2\},4)\\
&\quad=\alpha''(G_{2,3}-\{u_1,v_1,w_1\},4)-\alpha''(G_{2,4}-\{u_2,v_2,w_2\},4)\\
&\quad=2^{q-4}-2^{q-5}\\
&\quad>0, \qquad (8)
\end{aligned}$$

since $G_{2,3}+u_1v_1 \cong G_{2,4}+u_2v_2$, $G_{2,3}-\{u_1,v_1\} \cong G_{2,4}-\{u_2,v_2\}$, and both $G_{2,3}-\{u_1,v_1,w_1\}$ and $G_{2,4}-\{u_2,v_2,w_2\}$ belong to $\mathscr{B}(q-1,q-2,s-2,1)$.

(3) $\mathscr{B}(p,q,s,3)$: The set $\mathscr{B}(p,q,s,3)$ contains eight graphs by Theorem 2.1, $G_{3,1}$, $G_{3,2},\ldots,G_{3,8}$ (see Table 3). Notice that $\alpha''(G_{3,i},4)$ is odd when $1\leqslant i\leqslant 4$ and even when $i\geqslant 5$. Thus $\alpha''(G_{3,i},4)\neq\alpha''(G_{3,j},4)$ if $1\leqslant i\leqslant 4$ and $5\leqslant j\leqslant 8$. Observe that $\alpha''(G_{3,i},4)+7$ contains a factor 2^{q-3} for $i=1,2$, but no factor 8 for $i=3,4$. Thus $\alpha''(G_{3,i},4)\neq\alpha''(G_{3,j},4)$ for all $i=1,2$ and all $j=3,4$. When $p>q$, $\alpha''(G_{3,1},4)\neq\alpha''(G_{3,2},4)$ and $\alpha''(G_{3,3},4)\neq\alpha''(G_{3,4},4)$. When $p=q$, $G_{3,1}\cong G_{3,2}$ and $G_{3,3}\cong G_{3,4}$.

For $i=5,\ldots,8$, the $\alpha''(G_{4,i},4)$'s are distinct if $p>q$. If $p=q$, then $G_{3,5}\cong G_{3,8}$ and $G_{3,6}\cong G_{3,7}$, and by using the method in (8), we have

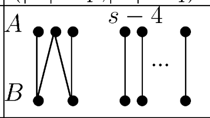
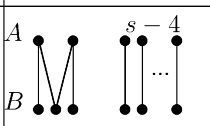
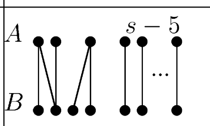
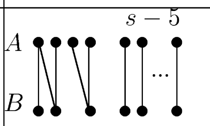
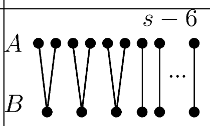
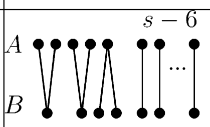
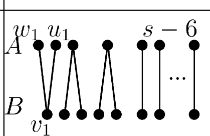
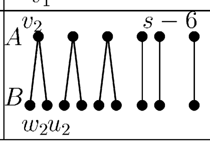
$$\begin{aligned}
\alpha(G_{3,7},5)-\alpha(G_{3,8},5)&=\alpha''(G_{3,7}-\{u_1,v_1,w_1\},4)-\alpha''(G_{3,8}-\{u_2,v_2,w_2\},4)\\
&=-2^{q-4}<0.
\end{aligned}$$

(4) $\mathscr{B}(p,q,s,4)$: The set $\mathscr{B}(p,q,s,4)$ has 16 graphs by Theorem 2.1, $G_{4,1},G_{4,2},\ldots,G_{4,16}$ (see Table 4). Partition $\mathscr{B}(p,q,s,4)$ into subsets:

$$\begin{aligned}
&\mathscr{S}_1=\{G_{4,1}\},\\
&\mathscr{S}_2=\{G_{4,2},G_{4,3},G_{4,4},G_{4,5}\},\\
&\mathscr{S}_3=\{G_{4,6},G_{4,7},G_{4,8}\},\\
&\mathscr{S}_4=\{G_{4,9}\},\\
&\mathscr{S}_5=\{G_{4,10},G_{4,11}\},\\
&\mathscr{S}_6=\{G_{4,12},G_{4,13},G_{4,14},G_{4,15},G_{4,16}\}.
\end{aligned}$$

For non-empty sets $W_1,\ldots,W_k$ of graphs, let $\eta(W_1,\ldots,W_k)=0$ if $\alpha(G_1,4)\neq\alpha(G_2,4)$ for every two graphs $G_1\in W_i$ and $G_2\in W_j$, where $i\neq j$, and let $\eta(W_1,\ldots,W_k)=1$ otherwise.

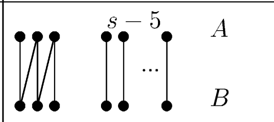
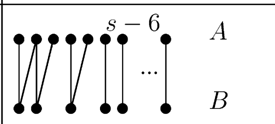
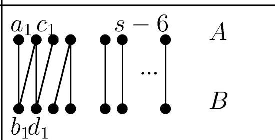
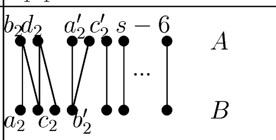
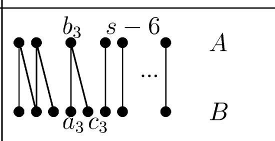
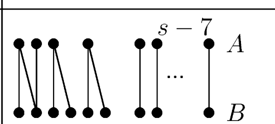
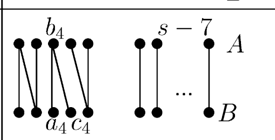
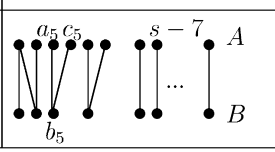
Table 3
$\mathscr{B}(p,q,s,3)$

name of graph	graphs $G'_{3,i}$ ($G'_{3,i} = K_{p,q} - G_{3,i}$) ($\|A\| = p, \|B\| = q$)	$\alpha''(G_{3,i},4)$	conditions on s
$G_{3,1}$		$(2^{p-3} - 2^{q-3})$ -7	$4 \le s \le q-1$
$G_{3,2}$		$2(2^{p-3} - 2^{q-3})$ -7	$4 \le s \le q-1$
$G_{3,3}$		$2(2^{p-3} - 2^{q-3})$ -3	$5 \le s \le q-1$
$G_{3,4}$		$(2^{p-3} - 2^{q-3})$ -3	$5 \le s \le q-1$
$G_{3,5}$		0	$6 \le s \le q-1$
$G_{3,6}$		$(2^{p-3} - 2^{q-3})$	$6 \le s \le q-1$
$G_{3,7}$		$2(2^{p-3} - 2^{q-3})$	$6 \le s \le q-1$
$G_{3,8}$		$3(2^{p-3} - 2^{q-3})$	$6 \le s \le q-1$

The values of $\alpha''(G_{4,10},4)$ and $\alpha''(G_{4,11},4)$ are not given by Lemma 3.4, but can be obtained by Lemma 3.1 and (7). We have

$$\begin{aligned}\alpha''(G_{4,10},4) = {} & s(2^{p-2}+2^{q-2}-2)+3(2^{p-3}+2^{q-2}-2) \\ & +(2^{p-4}+2^{q-2}-2)+\binom{s}{2}-3+4(s-3) \\ & -s(2^{p-2}+2^{q-2}-2)-4(2^{p-3}+2^{q-2}-2)-\binom{s+4}{2}+12 \\ = {} & -2^{p-4}-9.\end{aligned}$$

Table 4
$\mathscr{B}(p,q,s,4)$

name of graph	graphs $G'_{4,i}$ ($G'_{4,i}=K_{p,q}-G_{4,i}$) ($\vert A\vert=p,\vert B\vert=q$)	$\alpha''(G_{4,i},4)$	conditions on s
$G_{4,1}$		$2(2^{p-3}-2^{q-3})$ -11	$5\le s\le q-1$
$G_{4,2}$		$(2^{p-3}-2^{q-3})$ -7	$6\le s\le q-1$
$G_{4,3}$		$2(2^{p-3}-2^{q-3})$ -7	$6\le s\le q-1$
$G_{4,4}$		$2(2^{p-3}-2^{q-3})$ -7	$6\le s\le q-1$
$G_{4,5}$		$3(2^{p-3}-2^{q-3})$ -7	$6\le s\le q-1$
$G_{4,6}$		$3(2^{p-3}-2^{q-3})$ -3	$7\le s\le q-1$
$G_{4,7}$		$2(2^{p-3}-2^{q-3})$ -3	$7\le s\le q-1$
$G_{4,8}$		$(2^{p-3}-2^{q-3})$ -3	$7\le s\le q-1$

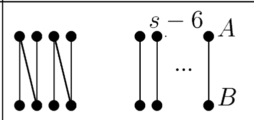
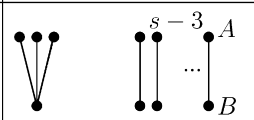
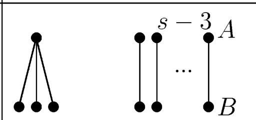
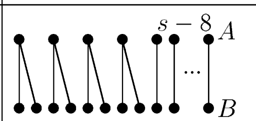
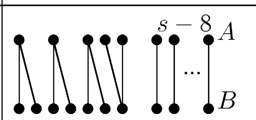
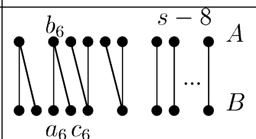
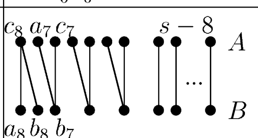
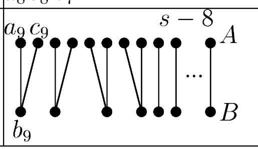
Similarly, we find $\alpha''(G_{4,11},4)=2^{p-1}-9\cdot 2^{q-4}-9$.

Claim 1. $\eta(\mathscr{S}_1,\ldots,\mathscr{S}_6)=0$:

(a) If $s\leqslant 4$, only $\mathscr{S}_5$ is non-empty.

(b) For $s\geqslant 5$, $\alpha''(G,4)$ is odd if $G\in\mathscr{S}_1\cup\mathscr{S}_2\cup\mathscr{S}_3\cup\mathscr{S}_5$ and even if $G\in\mathscr{S}_4\cup\mathscr{S}_6$. Hence $\eta(\mathscr{S}_1\cup\mathscr{S}_2\cup\mathscr{S}_3\cup\mathscr{S}_5,\mathscr{S}_4\cup\mathscr{S}_6)=0$.

Table 4. (*continued*)

name of graph	graphs $G'_{4,i}$ ($G'_{4,i} = K_{p,q} - G_{4,i}$) ($\|A\| = p, \|B\| = q$)	$\alpha''(G_{4,i}, 4)$	conditions on s
$G_{4,9}$		$2(2^{p-3} - 2^{q-3})$ -6	$6 \leq s \leq q-1$
$G_{4,10}$		-2^{p-4} -9	$3 \leq s \leq q-1$
$G_{4,11}$		$(2^{p-1} - 9 \cdot 2^{q-4})$ -9	$3 \leq s \leq q-1$
$G_{4,12}$		$4(2^{p-3} - 2^{q-3})$	$8 \leq s \leq q-1$
$G_{4,13}$		$3(2^{p-3} - 2^{q-3})$	$8 \leq s \leq q-1$
$G_{4,14}$		$2(2^{p-3} - 2^{q-3})$	$8 \leq s \leq q-1$
$G_{4,15}$		$(2^{p-3} - 2^{q-3})$	$8 \leq s \leq q-1$
$G_{4,16}$		0	$8 \leq s \leq q-1$

(c) For $s \geqslant 5$, we have $q \geqslant 6$ and 2^{q-4} is a factor of $\alpha''(G,4)+9$ for every $G \in \mathscr{S}_5$, but 4 is not a factor of $\alpha''(G,4)+9$ for every $G \in \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3$. Hence $\eta(\mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3, \mathscr{S}_5) = 0$.

(d) For $s \geqslant 5$, we have $q \geqslant 6$, and 2^{q-2} is a factor of $\alpha''(G,4)+11$ for $G \in \mathscr{S}_1$, 2^3 is not a factor of $\alpha''(G,4)+11$ for $G \in \mathscr{S}_2$, and 2^3 is a factor of $\alpha''(G,4)+11$ but 2^4 is not for $G \in \mathscr{S}_3$. Hence $\eta(\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3) = 0$.

(e) For $s\geqslant 6$, we have $q\geqslant 7$, and 2^2 is a factor of $\alpha''(G,4)$ for every $G\in\mathscr{S}_6$ but it is not for every $G\in\mathscr{S}_4$. Hence $\eta(\mathscr{S}_4,\mathscr{S}_6)=0$.

By (b)–(e), Claim 1 holds.

The remaining work is to compare every two graphs in each $\mathscr{S}_i$. Both $\mathscr{S}_1$ and $\mathscr{S}_4$ contain only one graph. For $\mathscr{S}_5$, when $p=q$, $G_{4,10}\cong G_{4,11}$; when $p>q$, $\alpha''(G_{4,10},4)\neq\alpha''(G_{4,11},4)$. In the following, we shall study the three sets $\mathscr{S}_2$, $\mathscr{S}_3$ and $\mathscr{S}_6$.

(4.1) $\mathscr{S}_3$: When $p>q$, $\alpha''(G_{4,6},4)>\alpha''(G_{4,7},4)>\alpha''(G_{4,8},4)$. When $p=q$, we have $G_{4,6}\cong G_{4,8}$ and by the method used in (8),

$$\alpha(G_{4,7},5)-\alpha(G_{4,8},5)$$
$$=\alpha(G_{4,7}-\{a_4,b_4,c_4\},4)-\alpha(G_{4,8}-\{a_5,b_5,c_5\},4)=-2^{q-5}\neq 0.$$

(4.2) $\mathscr{S}_6$: When $p>q$,

$$\alpha''(G_{4,12},4)>\alpha''(G_{4,13},4)>\alpha''(G_{4,14},4)>\alpha''(G_{4,15},4)>\alpha''(G_{4,16},4).$$

When $p=q$, $G_{4,12}\cong G_{4,16}$, $G_{4,13}\cong G_{4,15}$ and by the method used in (8),

$$\alpha(G_{4,14},5)-\alpha(G_{4,15},5)=-2^{q-5},$$

$$\alpha(G_{4,15},5)-\alpha(G_{4,16},5)=-3\times 2^{q-5}<0.$$

(4.3) $\mathscr{S}_2$: Observe that $\alpha''(G_{4,3},4)=\alpha''(G_{4,4},4)$. When $p>q$,

$$\alpha''(G_{4,2},4)<\alpha''(G_{4,3},4)<\alpha''(G_{4,5},4).$$

When $p=q$, $G_{4,2}\cong G_{4,5}$ and $G_{4,3}\cong G_{4,4}$. In the following, we shall compare $\alpha(G_{4,4},5)$ with $\alpha(G_{4,5},5)$ for $p=q$, and $\alpha(G_{4,3},5)$ with $\alpha(G_{4,4},5)$ for $p>q$.

By Lemma 3.5, when $p=q$, by the method used in (8),

$$\alpha(G_{4,4},5)-\alpha(G_{4,5},5)$$
$$=\alpha(G_{4,4}-\{a_2',b_2',c_2'\},4)-\alpha(G_{4,5}-\{a_3,b_3,c_3\},5)$$
$$=-\,2^{q-5}$$
$$<0.$$

For $G_{4,3}$ and $G_{4,4}$, we prove the following claim:

Claim 2. $\alpha(G_{4,3},5)-\alpha(G_{4,4},5)=3(2^{p-5}-2^{q-5})$.

By Lemma 3.5,

$$\alpha(G_{4,3},5)$$
$$=\alpha(G_{4,3}+a_1b_1,5)+\alpha(G_{4,3}-\{a_1,b_1\},4)+\alpha(G_{4,3}-\{a_1,b_1,c_1\},4)$$
$$=\alpha(G_{4,3}+a_1b_1+b_1c_1,5)+\alpha(G_{4,3}-\{b_1,c_1\},4)+\alpha(G_{4,3}-\{b_1,c_1,d_1\},4)$$
$$+\alpha(G_{4,3}-\{a_1,b_1\},4)+\alpha(G_{4,3}-\{a_1,b_1,c_1\},4),$$

and

$$\begin{aligned}
&\alpha(G_{4,4},5)\\
&\quad=\alpha(G_{4,4}+a_2b_2,5)+\alpha(G_{4,4}-\{a_2,b_2\},4)+\alpha(G_{4,4}-\{a_2,b_2,c_2\},4)\\
&\quad=\alpha(G_{4,4}+a_2b_2+b_2c_2,5)+\alpha(G_{4,4}-\{b_2,c_2\},4)+\alpha(G_{4,4}-\{b_2,c_2,d_2\},4)\\
&\qquad+\alpha(G_{4,4}-\{a_2,b_2\},4)+\alpha(G_{4,4}-\{a_2,b_2,c_2\},4).
\end{aligned}$$

Observe that

$$\begin{aligned}
&G_{4,3}+a_1b_1+b_1c_1\cong G_{4,4}+a_2b_2+b_2c_2,\\
&\alpha(G_{4,3}-\{b_1,c_1\},4)-\alpha(G_{4,4}-\{b_2,c_2\},4)=2^{p-4}-2^{q-4},\\
&G_{4,3}-\{a_1,b_1\}\cong G_{4,4}-\{a_2,b_2\}.
\end{aligned}\tag{9}$$

Since

$$\begin{aligned}
&G_{4,3}-\{a_1,b_1,c_1\}\in\mathscr{B}(p-2,q-1,s-3,1),\\
&G_{4,4}-\{b_2,c_2,d_2\}\in\mathscr{B}(p-2,q-1,s-4,1),
\end{aligned}$$

by Lemma 3.1, we have

$$\begin{aligned}
&\alpha(G_{4,3}-\{a_1,b_1,c_1\},4)-\alpha(G_{4,4}-\{b_2,c_2,d_2\},4)\\
&\quad=\alpha'(G_{4,3}-\{a_1,b_1,c_1\},4)-\alpha'(G_{4,4}-\{b_2,c_2,d_2\},4)\\
&\quad=(s-3)(2^{p-4}+2^{q-3}-2)+(2^{p-4}+2^{q-4}-2)+\binom{s-3}{2}-1+(s-5)\\
&\qquad-\Big((s-4)(2^{p-4}+2^{q-3}-2)+(2^{p-5}+2^{q-3}-2)\\
&\qquad+\binom{s-4}{2}-1+(s-6)\Big)\\
&\quad=2^{p-4}+2^{p-5}+2^{q-4}+s-5.
\end{aligned}\tag{10}$$

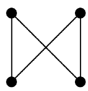
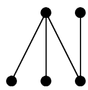
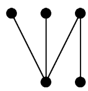
Similarly, since

$$\begin{aligned}
&G_{4,3}-\{b_1,c_1,d_1\}\in\mathscr{B}(p-1,q-2,s-4,1),\\
&G_{4,4}-\{a_2,b_2,c_2\}\in\mathscr{B}(p-1,q-2,s-3,1),
\end{aligned}$$

by Lemma 3.1, we have

$$\begin{aligned}
&\alpha(G_{4,3}-\{b_1,c_1,d_1\},4)-\alpha(G_{4,4}-\{a_2,b_2,c_2\},4)\\
&\quad=\alpha'(G_{4,3}-\{b_1,c_1,d_1\},4)-\alpha'(G_{4,4}-\{a_2,b_2,c_2\},4)\\
&\quad=(s-4)(2^{p-3}+2^{q-4}-2)+(2^{p-3}+2^{q-5}-2)+\binom{s-4}{2}-1+(s-6)
\end{aligned}$$

Table 5

name of graph	graphs R_i' $(R_i' = K_{p,q} - R_i)$ $(\lvert A\rvert = p,\ \lvert B\rvert = q)$	$\alpha'(R_i,4)$ $-4(2^{p-2}+2^{q-2}-2)$
R_1	 $\lvert A\rvert$ $\lvert B\rvert$	$2(2^{p-2}+2^{q-3}-2)$ $+2(2^{p-3}+2^{q-2}-2)$ $+(2^{p-3}+2^{q-3}-2)$ $+2$
R_2	 $\lvert A\rvert$ $\lvert B\rvert$	$3(2^{p-2}+2^{q-3}-2)$ $+(2^{p-3}+2^{q-2}-2)$ $+(2^{p-2}+2^{q-4}-2)$ $+3$
R_3	 $\lvert A\rvert$ $\lvert B\rvert$	$3(2^{p-3}+2^{q-2}-2)$ $+(2^{p-2}+2^{q-3}-2)$ $+(2^{p-4}+2^{q-2}-2)$ $+3$

$$
\begin{aligned}
&-\Big((s-3)(2^{p-3}+2^{q-4}-2)+(2^{p-4}+2^{q-4}-2)\\
&+\binom{s-3}{2}-1+(s-5)\Big)\\
=&-2^{p-4}-2^{q-4}-2^{q-5}-s+5. \qquad (11)
\end{aligned}
$$

By (9)–(11), Claim 2 is proved. □

Finally, we conclude that for every two graphs $G_1, G_2 \in \mathscr{B}(p,q,s,4)$, if $G_1 \not\cong G_2$, then either $\alpha''(G_1,4) \neq \alpha''(G_2,4)$ or $\alpha(G_1,5) \neq \alpha(G_2,5)$. This completes the proof of the result. □

Theorem 3.2. *For any $G \in \mathscr{K}_2^{-s}(p,q)$ with $p \geqslant q \geqslant 3$ and $0 \leqslant s \leqslant \min\{4, q-1\}$, G is χ-unique.*

Proof. Let $G \in \mathscr{K}_2^{-s}(p,q)$. If $s \leqslant 3$, then by Theorem 1.2, $\alpha'(G,3) \leqslant 2^s - 1 \leqslant s+4$. Thus by Theorem 3.1, G is χ-unique if $s \leqslant 3$. Now suppose that $s=4$. We have $q \geqslant 5$. If $\Delta(G') \in \{1,4\}$, then $\alpha'(G,3)=s$ or $\alpha'(G,3)=2^s-1$ and thus G is χ-unique by Theorem 1.4. If $\Delta(G')=2$ and $G' \not\cong K_{2,2}$, then $\alpha'(G,3) \leqslant s+3$ by Corollary (i) to Lemma 2.3, and thus G is χ-unique by Theorem 3.1. If $G' = K_{1,3} \cup K_2$, then $\alpha'(G,3)=8=s+4$, and thus G is χ-unique by Theorem 3.1. Otherwise, there are only two possible structures for G'. They are shown in Table 5. For $i=1,2,3$, $\alpha'(R_i,4)$ is obtained by Lemma 3.1. From Table 5, observe that $\alpha'(R_i,4)$ is even when $i=1$ and odd when

$i=2,3$. When $p=q$, $R_2 \cong R_3$; when $p>q$, $\alpha'(R_2,4)-\alpha'(R_3,4)=7(2^{p-4}-2^{q-4})>0$. Hence G is χ-unique when $\Delta(G')\in\{2,3\}$. This completes the proof. □

4. For further reading

The following references are also of interest to the reader: [3,4].

References

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