An attempt to classify bipartite graphs by chromatic polynomials

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Abstract

For integers $p, q, s$ with $p \geq q \geq 3$ and $1 \leq s \leq q - 1$, let $\mathcal{K}^{-s}(p, q)$ (resp. $\mathcal{K}_2^{-s}(p, q)$) denote the set of connected (resp. 2-connected) bipartite graphs which can be obtained from $K_{p, q}$ by deleting a set of $s$ edges. In this paper, we first find an upper bound for the 3-independent partition number of a graph $G_2\mathcal{K}^{-s}(p, q)$ with respect to the maximum degree of $G_0$, where $G_0 = K_{p, q} - G$. By using this result, we show that the set $\{G | G \in \mathcal{K}^{-s}_2(p, q), \Delta(G') = i\}$ is closed under the chromatic equivalence for every integer $i$ with $s \geq i \geq (s + 3)/2$. From this result, we prove that for any $G \in \mathcal{K}^{-s}_2(p, q)$ with $p \geq q \geq 3$, if $5 \leq s \leq q - 1$ and $\Delta(G') = s - 1$, or $7 \leq s \leq q - 1$ and $\Delta(G') = s - 2$, then $G$ is chromatically unique. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs considered here are simple graphs. For a graph $G$, let $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ and $P(G, \lambda)$ be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of $G$, respectively.

For integers $p, q, s$ with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p, q)$ (resp. $\mathcal{K}_2^{-s}(p, q)$) denote the connected (resp. 2-connected) bipartite graphs which can be obtained from $K_{p, q}$ by deleting a set of $s$ edges.

For a graph $G$ and a positive integer $k$, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a $k$-independent partition in $G$ if each $A_i$ is a non-empty independent set of $G$. Let $x(G, k)$ denote the number of $k$-independent partitions in $G$. For any bipartite graph $G = (A, B; E)$ with bipartition $A$ and $B$ and edge set $E$, let

$$x'(G, 3) = x(G, 3) - (2^{|A| - 1} + 2^{|B| - 1} - 2).$$

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For integers $s$ and $m$ with $m \geq 1$ and $s \geq 0$, define
\[
g(m,s) = 2^{a+m} + 2^{a+d} - 2^m - 2^{a+1} + 1,
\]
where $a$ and $d$ are integers determined by $s = am + d$, $a \geq 0$ and $0 \leq d \leq m - 1$. In [1, Theorem 3.1], we obtained the following result.

**Theorem 1.1.** For any graph $G \in \mathcal{X}^{-i}(p,q)$, where $p \geq q \geq 2$ and $0 \leq s \leq (p-1)(q-1)$,
\[
z'(G,3) \leq g(p-1,s).
\]

In this paper, we shall improve the upper bounds for $z'(G, 3)$, where $G \in \mathcal{X}^{-i}(p,q)$, under the following conditions for $p, q, s$:
\[
p \geq q \geq 3 \quad \text{and} \quad 1 \leq s \leq q - 1.
\]

Thus the above conditions are fixed throughout this paper. Note that they imply that $G$ is connected.

For a bipartite graph $H = (A, B; E)$, let $H' = (A', B'; E')$ be the graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $H' = K_{p,q} - H$, where $p = |A'|$ and $q = |B'|$.

The upper bound given in Theorem 1.1 is not good for bipartite graphs $G$ with low $\Delta(G')$. For example, when $\Delta(G') = 1$ and $s \leq q - 1 \leq p - 1$, $z'(G,3) = s$. But $g(p - 1, s) = 2^s - 1$, which is much larger than $z'(G,3)$ for large $s$. Thus it is necessary to study the relation between $z'(G, 3)$ and $\Delta(G')$. We first, in Theorem 2.1, give an upper bound for $z'(G,3)$ with respect to $\Delta(G')$:
\[
z'(G,3) \leq g(r,s)
\]
for any $G \in \mathcal{X}^{-i}(p,q)$, where $r = \max\{|\Delta(G')|, (s+1)/2\}$. From this result, we prove, in Theorem 2.2, that for any $G_1, G_2 \in \mathcal{X}^{-i}(p,q)$, if $\Delta(G'_1) \geq \max\{\Delta(G'_1)+1, (s+3)/2\}$, then $z'(G_2,3) > z'(G_1,3)$. Partition $\mathcal{X}^{-i}(p,q)$ into the following subsets:
\[
\mathcal{D}_i(p,q,s) = \{G \in \mathcal{X}^{-i}(p,q) \mid \Delta(G') = i\}, \quad i = 1, 2, \ldots, s.
\]

Then for any $H \in \bigcup_{1 \leq i \leq \lfloor(s+3)/2\rfloor} \mathcal{D}_i(p,q,s)$ and $H_1 \in \mathcal{D}_i(p,q,s)$, where $(s+3)/2 \leq i \leq s$, it follows from Theorem 2.2 that
\[
z'(H_1,3) > \cdots > z'(H_{\lfloor(s+3)/2\rfloor},3) > z'(H,3).
\]

We then use the above results to study the chromaticity of bipartite graphs. Two graphs $G$ and $H$ are said to be *chromatically equivalent* (or simply $\chi$-*equivalent*), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by $G$ under $\sim$ is denoted by $[G]$. A graph $G$ is *chromatically unique* (or simply $\chi$-*unique*) if $H \cong G$ whenever $H \sim G$, i.e., $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi$-*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1, \lambda) \neq P(G_2, \lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint*, or simply $\chi$-*disjoint*.
We shall show, in Theorem 3.1, that the following sets are pairwise \( \chi \)-disjoint:

\[
\mathcal{D}_i(p,q,s) = \bigcup_{2 < i < t} \mathcal{D}_i(p,q,s),
\]

where \( t = [(s + 3)/2] \). This result gives a rough classification of graphs in the set \( \mathcal{X}^{-1}(p,q) \) by chromatic polynomials.

We have proved in [1] that every 2-connected graph in \( \mathcal{D}_i(p,q,s) \) is \( \chi \)-unique. We shall, in Theorems 4.1 and 4.2, prove that \( G \) is \( \chi \)-unique for every \( G \in \mathcal{D}_{s-1}(p,q,s) \), where \( s \geq 5 \), or \( G \in \mathcal{D}_{s-2}(p,q,s) \), where \( s \geq 7 \).

2. An upper bound for \( x'(G,3) \)

For a graph \( G \) and \( x \in V(G) \), let \( N_G(x) \), or simply \( N(x) \), be the set of vertices in \( G \) adjacent to \( x \), and let \( d_G(x) \), or simply \( d(x) \), be the degree of \( x \) in \( G \).

For a bipartite graph \( G = (A,B;E) \) and two vertices \( x, y \) with \( x, y \in B \) (or similarly \( x, y \in A \)), we construct a new bipartite graph, denoted by \( F(G,x,y) \) or simply \( F \), from \( G - x - y \) by adding two new vertices \( w_1 \) and \( w_2 \) and edges joining \( w_1 \) to all vertices in \( N(x) \cup N(y) \) and \( w_2 \) to all vertices in \( N(x) \cap N(y) \). The graph \( F(G,x,y) \), say \( x, y \in B \), is also a bipartite graph, which can be written as \( (A,B';E') \), where \( B' = (B - \{x, y\}) \cup \{w_1, w_2\} \). Observe that \( F' = F(G',x,y) \) and \( A(F') \geq A(G') \).

For a bipartite graph \( G = (A,B;E) \), let

\[
\Phi(G) = \{x, y \mid x, y \in A \text{ or } x, y \in B, N(x) \not\subset N(y), \text{ and } N(y) \not\subset N(x)\}.
\]

In [1, Lemma 3.8], the following result was found.

**Lemma 2.1.** For \( G \in \mathcal{X}^{-1}(p,q) \) with \( \Phi(G) \neq \emptyset \), there is a sequence of graphs \( G_0 = G, G_1, \ldots, G_k \) in \( \mathcal{X}^{-1}(p,q) \) such that \( \Phi(G_i) = \emptyset \) and for \( i = 0, 1, \ldots, k-1 \),

(i) \( G_{i+1} = F(G_i, u_i, v_i) \) for some \( \{u_i, v_i\} \in \Phi(G_i) \) with \( N_G(u_i) \cap N_G(v_i) \neq \emptyset \),

(ii) \( |\Phi(G_{i+1})| < |\Phi(G_i)| \), and

(iii) \( x'(G_{i+1}, 3) \geq x'(G, 3) \).

For a bipartite graph \( G = (A,B;E) \), let \( \mathcal{I}(G) \) be the set of independent sets in \( G \) and

\[
\Omega(G) = \{Q \in \mathcal{I}(G) \mid Q \cap A \neq \emptyset, Q \cap B \neq \emptyset\}.
\]

In [2], we found the following result.

**Lemma 2.2.** For \( G \in \mathcal{X}^{-1}(p,q) \), \( x'(G,3) = |\Omega(G)| \geq 2^{\Delta(G)} + s - 1 - \Delta(G') \).

We now study the difference between \( x'(F,3) \) and \( x'(G,3) \).
Lemma 2.3. For $G = (A, B; E) \in \mathcal{H}^{-s}(p, q)$ with $|A| = p$ and $|B| = q$, and $x, y \in A$ or $x, y \in B$, we have

\[ x'(F, 3) - x'(G, 3) \geq 2^s(2^{a-c} - 1)(2^{b-c} - 1), \]

where $F = F(G, x, y)$, $a = d_G(x)$, $b = d_G(y)$ and $c = |N_G(x) \cap N_G(y)|$.

Proof. Without loss of generality, assume that $x, y \in A$. Let $N_1 = B - N_G(x)$, $N_2 = B - N_G(y)$ and $N_0 = B - (N_G(x) \cup N_G(y))$. Then $N_1 = N_G^c(x)$, $N_2 = N_G^c(y)$ and $N_0 = N_G^c(x) \cap N_G^c(y)$.

Let $\Omega_1(G) = \{Q \in \Omega(G) \mid Q \cap B \subseteq N_0\}$. We first show that $|\Omega_1(G)| = |\Omega_1(F)|$. Let $H$ be the subgraph of $G$ induced by $A \cup N_0$. Then $\Omega_1(G) = \Omega(H)$. Similarly, $\Omega_1(F) = \Omega(H')$, where $H'$ is the subgraph of $F$ induced by $(A - \{x, y\}) \cup \{w_1, w_2\} \cup N_0$. Obviously, $H' \cong H$. Thus $|\Omega_1(G)| = |\Omega_1(F)|$.

Let $\Omega_2(G) = \Omega(G) - \Omega_1(G)$, and let $Q \in \Omega_2(G)$. Since $Q \cap ((N_1 \cup N_2) - N_0) \neq \emptyset$, we have $\{x, y\} \not\subseteq Q$. We define a mapping $p$ from $\Omega_2(G)$ to $\Omega_2(F)$: for $Q \in \Omega_2(G)$,

\[ p(Q) = \begin{cases} Q & \text{if } Q \cap \{x, y\} = \emptyset, \\ (Q - \{x, y\}) \cup \{w_2\} & \text{otherwise}. \end{cases} \]

We observe that

(i) for $Q_1, Q_2 \in \Omega_2(G)$, if $Q_1 \neq Q_2$, then $p(Q_1) \neq p(Q_2)$;

(ii) for $Q \in \Omega_2(G)$, if $x \in Q$ or $y \in Q$, then $Q \cap B \subseteq N_1$ or $Q \cap B \subseteq N_2$, respectively.

Let $\Omega'(F)$ be the set of all $Q \subseteq A \cup N_1 \cup N_2 \cup \{w_2\}$ such that $w_2 \in Q$, $Q \cap (N_1 - N_0) \neq \emptyset$ and $Q \cap (N_2 - N_0) \neq \emptyset$. It is clear that $\Omega'(F)$ is a subset of $\Omega_2(F)$. By (ii), there exists no $Q \in \Omega_2(G)$ such that $p(Q) \in \Omega'(F)$. Then by (i),

\[ |\Omega_2(F)| - |\Omega_2(G)| \geq |\Omega'(F)|. \]

Observe that

\[ |\Omega'(F)| = 2^{N_0}(2^{N_1} - 1)(2^{N_2} - 1) = 2^s(2^{a-c} - 1)(2^{b-c} - 1). \]

Hence $x'(F, 3) - x'(G, 3) \geq 2^s(2^{a-c} - 1)(2^{b-c} - 1)$. \hfill \Box

Lemma 2.4. For $G = (A, B; E) \in \mathcal{H}^{-s}(p, q)$ and $F = (G, x, y)$ for $x, y \in A$ or $x, y \in B$, we have

\[ x'(F, 3) - x'(G, 3) \geq 2^s1(F') - 2^s2(F') - 2^{s - \Delta(G')} + 2^{s - \Delta(F')} \]

and

\[ x'(F', 3) - x'(G', 3) \geq 2^s1(F') - 2^s2(F') + 2^{s - \Delta(G')} - 2^{s - \Delta(F')} \]

Proof. Let $a = d_G(x)$, $b = d_G(y)$ and $c = |N_G(x) \cap N_G(y)|$. By Lemma 2.3,

\[ x'(F, 3) - x'(G, 3) \geq 2^s(2^{a-c} - 1)(2^{b-c} - 1) \geq 0. \]

Recall that $\Delta(F') \geq \Delta(G')$. The result holds when $\Delta(F') = \Delta(G')$. Now suppose that $\Delta(F') > \Delta(G')$. The
By the definition of $F = F(G, x, y)$, $A(F') = \max \{A(G'), |N_G'(x) \cup N_G'(y)|\}$. Since $A(F') > A(G')$, we have $A(F') = |N_G'(x) \cup N_G'(y)| = a + b - c$. It is obvious that $a, b \leq A(G')$ and $a + b \leq s$. Since $2^x$ is a convex function of $x$, it follows that $2^a + 2^b \leq 2^a(G') + 2^{a+b-A(G')}$. Therefore,

$$
2^c(2^{a-c} - 1)(2^{b-c} - 1) = 2^{a+b-c} - 2^a - 2^b + 2^c \\
\geq 2^a(G') - 2^{a+b} + 2^c(2^{a(G')} - 2^{A(G')}) \\
\geq 2^a(G') - 2^{a+b} + 2^c(2^{a-G'} - 2^{A(G')}) \\
= 2^a(G') - 2^{a+b} + 2^c(2^{a-G'} + 2^{s-A(G')})
$$

and

$$
2^c(2^{a-c} - 1)(2^{b-c} - 1) = 2^{e-a-b}(2^{a+b-c} - 2^a)(2^{a+b-c} - 2^b) \\
= 2^{e-A(G')}(2^a(G') - 2^a)(2^b(G') - 2^b) \\
\geq 2^{e-A(G')}(2^a(G') - 2^a)(2^b(G') - 2^b) \\
= 2^a(G') - 2^{a+b} + 2^c(2^{a-G'} + 2^{s-A(G')}).
$$

This completes the proof of the result. □

In [1] (the corollary to Lemma 3.10), we have the following result.

**Lemma 2.5.** For $G = (A, B; E) \in \mathscr{X}^{-s}(p, q)$, if $\Phi(G) = \emptyset$, then

$$
\lambda'(G, 3) \leq g(m,s)
$$

for each $m \geq A(G')$.

**Lemma 2.6.** For $G = (A, B; E) \in \mathscr{X}^{-s}(p, q)$,

$$
\lambda'(G, 3) \leq g(m,s) - (2^m - 2^{a(G')} - 2^{s-A(G')} + 2^{s-m})
$$

for some $m$ with $s \geq m \geq A(G')$.

**Proof.** If $\Phi(G) = \emptyset$, then by Lemma 2.5, the result holds by taking $m = A(G')$. Now assume that $\Phi(G) \neq \emptyset$. By Lemma 2.1, there is a sequence of graphs $G_0(=G), G_1, \ldots, G_k$ in $\mathscr{X}^{-s}(p, q)$ such that $\Phi(G_k) = \emptyset$ and for $i = 0, 1, \ldots, k - 1$, $G_{i+1} = F(G_i, u_i, v_i)$ for some $\{u_i, v_i\} \in F(G_i)$ with $N_{G_i}(u_i) \cap N_{G_i}(v_i) \neq \emptyset$.

By Lemma 2.4, for $i = 0, 1, \ldots, k - 1$,

$$
\lambda'(G_{i+1}, 3) - \lambda'(G_i, 3) \geq 2^{A(G_i)} - 2^{A(G_i')} - 2^{s-A(G_i')} + 2^{s-A(G_{i+1})}.
$$

Hence,

$$
\lambda'(G_k, 3) - \lambda'(G_0, 3) = \sum_{i=0}^{k-1}(\lambda'(G_{i+1}, 3) - \lambda'(G_i, 3)) \\
\geq 2^{A(G_0)} - 2^{A(G_0')} - 2^{s-A(G_0')} + 2^{s-A(G_k)}.
$$
Let $m = \Delta(G'_k)$. Then $m \geq \Delta(G')$ and $\varepsilon'(G_k, 3) \leq g(m, s)$ by Lemma 2.5, as $\Phi(G_k) = \emptyset$. The result is thus obtained. □

Lemma 2.7. For integers $m$ and $s$ with $s \geq 1$ and $s/2 \leq m \leq s$, we have
$$g(m, s) = 2^m + 2^{s-m+1} - 3.$$  

Proof. We have $s - m \leq m$. If $s - m < m$, then as $s = m + (s - m)$, we have
$$g(m, s) = 2^{m+1} + 2^{s-m+1} - 2^m - 2^2 + 1 = 2^m + 2^{s-m+1} - 3.$$  

If $s - m = m$, we have $s = 2m$ and
$$g(m, s) = 2^{m+2} + 2^2 - 2^m - 3 = 2^m + 2^{m+1} - 3 = 2^m + 2^{s-m+1} - 3.$$  

This completes the proof. □

Theorem 2.1. For $G \in \mathcal{K}^{-\mathcal{A}}(p, q)$,
$$\varepsilon'(G, 3) \leq g(r, s),$$
where $r = \max\{\Delta(G'), \lfloor(s + 1)/2\rfloor\}$.

Proof. Case 1: $\Delta(G') \geq \lfloor(s + 1)/2\rfloor$. Let $r = \Delta(G')$. By Lemma 2.6,
$$\varepsilon'(G, 3) \leq g(m, s) - (2^m - 2^{\Delta(G')} - 2^{s-\Delta(G')} + 2^{s-m})$$
for some $m$ with $s \geq m \geq \Delta(G')$. Since $m \geq \Delta(G') \geq \lfloor(s + 1)/2\rfloor$, by Lemma 2.7,
$$g(\Delta(G'), s) = 2^{\Delta(G')} + 2^{s-\Delta(G')} - 3,$$
$$g(m, s) = 2^m + 2^{s-m+1} - 3.$$  

Thus,
$$g(m, s) - g(\Delta(G'), s) = 2^m - 2^{\Delta(G')} - 2^{s-\Delta(G')} + 2^{s-m+1}$$
$$\leq 2^m - 2^{\Delta(G')} - 2^{s-\Delta(G')} + 2^{s-m}$$
$$\leq g(m, s) - \varepsilon'(G, 3),$$
which implies that $\varepsilon'(G, 3) \leq g(\Delta(G'), s) = g(r, s)$.

Case 2: $\Delta(G') < \lfloor(s + 1)/2\rfloor$. Let $r = \lfloor(s + 1)/2\rfloor$.

Subcase 2.1: $\Phi(G) = \emptyset$. By Lemma 2.5, $\varepsilon'(G, 3) \leq g(m, s)$ for each $m \geq \Delta(G')$. Since $\Delta(G') < \lfloor(s + 1)/2\rfloor = r$, we have $\varepsilon'(G, 3) \leq g(r, s)$.

Subcase 2.2: $\Phi(G) \neq \emptyset$. By Lemma 2.1, there is a sequence of graphs $G_0(=G)$, $G_1, \ldots, G_k$ in $\mathcal{K}^{-\mathcal{A}}(p, q)$ such that $\Phi(G_k) = \emptyset$ and for $i = 0, 1, \ldots, k - 1$,

(i) $G_{i+1} = F(G_i, u_i, v_i)$ for some $\{u_i, v_i\} \in \Phi(G_i)$ with $N_{G_i}(u_i) \cap N_{G_i}(v_i) \neq \emptyset$,
(ii) $\varepsilon(G_{i+1}, 3) \geq \varepsilon(G_i, 3)$.

Since $\varepsilon'(G_{i+1}, 3) - \varepsilon'(G_i, 3) = \varepsilon(G_{i+1}, 3) - \varepsilon(G_i, 3)$, we have $\varepsilon'(G_{i+1}, 3) \geq \varepsilon'(G_i, 3)$ for $i = 0, 1, \ldots, k - 1$. If $\Delta(G'_k) < \lfloor(s + 1)/2\rfloor = r$, then by the result in Subcase 2.1,
\[ \alpha'(G_k,3) \leq g(r,s). \] Thus \[ \alpha'(G,3) \leq g(r,s). \] Now assume that \( \Delta(G'_k) \geq r \). Since \( \Delta(G'_i) \leq \Delta(G'_{i+1}) \) for all \( i \) with \( 0 \leq i < k - 1 \), there is some \( i \) such that \( \Delta(G'_i) < r \) and \( \Delta(G'_{i+1}) \geq r \). Let \( m_1 = \Delta(G'_{i+1}) \) and \( m_2 = \Delta(G'_i) \). Since \( m_1 \geq r \), by the result in Case 1, we have
\[ \alpha'(G_{i+1},3) \leq g(m_1,s). \]

By Lemma 2.4, we have
\[ \alpha'(G_{i+1},3) - \alpha'(G_i,3) \geq 2^{m_1} - 2^{m_2+1} + 2^{2m_2-m_1}. \]

By Lemma 2.7, \( g(m_1,s) = 2^{m_1} + 2^{s-m_1+1} - 3 \). Thus
\[ \alpha'(G_i,3) \leq g(m_1,s) - (2^{m_1} - 2^{m_2+1} + 2^{2m_2-m_1}) \]
\[ = 2^{s-m_1+1} + 2^{m_1+1} - 3 - 2^{m_1-m_2} \]
\[ \leq 2^{s-m_1+1} + 2^{m_2+1} - 3. \]

By Lemma 2.7, \( g(r,s) = 2^r + 2^{s-r+1} - 3 \). Since \( m_1 \geq r \geq m_2 + 1 \), we have
\[ \alpha'(G_i,3) \leq 2^{s-r+1} + 2^r - 3 = g(r,s). \]

This completes the proof. \( \square \)

Define \( h(i,s) = 2^i + s - i - 1 \).

**Lemma 2.8.** For \( s - 1 \geq i \geq (s + 1)/2 \), \( h(i,s) < g(i,s) < h(i + 1,s) \).

**Proof.** Let \( s - 1 \geq i \geq (s + 1)/2 \). By Lemma 2.7, we have \( g(i,s) = 2^i + 2^{s-i+1} - 3 \). Therefore,
\[ h(i + 1,s) - g(i,s) = 2^i - 2^{s-i+1} + (s - i + 1) > 0 \]
and
\[ g(i,s) - h(i,s) = 2^{s-i+1} - 3 - (s - i - 1) \geq (s - i + 3) - (s - i + 2) > 0. \] \( \square \)

**Theorem 2.2.** For \( G_1, G_2 \in \mathcal{K}^{-i}(p,q) \), if \( \Delta(G'_2) \geq \max\{\Delta(G'_1) + 1, (s + 3)/2\} \), then
\[ \alpha'(G_2,3) > \alpha'(G_1,3). \]

**Proof.** By Lemma 2.7, it is clear that \( g(i,s) \leq g(i + 1,s) \) for any \( i \) with \( s/2 \leq i \leq s - 1 \). Thus \( g(i,s) \leq g(j,s) \) for any \( i, j \) with \( s/2 \leq i < j \leq s \).

By Lemma 2.2,
\[ \alpha'(G_2,3) \geq h(\Delta(G'_2),s). \]
By Theorem 2.1,
\[ \chi'(G_1, 3) \leq g(m, s) \]
for \( m = \max\{\Delta(G'_1), \lfloor (s + 1)/2 \rfloor \} \). We now prove that \( g(m, s) < h(\Delta(G'_2), s) \). Since \( \lfloor (s + 1)/2 \rfloor \leq m \leq \Delta(G'_2) - 1 \), it follows from Lemma 2.7 that
\[ g(m, s) \leq g(\Delta(G'_2) - 1, s). \]
Since \( \Delta(G'_2) - 1 \geq (s + 1)/2 \), by Lemma 2.8,
\[ g(\Delta(G'_2) - 1, s) < h(\Delta(G'_2), s). \]
Thus \( g(m, s) < h(\Delta(G'_2), s) \). Therefore \( \chi'(G_1, 3) < \chi'(G_2, 3) \). \( \square \)

**Corollary.** For any \( H \in \bigcup_{1 \leq i < (s+3)/2} D_i(p, q, s) \) and \( H_t \in D_t(p, q, s) \), where \( (s + 3)/2 \leq t \leq s \),
\[ \chi'(H_t, 3) > \chi'(H_{t-1}, 3) > \cdots > \chi'(H_{\lfloor (s+3)/2 \rfloor}, 3) > \chi'(H, 3). \]

**3. Chromaticity of bipartite graphs**

In this section, we use the results in Section 2 to study the chromaticity of bipartite graphs.

For any graph \( G \) of order \( n \), we have [3]:
\[ P(G, \lambda) = \sum_{k=1}^{n} \lambda(G, k) \lambda(\lambda - 1) \cdots (\lambda - k + 1). \]

**Lemma 3.1.** If \( G \sim H \), then \( \chi(G, k) = \chi(H, k) \) for \( k = 1, 2, \ldots \).

**Theorem 3.1.** Let \( p, q, s \) be integers with \( p \geq q \geq 3 \) and \( 1 \leq s \leq q - 1 \). The following sets are pairwise \( \chi \)-disjoint:
\[ D_1(p, q, s), \bigcup_{2 \leq i < t} D_i(p, q, s), D_t(p, q, s), D_{t+1}(p, q, s), \ldots, D_s(p, q, s), \]
where \( t = \lfloor (s + 3)/2 \rfloor \).

**Proof.** By Lemma 3.1 and the corollary to Theorem 2.2, the following sets are pairwise \( \chi \)-disjoint:
\[ \bigcup_{2 \leq i < t} D_i(p, q, s), D_t(p, q, s), D_{t+1}(p, q, s), \ldots, D_s(p, q, s). \]
The remaining work is to prove that \( D_1(p, q, s) \) and \( D_t(p, q, s) \) are \( \chi \)-disjoint for every \( t \geq 2 \). Observe that \( \chi'(G, 3) = s \) for any \( G \in D_1(p, q, s) \) by Lemma 2.2. But for any
by Lemma 2.2. This completes the proof. □

In [1], we obtained the following result.

**Theorem 3.2** (Dong et al. [1]). For \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \), \( \mathcal{K}^{-i}(p,q) \) is \( \chi \)-closed.

The following result follows immediately from Theorems 3.1 and 3.2.

**Theorem 3.3.** Each of the following sets is \( \chi \)-closed:

\[
\mathcal{K}^{-i}(p,q) \cap \mathcal{D}_1(p,q,s), \quad \mathcal{K}^{-i}(p,q) \cap \bigcup_{2 \leq i < (s+3)/2} \mathcal{D}_i(p,q,s),
\]

and

\[
\mathcal{K}^{-i}(p,q) \cap \mathcal{D}_i(p,q,s), \quad i = [(s+3)/2], \ldots, s.
\]

Which graphs in \( \mathcal{K}^{-i}(p,q) \) are 2-connected?

**Lemma 3.2** (Dong et al. [1]). If \( p \geq q \geq 3 \) and \( s \leq p+q-4 \), then for any \( G \in \mathcal{K}^{-i}(p,q) \) with \( \delta(G) \geq 2 \), \( G \) is 2-connected.

**Lemma 3.3.** If \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \), then

\[
\mathcal{K}^{-i}(p,q) - \mathcal{K}^{-i-1}(p,q) \subseteq \mathcal{D}_{q-i}(p,q,s).
\]

**Proof.** Since \( s \leq q - 1 \), we have \( s \leq p+q-4 \). For any \( G \in \mathcal{K}^{-i}(p,q) \), if \( \Delta(G') \leq q - 2 \), then \( \delta(G) \geq 2 \) and by Lemma 3.2, \( G \) is 2-connected. Hence, \( G \notin \mathcal{K}^{-i-1}(p,q) \) implies that \( G \in \mathcal{D}_{q-i}(p,q,s) \). □

By Theorem 3.3 and Lemma 3.3, the following result is obtained.

**Theorem 3.4.** Let \( p \geq q \geq 3 \) and \( 1 \leq s \leq q - 1 \).

(i) \( \mathcal{D}_1(p,q,s) \) is \( \chi \)-closed.

(ii) \( \bigcup_{2 \leq i < (s+3)/2} \mathcal{D}_i(p,q,s) \) is \( \chi \)-closed for \( s \geq 2 \).

(iii) \( \mathcal{D}_i(p,q,s) \) is \( \chi \)-closed for each \( i \) with \( [(s+3)/2] \leq i \leq \min\{s, q-2\} \).

(iv) \( \mathcal{D}_{q-i}(p,q,s) \cap \mathcal{K}^{-i}(p,q) \) is \( \chi \)-closed for \( s = q - 1 \).

4. \( \chi \)-unique bipartite graphs

We have proved in [1] that every 2-connected graph in \( \mathcal{D}_s(p,q,s) \) is \( \chi \)-unique. In this section, we shall search for \( \chi \)-unique graphs from \( \mathcal{D}_{q-i}(p,q,s) \cap \mathcal{D}_{q-i-2}(p,q,s) \).
For a bipartite graph \( G = (A, B; E) \), the number of 4-independent partitions \( \{A_1, A_2, A_3, A_4\} \) in \( G \) with \( A_i \subseteq A \) or \( A_i \subseteq B \) for all \( i = 1, 2, 3, 4 \) is

\[
(2^{|A|} - 1)(2^{|B|} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3)
\]

\[
= (2^{|A|} - 2)(2^{|B|} - 2) + \frac{1}{2}(3^{|A|} - 3^{|B|} - 1) - 2.
\]

(1)

Define \( \chi'(G, 4) = \chi(G, 4) - ((2^{|A|} - 2)(2^{|B|} - 2) + \frac{1}{2}(3^{|A|} - 3^{|B|} - 1) - 2) \). Observe that for \( G, H \in \mathcal{X}^{-s}(p, q) \), \( \chi(G, 4) = \chi(H, 4) \) iff \( \chi'(G, 4) = \chi'(H, 4) \). In [2], we found the following two results.

**Lemma 4.1.** For \( G = (A, B; E) \in \mathcal{X}^{-s}(p, q) \) with \( |A| = p \) and \( |B| = q \),

\[
\chi'(G, 4) = \sum_{Q \in \Omega(G)} (2^{p-1-|Q\cap A|} + 2^{q-1-|Q\cap B|} - 2) + |\{\{Q_1, Q_2\} | Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset\}|. \quad \square
\]

**Lemma 4.2.** For a bipartite graph \( G = (A, B; E) \), if \( uvw \) is a path in \( G' \) with \( d_{G'}(u) = 1 \) and \( d_{G'}(v) = 2 \), then for any \( k \geq 2 \),

\[
\chi(G, k) = \chi(G + uv, k) + \chi(G - \{u, v\}, k - 1) + \chi(G - \{u, v, w\}, k - 1).
\]

**Theorem 4.1.** For any \( G \in \mathcal{X}^{-s}(p, q) \) with \( p \geq q \geq s + 1 \geq 6 \), if \( \Delta(G') = s - 1 \), then \( G \) is \( \chi \)-unique.

**Proof.** Since \( s \geq 5 \), we have \( (s + 3)/2 \leq s - 1 \leq \min\{s, q - 2\} \). By Theorem 3.4, \( \mathcal{D}_{s-1}(p, q, s) \) is \( \chi \)-closed. It suffices to prove that for any \( G_1, G_2 \in \mathcal{D}_{s-1}(p, q, s) \), if \( G_1 \neq G_2 \), then either \( \chi'(G_1, 3) \neq \chi'(G_2, 3) \) or \( \chi'(G_1, 4) \neq \chi'(G_2, 4) \).

There are only two bipartite graphs with size \( s \) and maximum degree \( s - 1 \), and they are shown in Fig. 1. Thus, there are four graphs in the set \( \mathcal{D}_{s-1}(p, q, s) \), which are named as \( T_1, T_2, T_3 \) and \( T_4 \), displayed in Table 1.
Table 1

| name of graph | graphs $T_i'$  
|----------------|---------------------------------------------|
| $(T_i' = K_{p,q} - T_i)$  
| $|A| = p, |B| = q$ | $\alpha'(T_i, 3)$  
| $\alpha'(T_i, 4)$ |
| $T_1$  
| $x_1 x_2 \ldots x_{s-1}$ | $A$  
| $B$ | $2^{s-1} + 1$  
| $\sum_{i=1}^{s-1} \binom{s-1}{i} (2^{p-i-1} + 2^{q-i-2} - 2)$  
| $+(2^{p-2} + 2^{q-2} - 2)$  
| $+(2^{p-3} + 2^{q-3} - 2)$  
| $+2^{q-2} - 1$ |
| $T_2$  
| $\ldots$ | $A$  
| $B$ | $2^{s-1} + 1$  
| $\sum_{i=1}^{s-1} \binom{s-1}{i} (2^{p-i-1} + 2^{q-i-2} - 2)$  
| $+(2^{p-2} + 2^{q-2} - 2)$  
| $+(2^{p-3} + 2^{q-3} - 2)$  
| $+2^{q-2} - 1$ |
| $T_3$  
| $\ldots$ | $A$  
| $B$ | $2^{s-1}$  
| $\sum_{i=1}^{s-1} \binom{s-1}{i} (2^{p-i-1} + 2^{q-i-2} - 2)$  
| $+(2^{p-2} + 2^{q-2} - 2)$  
| $+2^{q-1} - 1$ |
| $T_4$  
| $\ldots$ | $A$  
| $B$ | $2^{s-1}$  
| $\sum_{i=1}^{s-1} \binom{s-1}{i} (2^{p-i-1} + 2^{q-i-2} - 2)$  
| $+(2^{p-2} + 2^{q-2} - 2)$  
| $+2^{q-1} - 1$ |

For each $T_i$, we can find $\alpha'(T_i, 3)$ and $\alpha'(T_i, 4)$ by Lemmas 2.2 and 4.1, respectively. These values are also displayed in Table 1.

Observe that for any $i = 1, 2$ and $j = 3, 4$, $\alpha'(T_i, 3) > \alpha'(T_j, 3)$. If $p = q$, then $T_1 \cong T_2$ and $T_3 \cong T_4$. If $p > q$, then

$$\alpha'(T_1, 4) - \alpha'(T_2, 4) = 2^{p-3} - 2^{q-3} + \sum_{i=1}^{s-1} \binom{s-1}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1})$$

$$= \sum_{i=3}^{s-1} \binom{s-1}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1})$$

$$+ \left(1 - \binom{s-1}{2}\right) (2^{p-3} - 2^{q-3})$$

$$< 0$$  \hspace{1cm} (2)$$

and

$$\alpha'(T_3, 4) - \alpha'(T_4, 4) = \sum_{i=1}^{s-1} \binom{s-1}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1}) < 0.$$  

This completes the proof of the result. \qed
Lemma 4.3. For any $G \in \mathcal{D}_{s-2}(p,q,s)$, where $s \geq 4$, $G'$ is one of the graphs in Fig. 2.

Theorem 4.2. For any $G \in \mathcal{K}_{s-2}(p,q)$ with $p \geq q \geq s + 1 \geq 8$, if $\Delta(G') = s - 2$, then $G$ is $\chi$-unique.

Proof. Since $s \geq 7$, $(s + 3)/2 \leq s - 2$. By Theorem 3.4, $\mathcal{D}_{s-2}(p,q,s)$ is $\chi$-closed.

By Lemma 4.3, if $G \in \mathcal{D}_{s-2}(p,q,s)$, then $G'$ is one of the graphs in Fig. 2. Thus $\mathcal{D}_{s-2}(p,q,s)$ contains 16 graphs, which are named as $W_1, W_2, \ldots, W_{16}$. (See Table 2, parts 1 and 2.) Let

$$\mathcal{S}_1 = \{W_1, W_2, W_3, W_4\},$$

$$\mathcal{S}_2 = \{W_5, W_6, W_7, W_8\},$$

$$\mathcal{S}_3 = \{W_9, W_{10}, W_{11}, W_{12}, W_{13}, W_{14}\},$$

$$\mathcal{S}_4 = \{W_{15}, W_{16}\}.$$

Observe that for any $i, j$ with $1 \leq i < j \leq 4$, $\chi'(W_i, 3) > \chi'(W_j, 3)$ if $W_i \in \mathcal{S}_i$ and $W_j \in \mathcal{S}_j$. Thus each $\mathcal{S}_i$ is $\chi$-closed. Hence, for each $i$, to show that all graphs in $\mathcal{S}_i$ are $\chi$-unique, it suffices to show that for any two graphs $W_{i_1}, W_{i_2} \in \mathcal{S}_i$, if $W_{i_1} \neq W_{i_2}$, then either $\chi'(W_{i_1}, 4) \neq \chi'(W_{i_2}, 4)$ or $\chi(W_{i_1}, 5) \neq \chi(W_{i_2}, 5)$.

The values of $\chi'(W_i, 4)$ can be obtained by Lemma 4.1. We shall establish several inequalities of the form $\chi'(W_i, 4) < \chi'(W_j, 4)$ for some $i, j$. As an example, we use a method similar to the one for (2) and the fact that $8 \leq s + 1 \leq q$ to show that
Table 2

<table>
<thead>
<tr>
<th>name of graph</th>
<th>graphs $W_i'$</th>
<th>$\alpha'(W_i, 3)$</th>
<th>$\alpha'(W_i, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>$W_i' = K_{p,q} - W_i$ ($</td>
<td>A</td>
<td>= p$, $</td>
</tr>
<tr>
<td>$W_2$</td>
<td>$2^s-2 + 5$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q + 2^{p-3} + 2^{q-2}$ $+ 5 \cdot 2^{s-4} - 15$</td>
<td></td>
</tr>
<tr>
<td>$W_3$</td>
<td>$2^s-2 + 5$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q + 2^{p-3} - 2^{q-4}$ $+ 3 \cdot 2^{s-4} - 15$</td>
<td></td>
</tr>
<tr>
<td>$W_4$</td>
<td>$2^s-2 + 5$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q - 2^{p-4} + 2^{q-1}$ $+ 3 \cdot 2^{s-3} - 15$</td>
<td></td>
</tr>
<tr>
<td>$W_5$</td>
<td>$2^s-2 + 3$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q - 2^{p-3} - 2^{q-3}$ $+ 2^{s-1} - 11$</td>
<td></td>
</tr>
<tr>
<td>$W_6$</td>
<td>$2^s-2 + 3$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q - 2^{p-3} - 2^{q-3}$ $+ 2^{s-1} - 11$</td>
<td></td>
</tr>
<tr>
<td>$W_7$</td>
<td>$2^s-2 + 3$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q - 2^{p-2}$ $+ 2^{s-2} - 7$</td>
<td></td>
</tr>
<tr>
<td>$W_8$</td>
<td>$2^s-2 + 3$</td>
<td>$\sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} + 2^{q-2} - 2)$ $+ 2^p + 2^q - 2^{p-2}$ $+ 2^{s-2} - 7$</td>
<td></td>
</tr>
</tbody>
</table>
Table 2 (continued)

| name of graph | graphs $W'_i$ $(W'_i = K_{p,q} - W_i)$ $(|A| = p, |B| = q)$ | $\alpha'(W_i, 3)$ | $\alpha'(W_i, 4)$ |
|---------------|---------------------------------------------------------|------------------|------------------|
| $W_9$ | \[
\begin{array}{c}
\bullet \ \bullet \\
\bullet \ \bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 2$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[5(2^{p-3} + 2^{q-3}) + 2^{p-3} + 3 \cdot 2^{q-2} - 9\] |
| $W_{10}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 2$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[5(2^{p-3} + 2^{q-3}) + 2^{p-3} + 3 \cdot 2^{q-2} - 9\] |
| $W_{11}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 2$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[5(2^{p-3} + 2^{q-3}) + 2^{p-3} + 3 \cdot 2^{q-2} - 9\] |
| $W_{12}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 2$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[5(2^{p-3} + 2^{q-3}) + 2^{p-3} + 3 \cdot 2^{q-2} - 9\] |
| $W_{13}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 2$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[5(2^{p-3} + 2^{q-3}) + 2^{p-3} + 3 \cdot 2^{q-2} - 9\] |
| $W_{14}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 2$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[5(2^{p-3} + 2^{q-3}) + 2^{p-3} + 3 \cdot 2^{q-2} - 9\] |
| $W_{15}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 1$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[2(2^{p-2} + 2^{q-2} - 2) + 2(2^{s-2} - 1) + 1\] |
| $W_{16}$ | \[
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\] $A$ $B$ | $2^{s-2} + 1$ | $\sum_{i=1}^{s-2} \left( \frac{s-2}{i} \right) (2^{p-1} + 2^{q-2} - 2)$ + \[2(2^{p-2} + 2^{q-2} - 2) + 2(2^{s-2} - 1) + 1\] |
\(\varepsilon'(W_{10}, 4) < \varepsilon'(W_{14}, 4)\) when \(p > q\).

\[
\varepsilon'(W_{10}, 4) - \varepsilon'(W_{14}, 4) = \\
= \sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} - 2^{q-i-1} + 2^{i-2} - 2^{p-2}) \\
+ 2^{p-3} - 2^{q-3} + 3 \cdot 2^{s-2} - 2^{s-3} - 3 \\
= \sum_{i=1}^{s-2} \binom{s-2}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1}) \\
+ 2^{p-3} - 2^{q-3} + 3 \cdot 2^{s-3} - 3 \\
< -\left(\frac{s-2}{2}\right) (2^{p-3} - 2^{q-3}) + 2^{p-3} - 2^{q-3} + 3 \cdot 2^{s-3} \\
< -3 \cdot (2^{p-3} - 2^{q-3}) + 3 \cdot 2^{s-3} \\
< 0.
\]

(1) \(\mathcal{S}_1\).

(1.1) When \(p = q\), \(W_1 \cong W_2\), \(W_3 \cong W_4\), and

\(\varepsilon'(W_2, 4) < \varepsilon'(W_3, 4)\).

(1.2) When \(p > q\),

\(\varepsilon'(W_1, 4) < \varepsilon'(W_3, 4) < \varepsilon'(W_4, 4) < \varepsilon'(W_2, 4)\).

(2) \(\mathcal{S}_2\)

(2.1) When \(p = q\), \(W_5 \cong W_6\), \(W_7 \cong W_8\), and

\(\varepsilon'(W_7, 4) < \varepsilon'(W_6, 4)\).

(2.2) When \(p > q\),

\(\varepsilon'(W_5, 4) < \varepsilon'(W_7, 4) < \varepsilon'(W_8, 4) < \varepsilon'(W_6, 4)\).

(3) \(\mathcal{S}_3\)

(3.1) When \(p = q\), \(W_9 \cong W_{12}, W_{10} \cong W_{11}, W_{13} \cong W_{14}\),

\(\varepsilon'(W_{11}, 4) = \varepsilon'(W_{12}, 4) > \varepsilon'(W_{13}, 4)\),

and by Lemma 4.2,

\[
\varepsilon(W_{11}, 5) - \varepsilon(W_{12}, 5) = \\
= \varepsilon(W_{11} + a_1b_1, 5) + \varepsilon(W_{11} - \{a_1, b_1\}, 4) + \varepsilon(W_{11} - \{a_1, b_1, c_1\}, 4) \\
- (\varepsilon(W_{12} + a_2b_2, 5) + \varepsilon(W_{12} - \{a_2, b_2\}, 4) + \varepsilon(W_{12} - \{a_2, b_2, c_2\}, 4)) \\
= \varepsilon(W_{11} - \{a_1, b_1, c_1\}, 4) - \varepsilon(W_{12} - \{a_2, b_2, c_2\}, 4)
\]
\[
= \chi'(W_{11} - \{a_1, b_1, c_1\}, 4) - \chi'(W_{12} - \{a_2, b_2, c_2\}, 4)
\]
\[
= \sum_{i=1}^{s-2} \binom{s-2}{i} (2^p - 4 + 2^{q-2-i} - 2) - \sum_{i=1}^{s-2} \binom{s-2}{i} (2^p - 3 + 2^{q-3-i} - 2)
\]
\[
= \sum_{i=1}^{s-2} \binom{s-2}{i} (2^{q-3-i} - 2^p - 4)
\]
\[
< 0,
\]
since \(W_{11} + a_1 b_1 \cong W_{12} + a_2 b_2\) and \(W_{11} - \{a_1, b_1\} \cong W_{12} - \{a_2, b_2\}\).

(3.2) When \(p > q\),
\[
\chi'(W_9, 4) < \chi'(W_{10}, 4) < \chi'(W_{14}, 4) < \chi'(W_{11}, 4) < \chi'(W_{12}, 4)
\]
\[
\chi'(W_{13}, 4) < \chi'(W_{10}, 4),
\]
and
\[
\chi'(W_{13}, 4) - \chi'(W_9, 4) = 2^{p-3} - 2^{q-3} - 2^s - 1 + 2^{s-3} + 3
\]
\[
\begin{cases} 
< 0, & \text{if } p = q + 1, q = s + 1 \\
> 0, & \text{if } p > q + 2 \text{ or } p = q + 1 > s + 3.
\end{cases}
\]

(4) \(\mathcal{S}_4\).

(4.1) When \(p = q\), \(W_{15} \cong W_{16}\).

(4.2) When \(p > q\), \(\chi'(W_{15}, 4) < \chi'(W_{16}, 4)\).

This completes the proof. \(\square\)

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References

