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# An attempt to classify bipartite graphs by chromatic polynomials

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**Abstract**

For integers $p,q,s$ with $p\geqslant q\geqslant 3$ and $1\leqslant s\leqslant q-1$, let $\mathscr{K}^{-s}(p,q)$ (resp. $\mathscr{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2-connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of $s$ edges. In this paper, we first find an upper bound for the 3-independent partition number of a graph $G\in\mathscr{K}^{-s}(p,q)$ with respect to the maximum degree $\Delta(G')$ of $G'$, where $G'=K_{p,q}-G$. By using this result, we show that the set $\{G\,|\,G\in\mathscr{K}_2^{-s}(p,q),\Delta(G')=i\}$ is closed under the chromatic equivalence for every integer $i$ with $s\geqslant i\geqslant (s+3)/2$. From this result, we prove that for any $G\in\mathscr{K}_2^{-s}(p,q)$ with $p\geqslant q\geqslant 3$, if $5\leqslant s\leqslant q-1$ and $\Delta(G')=s-1$, or $7\leqslant s\leqslant q-1$ and $\Delta(G')=s-2$, then $G$ is chromatically unique. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

All graphs considered here are simple graphs. For a graph $G$, let $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of $G$, respectively.

For integers $p,q,s$ with $p\geqslant q\geqslant 2$ and $s\geqslant 0$, let $\mathscr{K}^{-s}(p,q)$ (resp. $\mathscr{K}_2^{-s}(p,q)$) denote the connected (resp. 2-connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of $s$ edges.

For a graph $G$ and a positive integer $k$, a partition $\{A_1,A_2,\ldots,A_k\}$ of $V(G)$ is called a *$k$-independent partition* in $G$ if each $A_i$ is a non-empty independent set of $G$. Let $\alpha(G,k)$ denote the number of $k$-independent partitions in $G$. For any bipartite graph $G=(A,B;E)$ with bipartition $A$ and $B$ and edge set $E$, let

$$\alpha'(G,3)=\alpha(G,3)-(2^{|A|-1}+2^{|B|-1}-2).$$

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For integers $s$ and $m$ with $m \geqslant 1$ and $s \geqslant 0$, define

$$g(m,s) = 2^{a+m} + 2^{a+d} - 2^m - 2^{a+1} + 1,$$

where $a$ and $d$ are integers determined by $s = am + d$, $a \geqslant 0$ and $0 \leqslant d \leqslant m-1$. In [1, Theorem 3.1], we obtained the following result.

**Theorem 1.1.** *For any graph* $G \in \mathcal{K}^{-s}(p,q)$, *where* $p \geqslant q \geqslant 2$ *and* $0 \leqslant s \leqslant (p-1)(q-1)$,

$$\alpha'(G,3) \leqslant g(p-1,s).$$

In this paper, we shall improve the upper bounds for $\alpha'(G,3)$, where $G \in \mathcal{K}^{-s}(p,q)$, under the following conditions for $p,q,s$:

$$p \geqslant q \geqslant 3 \quad \text{and} \quad 1 \leqslant s \leqslant q-1.$$

Thus the above conditions are fixed throughout this paper. Note that they imply that $G$ is connected.

For a bipartite graph $H = (A,B;E)$, let $H' = (A',B';E')$ be the graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $H' = K_{p,q} - H$, where $p = |A|$ and $q = |B|$.

The upper bound given in Theorem 1.1 is not good for bipartite graphs $G$ with low $\Delta(G')$. For example, when $\Delta(G') = 1$ and $s \leqslant q-1 \leqslant p-1$, $\alpha'(G,3) = s$. But $g(p-1,s) = 2^s - 1$, which is much larger than $\alpha'(G,3)$ for large $s$. Thus it is necessary to study the relation between $\alpha'(G,3)$ and $\Delta(G')$. We first, in Theorem 2.1, give an upper bound for $\alpha'(G,3)$ with respect to $\Delta(G')$:

$$\alpha'(G,3) \leqslant g(r,s)$$

for any $G \in \mathcal{K}^{-s}(p,q)$, where $r = \max\{\Delta(G'), \lfloor (s+1)/2 \rfloor\}$. From this result, we prove, in Theorem 2.2, that for any $G_1, G_2 \in \mathcal{K}^{-s}(p,q)$, if $\Delta(G_2') \geqslant \max\{\Delta(G_1')+1, (s+3)/2\}$, then $\alpha'(G_2,3) > \alpha'(G_1,3)$. Partition $\mathcal{K}^{-s}(p,q)$ into the following subsets:

$$\mathcal{D}_i(p,q,s) = \{G \in \mathcal{K}^{-s}(p,q) \mid \Delta(G') = i\}, \quad i = 1,2,\ldots,s.$$

Then for any $H \in \bigcup_{1 \leqslant i < (s+3)/2} \mathcal{D}_i(p,q,s)$ and $H_i \in \mathcal{D}_i(p,q,s)$, where $(s+3)/2 \leqslant i \leqslant s$, it follows from Theorem 2.2 that

$$\alpha'(H_s,3) > \cdots > \alpha'(H_{\lceil (s+3)/2 \rceil},3) > \alpha'(H,3).$$

We then use the above results to study the chromaticity of bipartite graphs. Two graphs $G$ and $H$ are said to be *chromatically equivalent* (or simply $\chi$-*equivalent*), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by $G$ under $\sim$ is denoted by $[G]$. A graph $G$ is *chromatically unique* (or simply $\chi$-unique) if $H \cong G$ whenever $H \sim G$, i.e., $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi$-*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint*, or simply $\chi$-*disjoint*.

We shall show, in Theorem 3.1, that the following sets are pairwise $\chi$-disjoint:

$$\mathscr{D}_1(p,q,s), \bigcup_{2\leqslant i<t} \mathscr{D}_i(p,q,s), \mathscr{D}_t(p,q,s), \mathscr{D}_{t+1}(p,q,s), \ldots, \mathscr{D}_s(p,q,s),$$

where $t=\lceil (s+3)/2\rceil$. This result gives a rough classification of graphs in the set $\mathscr{K}^{-s}(p,q)$ by chromatic polynomials.

We have proved in [1] that every 2-connected graph in $\mathscr{D}_s(p,q,s)$ is $\chi$-unique. We shall, in Theorems 4.1 and 4.2, prove that $G$ is $\chi$-unique for every $G\in\mathscr{D}_{s-1}(p,q,s)$, where $s\geqslant 5$, or $G\in\mathscr{D}_{s-2}(p,q,s)$, where $s\geqslant 7$.

## 2. An upper bound for $\alpha'(G,3)$

For a graph $G$ and $x\in V(G)$, let $N_G(x)$, or simply $N(x)$, be the set of vertices in $G$ adjacent to $x$, and let $d_G(x)$, or simply $d(x)$, be the degree of $x$ in $G$.

For a bipartite graph $G=(A,B;E)$ and two vertices $x,y$ with $x,y\in B$ (or similarly $x,y\in A$), we construct a new bipartite graph, denoted by $F(G,x,y)$ or simply $F$, from $G-x-y$ by adding two new vertices $w_1$ and $w_2$ and edges joining $w_1$ to all vertices in $N(x)\cup N(y)$ and $w_2$ to all vertices in $N(x)\cap N(y)$. The graph $F(G,x,y)$, say $x,y\in B$, is also a bipartite graph, which can be written as $(A,B';E')$, where $B'=(B-\{x,y\})\cup\{w_1,w_2\}$. Observe that $F'=F(G',x,y)$ and $\Delta(F')\geqslant\Delta(G')$.

For a bipartite graph $G=(A,B;E)$, let

$$\Phi(G)=\{\{x,y\}\mid x,y\in A \text{ or } x,y\in B, N(x)\nsubseteq N(y), \text{ and } N(y)\nsubseteq N(x)\}.$$

In [1, Lemma 3.8], the following result was found.

**Lemma 2.1.** *For $G\in\mathscr{K}^{-s}(p,q)$ with $\Phi(G)\neq\emptyset$, there is a sequence of graphs $G_0(=G),G_1,\ldots,G_k$ in $\mathscr{K}^{-s}(p,q)$ such that $\Phi(G_k)=\emptyset$ and for $i=0,1,\ldots,k-1$,*

(i) *$G_{i+1}=F(G_i,u_i,v_i)$ for some $\{u_i,v_i\}\in\Phi(G_i)$ with $N_{G_i}(u_i)\cap N_{G_i}(v_i)\neq\emptyset$,*
(ii) *$|\Phi(G_{i+1})|<|\Phi(G_i)|$, and*
(iii) *$\alpha'(G_{i+1},3)\geqslant\alpha'(G_i,3)$.*

For a bipartite graph $G=(A,B;E)$, let $\mathscr{I}(G)$ be the set of independent sets in $G$ and

$$\Omega(G)=\{Q\in\mathscr{I}(G)\mid Q\cap A\neq\emptyset, Q\cap B\neq\emptyset\}.$$

In [2], we found the following result.

**Lemma 2.2.** *For $G\in\mathscr{K}^{-s}(p,q)$, $\alpha'(G,3)=|\Omega(G)|\geqslant 2^{\Delta(G')}+s-1-\Delta(G')$.*

We now study the difference between $\alpha'(F,3)$ and $\alpha'(G,3)$.

**Lemma 2.3.** *For $G=(A,B;E)\in \mathscr{K}^{-s}(p,q)$ with $|A|=p$ and $|B|=q$, and $x,y\in A$ or $x,y\in B$, we have*

$$\alpha'(F,3)-\alpha'(G,3)\geqslant 2^c(2^{a-c}-1)(2^{b-c}-1),$$

*where $F=F(G,x,y)$, $a=d_{G'}(x)$, $b=d_{G'}(y)$ and $c=|N_{G'}(x)\cap N_{G'}(y)|$.*

**Proof.** Without loss of generality, assume that $x,y\in A$. Let $N_1=B-N_G(x)$, $N_2=B-N_G(y)$ and $N_0=B-(N_G(x)\cup N_G(y))$. Then $N_1=N_{G'}(x)$, $N_2=N_{G'}(y)$ and $N_0=N_{G'}(x)\cap N_{G'}(y)$.

Let $\Omega_1(G)=\{Q\in\Omega(G)\,|\,Q\cap B\subseteq N_0\}$. We first show that $|\Omega_1(G)|=|\Omega_1(F)|$. Let $H$ be the subgraph of $G$ induced by $A\cup N_0$. Then $\Omega_1(G)=\Omega(H)$. Similarly, $\Omega_1(F)=\Omega(H')$, where $H'$ is the subgraph of $F$ induced by $(A-\{x,y\})\cup\{w_1,w_2\}\cup N_0$. Obviously, $H'\cong H$. Thus $|\Omega_1(G)|=|\Omega_1(F)|$.

Let $\Omega_2(G)=\Omega(G)-\Omega_1(G)$, and let $Q\in\Omega_2(G)$. Since $Q\cap((N_1\cup N_2)-N_0)\neq\emptyset$, we have $\{x,y\}\nsubseteq Q$. We define a mapping $p$ from $\Omega_2(G)$ to $\Omega_2(F)$: for $Q\in\Omega_2(G)$,

$$p(Q)=\begin{cases} Q & \text{if } Q\cap\{x,y\}=\emptyset,\\ (Q-\{x,y\})\cup\{w_2\} & \text{otherwise.}\end{cases}$$

We observe that

(i) for $Q_1,Q_2\in\Omega_2(G)$, if $Q_1\neq Q_2$, then $p(Q_1)\neq p(Q_2)$;
(ii) for $Q\in\Omega_2(G)$, if $x\in Q$ or $y\in Q$, then $Q\cap B\subseteq N_1$ or $Q\cap B\subseteq N_2$, respectively.

Let $\Omega'(F)$ be the set of all $\bar{Q}\subseteq N_1\cup N_2\cup\{w_2\}$ such that $w_2\in\bar{Q}$, $\bar{Q}\cap(N_1-N_0)\neq\emptyset$ and $\bar{Q}\cap(N_2-N_0)\neq\emptyset$. It is clear that $\Omega'(F)$ is a subset of $\Omega_2(F)$. By (ii), there exists no $Q\in\Omega_2(G)$ such that $p(Q)\in\Omega'(F)$. Then by (i),

$$|\Omega_2(F)|-|\Omega_2(G)|\geqslant|\Omega'(F)|.$$

Observe that

$$|\Omega'(F)|=2^{|N_0|}(2^{|N_1-N_0|}-1)(2^{|N_2-N_0|}-1)=2^c(2^{a-c}-1)(2^{b-c}-1).$$

Hence $\alpha'(F,3)-\alpha'(G,3)\geqslant 2^c(2^{a-c}-1)(2^{b-c}-1)$. □

**Lemma 2.4.** *For $G=(A,B;E)\in\mathscr{K}^{-s}(p,q)$ and $F=(G,x,y)$ for $x,y\in A$ or $x,y\in B$, we have*

$$\alpha'(F,3)-\alpha'(G,3)\geqslant 2^{\Delta(F')}-2^{\Delta(G')}-2^{s-\Delta(G')}+2^{s-\Delta(F')}$$

*and*

$$\alpha'(F,3)-\alpha'(G,3)\geqslant 2^{\Delta(F')}-2^{\Delta(G')+1}+2^{2\Delta(G')-\Delta(F')}.$$

**Proof.** Let $a=d_{G'}(x)$, $b=d_{G'}(y)$ and $c=|N_{G'}(x)\cap N_{G'}(y)|$. By Lemma 2.3,

$$\alpha'(F,3)-\alpha'(G,3)\geqslant 2^c(2^{a-c}-1)(2^{b-c}-1)\geqslant 0.$$

Recall that $\Delta(F')\geqslant\Delta(G')$. The result holds when $\Delta(F')=\Delta(G')$. Now suppose that $\Delta(F')>\Delta(G')$.

By the definition of $F = F(G,x,y)$, $\Delta(F') = \max\{\Delta(G'), |N_{G'}(x) \cup N_{G'}(y)|\}$. Since $\Delta(F') > \Delta(G')$, we have $\Delta(F') = |N_{G'}(x) \cup N_{G'}(y)| = a + b - c$. It is obvious that $a, b \leqslant \Delta(G')$ and $a + b \leqslant s$. Since $2^x$ is a convex function of $x$, it follows that $2^a + 2^b \leqslant 2^{\Delta(G')} + 2^{a+b-\Delta(G')}$. Therefore,

$$\begin{aligned} 2^c(2^{a-c}-1)(2^{b-c}-1) &= 2^{a+b-c} - 2^a - 2^b + 2^c \\ &\geqslant 2^{\Delta(F')} - 2^{\Delta(G')} - 2^{a+b}(2^{-\Delta(G')} - 2^{-\Delta(F')}) \\ &\geqslant 2^{\Delta(F')} - 2^{\Delta(G')} - 2^{s}(2^{-\Delta(G')} - 2^{-\Delta(F')}) \\ &= 2^{\Delta(F')} - 2^{\Delta(G')} - 2^{s-\Delta(G')} + 2^{s-\Delta(F')} \end{aligned}$$

and

$$\begin{aligned} 2^c(2^{a-c}-1)(2^{b-c}-1) &= 2^{c-a-b}(2^{a+b-c} - 2^a)(2^{a+b-c} - 2^b) \\ &= 2^{-\Delta(F')}(2^{\Delta(F')} - 2^a)(2^{\Delta(F')} - 2^b) \\ &\geqslant 2^{-\Delta(F')}(2^{\Delta(F')} - 2^{\Delta(G')})(2^{\Delta(F')} - 2^{\Delta(G')}) \\ &= 2^{\Delta(F')} - 2^{\Delta(G')+1} + 2^{2\Delta(G')-\Delta(F')}. \end{aligned}$$

This completes the proof of the result. □

In [1] (the corollary to Lemma 3.10), we have the following result.

**Lemma 2.5.** *For $G = (A,B;E) \in \mathscr{K}^{-s}(p,q)$, if $\Phi(G) = \emptyset$, then*

$$\alpha'(G,3) \leqslant g(m,s)$$

*for each $m \geqslant \Delta(G')$.*

**Lemma 2.6.** *For $G = (A,B;E) \in \mathscr{K}^{-s}(p,q)$,*

$$\alpha'(G,3) \leqslant g(m,s) - (2^m - 2^{\Delta(G')} - 2^{s-\Delta(G')} + 2^{s-m})$$

*for some $m$ with $s \geqslant m \geqslant \Delta(G')$.*

**Proof.** If $\Phi(G) = \emptyset$, then by Lemma 2.5, the result holds by taking $m = \Delta(G')$. Now assume that $\Phi(G) \neq \emptyset$. By Lemma 2.1, there is a sequence of graphs $G_0(=G), G_1, \ldots, G_k$ in $\mathscr{K}^{-s}(p,q)$ such that $\Phi(G_k) = \emptyset$ and for $i = 0, 1, \ldots, k-1$, $G_{i+1} = F(G_i, u_i, v_i)$ for some $\{u_i, v_i\} \in \Phi(G_i)$ with $N_{G_i}(u_i) \cap N_{G_i}(v_i) \neq \emptyset$.

By Lemma 2.4, for $i = 0, 1, \ldots, k-1$,

$$\alpha'(G_{i+1},3) - \alpha'(G_i,3) \geqslant 2^{\Delta(G'_{i+1})} - 2^{\Delta(G'_i)} - 2^{s-\Delta(G'_i)} + 2^{s-\Delta(G'_{i+1})}.$$

Hence,

$$\begin{aligned} \alpha'(G_k,3) - \alpha'(G_0,3) &= \sum_{i=0}^{k-1} (\alpha'(G_{i+1},3) - \alpha'(G_i,3)) \\ &\geqslant 2^{\Delta(G'_k)} - 2^{\Delta(G'_0)} - 2^{s-\Delta(G'_0)} + 2^{s-\Delta(G'_k)}. \end{aligned}$$

Let $m=\Delta(G_k')$. Then $m \geqslant \Delta(G')$ and $\alpha'(G_k,3) \leqslant g(m,s)$ by Lemma 2.5, as $\Phi(G_k)=\emptyset$. The result is thus obtained. □

**Lemma 2.7.** *For integers m and s with* $s \geqslant 1$ *and* $s/2 \leqslant m \leqslant s$, *we have*

$$g(m,s)=2^m+2^{s-m+1}-3.$$

**Proof.** We have $s-m \leqslant m$. If $s-m<m$, then as $s=m+(s-m)$, we have

$$g(m,s)=2^{m+1}+2^{s-m+1}-2^m-2^2+1=2^m+2^{s-m+1}-3.$$

If $s-m=m$, we have $s=2m$ and

$$g(m,s)=2^{m+2}+2^2-2^m-2^3+1=2^m+2^{m+1}-3=2^m+2^{s-m+1}-3.$$

This completes the proof. □

**Theorem 2.1.** *For* $G \in \mathscr{K}^{-s}(p,q)$,

$$\alpha'(G,3) \leqslant g(r,s),$$

*where* $r=\max\{\Delta(G'), \lfloor (s+1)/2 \rfloor\}$.

**Proof.** *Case* 1: $\Delta(G') \geqslant \lfloor (s+1)/2 \rfloor$. Let $r=\Delta(G')$. By Lemma 2.6,

$$\alpha'(G,3) \leqslant g(m,s)-(2^m-2^{\Delta(G')}-2^{s-\Delta(G')}+2^{s-m})$$

for some $m$ with $s \geqslant m \geqslant \Delta(G')$. Since $m \geqslant \Delta(G') \geqslant \lfloor (s+1)/2 \rfloor$, by Lemma 2.7,

$$g(\Delta(G'),s)=2^{\Delta(G')}+2^{s-\Delta(G')+1}-3,$$
$$g(m,s)=2^m+2^{s-m+1}-3.$$

Thus,

$$\begin{aligned} g(m,s)-g(\Delta(G'),s) &= 2^m-2^{\Delta(G')}-2^{s-\Delta(G')+1}+2^{s-m+1} \\ &\leqslant 2^m-2^{\Delta(G')}-2^{s-\Delta(G')}+2^{s-m} \\ &\leqslant g(m,s)-\alpha'(G,3), \end{aligned}$$

which implies that $\alpha'(G,3) \leqslant g(\Delta(G'),s)=g(r,s)$.

*Case* 2: $\Delta(G')<\lfloor (s+1)/2 \rfloor$. Let $r=\lfloor (s+1)/2 \rfloor$.

*Subcase* 2.1: $\Phi(G)=\emptyset$. By Lemma 2.5, $\alpha'(G,3) \leqslant g(m,s)$ for each $m \geqslant \Delta(G')$. Since $\Delta(G')<\lfloor (s+1)/2 \rfloor=r$, we have $\alpha'(G,3) \leqslant g(r,s)$.

*Subcase* 2.2: $\Phi(G) \neq \emptyset$. By Lemma 2.1, there is a sequence of graphs $G_0(=G)$, $G_1,\ldots,G_k$ in $\mathscr{K}^{-s}(p,q)$ such that $\Phi(G_k)=\emptyset$ and for $i=0,1,\ldots,k-1$,

(i) $G_{i+1}=F(G_i,u_i,v_i)$ for some $\{u_i,v_i\} \in \Phi(G_i)$ with $N_{G_i}(u_i) \cap N_{G_i}(v_i) \neq \emptyset$,
(ii) $\alpha(G_{i+1},3) \geqslant \alpha(G_i,3)$.

Since $\alpha'(G_{i+1},3)-\alpha'(G_i,3)=\alpha(G_{i+1},3)-\alpha(G_i,3)$, we have $\alpha'(G_{i+1},3) \geqslant \alpha'(G_i,3)$ for $i=0,1,\ldots,k-1$. If $\Delta(G_k')<\lfloor (s+1)/2 \rfloor=r$, then by the result in Subcase 2.1,

$\alpha'(G_k,3)\leqslant g(r,s)$. Thus $\alpha'(G,3)\leqslant g(r,s)$. Now assume that $\Delta(G_k')\geqslant r$. Since $\Delta(G_i')\leqslant\Delta(G_{i+1}')$ for all $i$ with $0\leqslant i\leqslant k-1$, there is some $i$ such that $\Delta(G_i')<r$ and $\Delta(G_{i+1}')\geqslant r$. Let $m_1=\Delta(G_{i+1}')$ and $m_2=\Delta(G_i')$. Since $m_1\geqslant r$, by the result in Case 1, we have

$$\alpha'(G_{i+1},3)\leqslant g(m_1,s).$$

By Lemma 2.4, we have

$$\alpha'(G_{i+1},3)-\alpha'(G_i,3)\geqslant 2^{m_1}-2^{m_2+1}+2^{2m_2-m_1}.$$

By Lemma 2.7, $g(m_1,s)=2^{m_1}+2^{s-m_1+1}-3$. Thus

$$\begin{aligned}\alpha'(G_i,3) &\leqslant g(m_1,s)-(2^{m_1}-2^{m_2+1}+2^{2m_2-m_1})\\ &= 2^{m_1}+2^{s-m_1+1}-3-(2^{m_1}-2^{m_2+1}+2^{2m_2-m_1})\\ &= 2^{s-m_1+1}+2^{m_2+1}-3-2^{2m_2-m_1}\\ &\leqslant 2^{s-m_1+1}+2^{m_2+1}-3.\end{aligned}$$

By Lemma 2.7, $g(r,s)=2^r+2^{s-r+1}-3$. Since $m_1\geqslant r\geqslant m_2+1$, we have

$$\alpha'(G_i,3)\leqslant 2^{s-r+1}+2^r-3=g(r,s).$$

This completes the proof. □

Define $h(i,s)=2^i+s-i-1$.

**Lemma 2.8.** *For* $s-1\geqslant i\geqslant(s+1)/2$, $h(i,s)<g(i,s)<h(i+1,s)$.

**Proof.** Let $s-1\geqslant i\geqslant(s+1)/2$. By Lemma 2.7, we have $g(i,s)=2^i+2^{s-i+1}-3$. Therefore,

$$h(i+1,s)-g(i,s)=2^i-2^{s-i+1}+(s-i+1)>0$$

and

$$g(i,s)-h(i,s)=2^{s-i+1}-3-(s-i-1)\geqslant(s-i+3)-(s-i+2)>0. \qquad \square$$

**Theorem 2.2.** *For* $G_1,G_2\in\mathscr{K}^{-s}(p,q)$, *if* $\Delta(G_2')\geqslant\max\{\Delta(G_1')+1,(s+3)/2\}$, *then*

$$\alpha'(G_2,3)>\alpha'(G_1,3).$$

**Proof.** By Lemma 2.7, it is clear that $g(i,s)\leqslant g(i+1,s)$ for any $i$ with $s/2\leqslant i\leqslant s-1$. Thus $g(i,s)\leqslant g(j,s)$ for any $i,j$ with $s/2\leqslant i<j\leqslant s$.

By Lemma 2.2,

$$\alpha'(G_2,3)\geqslant h(\Delta(G_2'),s).$$

By Theorem 2.1,

$$\alpha'(G_1,3) \leqslant g(m,s)$$

for $m = \max\{\Delta(G_1'), \lfloor (s+1)/2 \rfloor\}$. We now prove that $g(m,s) < h(\Delta(G_2'),s)$. Since $\lfloor (s+1)/2 \rfloor \leqslant m \leqslant \Delta(G_2') - 1$, it follows from Lemma 2.7 that

$$g(m,s) \leqslant g(\Delta(G_2') - 1, s).$$

Since $\Delta(G_2') - 1 \geqslant (s+1)/2$, by Lemma 2.8,

$$g(\Delta(G_2') - 1, s) < h(\Delta(G_2'), s).$$

Thus $g(m,s) < h(\Delta(G_2'),s)$. Therefore $\alpha'(G_1,3) < \alpha'(G_2,3)$. □

**Corollary.** *For any* $H \in \bigcup_{1 \leqslant i < (s+3)/2} \mathscr{D}_i(p,q,s)$, *and* $H_i \in \mathscr{D}_i(p,q,s)$, *where* $(s+3)/2 \leqslant i \leqslant s$,

$$\alpha'(H_s,3) > \alpha'(H_{s-1},3) > \cdots > \alpha'(H_{\lceil (s+3)/2 \rceil},3) > \alpha'(H,3).$$

## 3. Chromaticity of bipartite graphs

In this section, we use the results in Section 2 to study the chromaticity of bipartite graphs.

For any graph $G$ of order $n$, we have [3]:

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

**Lemma 3.1.** *If* $G \sim H$, *then* $\alpha(G,k) = \alpha(H,k)$ *for* $k = 1,2,\ldots$ .

**Theorem 3.1.** *Let* $p,q,s$ *be integers with* $p \geqslant q \geqslant 3$ *and* $1 \leqslant s \leqslant q-1$. *The following sets are pairwise* $\chi$-*disjoint*:

$$\mathscr{D}_1(p,q,s), \bigcup_{2 \leqslant i < t} \mathscr{D}_i(p,q,s), \mathscr{D}_t(p,q,s), \mathscr{D}_{t+1}(p,q,s), \ldots, \mathscr{D}_s(p,q,s),$$

*where* $t = \lceil (s+3)/2 \rceil$.

**Proof.** By Lemma 3.1 and the corollary to Theorem 2.2, the following sets are pairwise $\chi$-disjoint:

$$\bigcup_{2 \leqslant i < t} \mathscr{D}_i(p,q,s), \mathscr{D}_t(p,q,s), \mathscr{D}_{t+1}(p,q,s), \ldots, \mathscr{D}_s(p,q,s).$$

The remaining work is to prove that $\mathscr{D}_1(p,q,s)$ and $\mathscr{D}_i(p,q,s)$ are $\chi$-disjoint for every $i \geqslant 2$. Observe that $\alpha'(G,3) = s$ for any $G \in \mathscr{D}_1(p,q,s)$ by Lemma 2.2. But for any

$H \in \mathscr{D}_i(p,q,s)$, where $i \geqslant 2$, we have

$$\alpha'(H,3) \geqslant 2^i + s - 1 - i > s,$$

by Lemma 2.2. This completes the proof. □

In [1], we obtained the following result.

**Theorem 3.2** (Dong et al. [1]). *For* $p \geqslant q \geqslant 3$ *and* $0 \leqslant s \leqslant q-1$, $\mathscr{K}_2^{-s}(p,q)$ is $\chi$-*closed.*

The following result follows immediately from Theorems 3.1 and 3.2.

**Theorem 3.3.** *Each of the following sets is* $\chi$-*closed*:

$$\mathscr{K}_2^{-s}(p,q) \cap \mathscr{D}_1(p,q,s),\ \mathscr{K}_2^{-s}(p,q) \cap \bigcup_{2 \leqslant i < (s+3)/2} \mathscr{D}_i(p,q,s),$$

*and*

$$\mathscr{K}_2^{-s}(p,q) \cap \mathscr{D}_i(p,q,s), \quad i = \lceil (s+3)/2 \rceil, \ldots, s.$$

Which graphs in $\mathscr{K}^{-s}(p,q)$ are 2-connected?

**Lemma 3.2** (Dong et al. [1]). *If* $p \geqslant q \geqslant 3$ *and* $s \leqslant p+q-4$, *then for any* $G \in \mathscr{K}^{-s}(p,q)$ *with* $\delta(G) \geqslant 2$, *G is* 2-*connected.*

**Lemma 3.3.** *If* $p \geqslant q \geqslant 3$ *and* $0 \leqslant s \leqslant q-1$, *then*

$$\mathscr{K}^{-s}(p,q) - \mathscr{K}_2^{-s}(p,q) \subseteq \mathscr{D}_{q-1}(p,q,s).$$

**Proof.** *Since* $s \leqslant q-1$, *we have* $s \leqslant p+q-4$. *For any* $G \in \mathscr{K}^{-s}(p,q)$, *if* $\Delta(G') \leqslant q-2$, *then* $\delta(G) \geqslant 2$ and by Lemma 3.2, $G$ is 2-*connected.* Hence, $G \notin \mathscr{K}_2^{-s}(p,q)$ implies that $G \in \mathscr{D}_{q-1}(p,q,s)$. □

By Theorem 3.3 and Lemma 3.3, the following result is obtained.

**Theorem 3.4.** *Let* $p \geqslant q \geqslant 3$ *and* $1 \leqslant s \leqslant q-1$.

(i) $\mathscr{D}_1(p,q,s)$ *is* $\chi$-*closed.*
(ii) $\bigcup_{2 \leqslant i < (s+3)/2} \mathscr{D}_i(p,q,s)$ *is* $\chi$-*closed for* $s \geqslant 2$.
(iii) $\mathscr{D}_i(p,q,s)$ *is* $\chi$-*closed for each i with* $\lceil (s+3)/2 \rceil \leqslant i \leqslant \min\{s, q-2\}$.
(iv) $\mathscr{D}_{q-1}(p,q,s) \cap \mathscr{K}_2^{-s}(p,q)$ *is* $\chi$-*closed for* $s = q-1$.

## 4. $\chi$ -unique bipartite graphs

We have proved in [1] that every 2-connected graph in $\mathscr{D}_s(p,q,s)$ is $\chi$-unique. In this section, we shall search for $\chi$-unique graphs from $\mathscr{D}_{s-1}(p,q,s) \cup \mathscr{D}_{s-2}(p,q,s)$.

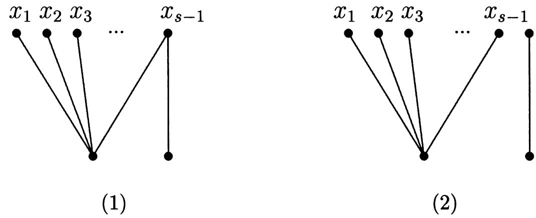



Fig. 1.

For a bipartite graph $G=(A,B;E)$, the number of 4-independent partitions $\{A_1,A_2,A_3,A_4\}$ in $G$ with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i=1,2,3,4$ is

$$\begin{aligned}&(2^{|A|-1}-1)(2^{|B|-1}-1)+\frac{1}{3!}(3^{|A|}-3\cdot 2^{|A|}+3)+\frac{1}{3!}(3^{|B|}-3\cdot 2^{|B|}+3)\\ &=(2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2. \qquad (1)\end{aligned}$$

Define $\alpha'(G,4)=\alpha(G,4)-((2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2)$. Observe that for $G,H\in\mathscr{K}^{-s}(p,q)$, $\alpha(G,4)=\alpha(H,4)$ iff $\alpha'(G,4)=\alpha'(H,4)$. In [2], we found the following two results.

**Lemma 4.1.** *For $G=(A,B;E)\in\mathscr{K}^{-s}(p,q)$ with $|A|=p$ and $|B|=q$,*

$$\begin{aligned}\alpha'(G,4)=&\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)\\ &+|\{\{Q_1,Q_2\}\mid Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\emptyset\}|. \qquad \square\end{aligned}$$

**Lemma 4.2.** *For a bipartite graph $G=(A,B;E)$, if $uvw$ is a path in $G'$ with $d_{G'}(u)=1$ and $d_{G'}(v)=2$, then for any $k\geqslant 2$,*

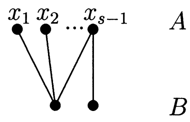
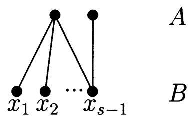
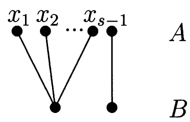
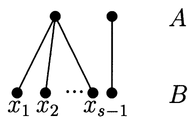
$$\alpha(G,k)=\alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1).$$

**Theorem 4.1.** *For any $G\in\mathscr{K}_2^{-s}(p,q)$ with $p\geqslant q\geqslant s+1\geqslant 6$, if $\Delta(G')=s-1$, then $G$ is $\chi$-unique.*

**Proof.** Since $s\geqslant 5$, we have $(s+3)/2\leqslant s-1\leqslant\min\{s,q-2\}$. By Theorem 3.4, $\mathscr{D}_{s-1}(p,q,s)$ is $\chi$-closed. It suffices to prove that for any $G_1,G_2\in\mathscr{D}_{s-1}(p,q,s)$, if $G_1\not\cong G_2$, then either $\alpha'(G_1,3)\neq\alpha'(G_2,3)$ or $\alpha'(G_1,4)\neq\alpha'(G_2,4)$.

There are only two bipartite graphs with size $s$ and maximum degree $s-1$, and they are shown in Fig. 1. Thus, there are four graphs in the set $\mathscr{D}_{s-1}(p,q,s)$, which are named as $T_1,T_2,T_3$ and $T_4$, displayed in Table 1.

Table 1

| name of graph | graphs $T_i'$ ($T_i' = K_{p,q} - T_i$) ($\lvert A\rvert = p$, $\lvert B\rvert = q$) | $\alpha'(T_i,3)$ | $\alpha'(T_i,4)$ |
|---|---|---|---|
| $T_1$ |  | $2^{s-1}+1$ | $\sum_{i=1}^{s-1}\binom{s-1}{i}(2^{p-i-1}+2^{q-2}-2)$ <br> $+(2^{p-2}+2^{q-2}-2)$ <br> $+(2^{p-2}+2^{q-3}-2)$ <br> $+2^{s-2}-1$ |
| $T_2$ |  | $2^{s-1}+1$ | $\sum_{i=1}^{s-1}\binom{s-1}{i}(2^{q-i-1}+2^{p-2}-2)$ <br> $+(2^{p-2}+2^{q-2}-2)$ <br> $+(2^{p-3}+2^{q-2}-2)$ <br> $+2^{s-2}-1$ |
| $T_3$ |  | $2^{s-1}$ | $\sum_{i=1}^{s-1}\binom{s-1}{i}(2^{p-i-1}+2^{q-2}-2)$ <br> $+(2^{p-2}+2^{q-2}-2)$ <br> $+2^{s-1}-1$ |
| $T_4$ |  | $2^{s-1}$ | $\sum_{i=1}^{s-1}\binom{s-1}{i}(2^{q-i-1}+2^{p-2}-2)$ <br> $+(2^{p-2}+2^{q-2}-2)$ <br> $+2^{s-1}-1$ |

For each $T_i$, we can find $\alpha'(T_i,3)$ and $\alpha'(T_i,4)$ by Lemmas 2.2 and 4.1, respectively. These values are also displayed in Table 1.

Observe that for any $i=1,2$ and $j=3,4$, $\alpha'(T_i,3) > \alpha'(T_j,3)$. If $p=q$, then $T_1 \cong T_2$ and $T_3 \cong T_4$. If $p>q$, then

$$\begin{aligned}\alpha'(T_1,4)-\alpha'(T_2,4) &= 2^{p-3}-2^{q-3}+\sum_{i=1}^{s-1}\binom{s-1}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\\ &= \sum_{i=3}^{s-1}\binom{s-1}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\\ &\quad+\left(1-\binom{s-1}{2}\right)(2^{p-3}-2^{q-3})\\ &<0 \qquad (2)\end{aligned}$$

and

$$\alpha'(T_3,4)-\alpha'(T_4,4)=\sum_{i=1}^{s-1}\binom{s-1}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})<0.$$

This completes the proof of the result. □

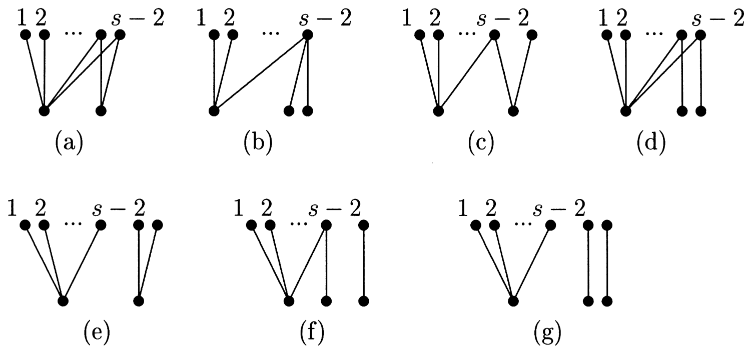



Fig. 2.

**Lemma 4.3.** *For any* $G \in \mathscr{D}_{s-2}(p,q,s)$, *where* $s \geqslant 4$, $G'$ *is one of the graphs in Fig.* 2.

**Theorem 4.2.** *For any* $G \in \mathscr{K}_2^{-s}(p,q)$ *with* $p \geqslant q \geqslant s+1 \geqslant 8$, *if* $\Delta(G') = s-2$, *then* $G$ *is* $\chi$*-unique.*

**Proof.** Since $s \geqslant 7$, $(s+3)/2 \leqslant s-2$. By Theorem 3.4 , $\mathscr{D}_{s-2}(p,q,s)$ is $\chi$-closed.

By Lemma 4.3, if $G \in \mathscr{D}_{s-2}(p,q,s)$, then $G'$ is one of the graphs in Fig. 2. Thus $\mathscr{D}_{s-2}(p,q,s)$ contains 16 graphs, which are named as $W_1, W_2, \ldots, W_{16}$. (See Table 2, parts 1 and 2.) Let

$$\mathscr{S}_1 = \{W_1, W_2, W_3, W_4\},$$

$$\mathscr{S}_2 = \{W_5, W_6, W_7, W_8\},$$

$$\mathscr{S}_3 = \{W_9, W_{10}, W_{11}, W_{12}, W_{13}, W_{14}\},$$

$$\mathscr{S}_4 = \{W_{15}, W_{16}\}.$$

Observe that for any $i,j$ with $1 \leqslant i < j \leqslant 4$, $\alpha'(W_{i_1},3) > \alpha'(W_{j_1},3)$ if $W_{i_1} \in \mathscr{S}_i$ and $W_{j_1} \in \mathscr{S}_j$. Thus each $\mathscr{S}_i$ is $\chi$-closed. Hence, for each $i$, to show that all graphs in $\mathscr{S}_i$ are $\chi$-unique, it suffices to show that for any two graphs $W_{i_1}, W_{i_2} \in \mathscr{S}_i$, if $W_{i_1} \not\cong W_{i_2}$, then either $\alpha'(W_{i_1},4) \neq \alpha'(W_{i_2},4)$ or $\alpha(W_{i_1},5) \neq \alpha(W_{i_2},5)$.

The values of $\alpha'(W_i,4)$ can be obtained by Lemma 4.1. We shall establish several inequalities of the form $\alpha'(W_i,4) < \alpha'(W_j,4)$ for some $i,j$. As an example, we use a method similar to the one for (2) and the fact that $8 \leqslant s+1 \leqslant q$ to show that

Table 2

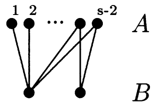
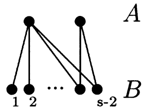
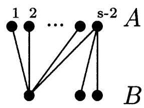
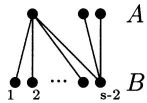
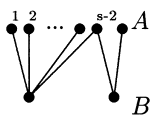
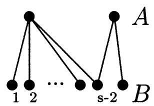
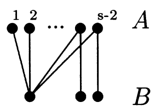
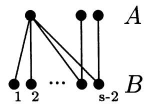
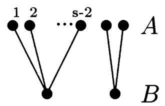
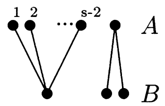
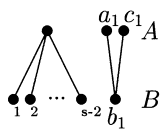
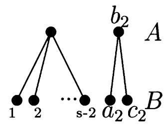
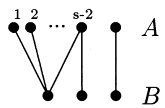
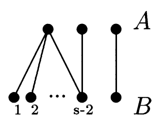
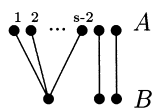
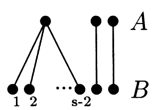
| name of graph | graphs $W_i'$ ($W_i' = K_{p,q} - W_i$) ($\lvert A\rvert = p$, $\lvert B\rvert = q$) | $\alpha'(W_i, 3)$ | $\alpha'(W_i, 4)$ |
|---|---|---|---|
| $W_1$ |  | $2^{s-2}+5$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+2^p+2^q+2^{p-2}+2^{q-3}$ $+5\cdot 2^{s-4}-15$ |
| $W_2$ |  | $2^{s-2}+5$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+2^p+2^q+2^{p-3}+2^{q-2}$ $+5\cdot 2^{s-4}-15$ |
| $W_3$ |  | $2^{s-2}+5$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+2^p+2^q+2^{p-1}-2^{q-4}$ $+3\cdot 2^{s-3}-15$ |
| $W_4$ |  | $2^{s-2}+5$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+2^p+2^q-2^{p-4}+2^{q-1}$ $+3\cdot 2^{s-3}-15$ |
| $W_5$ |  | $2^{s-2}+3$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+2^p+2^q-2^{p-3}-2^{q-3}$ $+2^{s-1}-11$ |
| $W_6$ |  | $2^{s-2}+3$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+2^p+2^q-2^{p-3}-2^{q-3}$ $+2^{s-1}-11$ |
| $W_7$ |  | $2^{s-2}+3$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+2^p+2^q-2^{q-2}$ $+2^{s-2}-7$ |
| $W_8$ |  | $2^{s-2}+3$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+2^p+2^q-2^{p-2}$ $+2^{s-2}-7$ |

Table 2 (*continued*)

| name of graph | graphs $W_i'$ ($W_i' = K_{p,q} - W_i$) ($\|A\| = p, \|B\| = q$) | $\alpha'(W_i, 3)$ | $\alpha'(W_i, 4)$ |
|---|---|---|---|
| $W_9$ |  | $2^{s-2}+2$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+5(2^{p-3}+2^{q-3})$ $+2^{q-3}+3\cdot 2^{s-2}-9$ |
| $W_{10}$ |  | $2^{s-2}+2$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+5(2^{p-3}+2^{q-3})$ $+2^{p-3}+3\cdot 2^{s-2}-9$ |
| $W_{11}$ |  | $2^{s-2}+2$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+5(2^{p-3}+2^{q-3})$ $+2^{q-3}+3\cdot 2^{s-2}-9$ |
| $W_{12}$ |  | $2^{s-2}+2$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+5(2^{p-3}+2^{q-3})$ $+2^{p-3}+3\cdot 2^{s-2}-9$ |
| $W_{13}$ |  | $2^{s-2}+2$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+5(2^{p-3}+2^{q-3})$ $+2^{p-3}+2^{s-2}+2^{s-3}-6$ |
| $W_{14}$ |  | $2^{s-2}+2$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+5(2^{p-3}+2^{q-3})$ $+2^{q-3}+2^{s-2}+2^{s-3}-6$ |
| $W_{15}$ |  | $2^{s-2}+1$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-i-1}+2^{q-2}-2)$ $+2(2^{p-2}+2^{q-2}-2)$ $+2(2^{s-2}-1)+1$ |
| $W_{16}$ |  | $2^{s-2}+1$ | $\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-i-1}+2^{p-2}-2)$ $+2(2^{p-2}+2^{q-2}-2)$ $+2(2^{s-2}-1)+1$ |

$\alpha'(W_{10},4) < \alpha'(W_{14},4)$ when $p > q$.

$$\begin{aligned}
&\alpha'(W_{10},4) - \alpha'(W_{14},4)\\
&\quad = \sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} - 2^{q-i-1} + 2^{q-2} - 2^{p-2})\\
&\qquad + 2^{p-3} - 2^{q-3} + 3\cdot 2^{s-2} - 2^{s-2} - 2^{s-3} - 3\\
&\quad = \sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1})\\
&\qquad + 2^{p-3} - 2^{q-3} + 3\cdot 2^{s-3} - 3\\
&\quad < -\binom{s-2}{2}(2^{p-3} - 2^{q-3}) + 2^{p-3} - 2^{q-3} + 3\cdot 2^{s-3}\\
&\quad < -3\cdot(2^{p-3} - 2^{q-3}) + 3\cdot 2^{s-3}\\
&\quad < 0.
\end{aligned}$$

(1) $\mathscr{S}_1$.

(1.1) When $p = q$, $W_1 \cong W_2$, $W_3 \cong W_4$, and

$$\alpha'(W_2,4) < \alpha'(W_3,4).$$

(1.2) When $p > q$,

$$\alpha'(W_1,4) < \alpha'(W_3,4) < \alpha'(W_4,4) < \alpha'(W_2,4).$$

(2) $\mathscr{S}_2$

(2.1) When $p = q$, $W_5 \cong W_6$, $W_7 \cong W_8$, and

$$\alpha'(W_7,4) < \alpha'(W_6,4).$$

(2.2) When $p > q$,

$$\alpha'(W_5,4) < \alpha'(W_7,4) < \alpha'(W_8,4) < \alpha'(W_6,4).$$

(3) $\mathscr{S}_3$

(3.1) When $p = q$, $W_9 \cong W_{12}$, $W_{10} \cong W_{11}$, $W_{13} \cong W_{14}$,

$$\alpha'(W_{11},4) = \alpha'(W_{12},4) > \alpha'(W_{13},4),$$

and by Lemma 4.2,

$$\begin{aligned}
&\alpha(W_{11},5) - \alpha(W_{12},5)\\
&\quad = \alpha(W_{11} + a_1b_1,5) + \alpha(W_{11} - \{a_1,b_1\},4) + \alpha(W_{11} - \{a_1,b_1,c_1\},4)\\
&\qquad - (\alpha(W_{12} + a_2b_2,5) + \alpha(W_{12} - \{a_2,b_2\},4) + \alpha(W_{12} - \{a_2,b_2,c_2\},4))\\
&\quad = \alpha(W_{11} - \{a_1,b_1,c_1\},4) - \alpha(W_{12} - \{a_2,b_2,c_2\},4)
\end{aligned}$$

$$=\alpha'(W_{11}-\{a_1,b_1,c_1\},4)-\alpha'(W_{12}-\{a_2,b_2,c_2\},4)$$
$$=\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-4}+2^{q-2-i}-2)-\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{p-3}+2^{q-3-i}-2)$$
$$=\sum_{i=1}^{s-2}\binom{s-2}{i}(2^{q-3-i}-2^{p-4})$$
$$<0,$$

since $W_{11}+a_1b_1\cong W_{12}+a_2b_2$ and $W_{11}-\{a_1,b_1\}\cong W_{12}-\{a_2,b_2\}$.

(3.2) When $p>q$,

$$\alpha'(W_9,4)<\alpha'(W_{10},4)<\alpha'(W_{14},4)<\alpha'(W_{11},4)<\alpha'(W_{12},4)$$
$$\alpha'(W_{13},4)<\alpha'(W_{10},4),$$

and

$$\alpha'(W_{13},4)-\alpha'(W_9,4)=2^{p-3}-2^{q-3}-2^{s-1}+2^{s-3}+3$$
$$\begin{cases}<0, & \text{if } p=q+1, q=s+1\\ >0, & \text{if } p\geqslant q+2 \text{ or } p=q+1\geqslant s+3.\end{cases}$$

(4) $\mathscr{S}_4$.

(4.1) When $p=q$, $W_{15}\cong W_{16}$.

(4.2) When $p>q$, $\alpha'(W_{15},4)<\alpha'(W_{16},4)$.

This completes the proof. □

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