Personal list of open problems related to chromatic polynomials

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For any graph G, let $P(G,x)$ denote the *chromatic polynomial* of G, i.e., the polynomial which counts the number of proper k-colorings whenever $x = k$ is a positive integer.

Problem 1: For any simple graph G of order n, does $P(G,x)^2 \geq P(G,x-1)P(G,x+1)$ hold for all real $x \geq n$?

Background: In the early 1970's, Welsh conjectured that $P(G,k)^2 \geq P(G,k-1)P(G,k+1)$ holds for any graph G and any positive integer k.

In 1992, Brenti [2][1] proposed this conjecture again.

In 1997, Seymour [9][2] disproved this conjecture for $k = 6$.

In 2005, Dong, Koh and Teo [4][3] showed that for any connected (n,m)-graph G, if x is real and $x \geq \max\{n-1, \sqrt{2}(m-n+2.5)\}$, then $P(G,x)^2 \geq P(G,x-1)P(G,x+1)$ holds.

Problem 2: For any planar and loopless graph G, does $P(G,x) > 0$ hold for all real $x \in (4,5)$?

Background: In 1946, Birkhoff and Lewis [1][4] conjectured that for any planar and loopless graph G, $P(G,x) > 0$ holds for all real numbers $x \in [4,5)$.

[1]F. Brenti, Expansion of chromatic polynomials and log-concavity, *Trans. Amer. Math. Soc.* **332** (1992), 729–755.

[2]P. Seymour, two chromatic polynomial conjectures, *J. Combin. Theory Ser. B* **70** (1997), 184–196.

[3]Fengming Dong, K.M. Koh and K.L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, Singapore, 2005.

[4]G.D. Birkhoff and D.C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355–451.

So far, the four-color theorem (i.e., for $x = 4$) is the only result on Birkhoff and Lewis's conjecture.

Problem 3: Is there a combinatorial proof for the unimodal conjecture on the coefficients of chromatic polynomials?

Background: In 1968, Read [8][5] conjectured that for a graph G of order n, if $P(G, x) = \sum_{i=1}^{n} (-1)^{n-i} h_i x^i$, then there always exists an integer k with $2 \leq k \leq n-1$ such that

$$h_1 \leq h_2 \leq \cdots \leq h_{k-1} \leq h_k \geq h_{k+1} \geq \cdots \geq h_n.$$

This is the **Unimodal Conjecture** on the coefficients of chromatic polynomials.

In 2012, Huh [7][6] proved the Unimodal Conjecture by an approach of algebraic geometry.

An elementary proof of the Unimodal Conjecture is expected.

Problem 4: Are there a complex number z and a graph G such that $P(G, z) = 0$, $Re(z) = 0$ and $Im(z) \neq 0$?

Background: This problem was proposed by the author when he attended the program on "Combinatorics and Statistical Mechanics" at the Newton Institute of Cambridge in 2008. No result on the problem was found.

Problem 5: For any graph G with maximum degree Δ, does $P(G, x) > 0$ holds for all real $x > \Delta$?

Background: In 2001, Sokal [10][7] showed that there is a constant C with $C \leq 7.963907$ such that $|z - 1| \leq C\Delta$ holds for all chromatic roots z of G.

In 2005, Sokal [11][8] conjectured that for any G with maximum degree Δ, $P(G, x) > 0$ holds for all real $x > \Delta$.

[5] R.C. Read, An introduction to chromatic polynomials, *J. Combin. Theory Ser. B* **4** (1968), 52–71.

[6] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, *J. Amer. Math. Soc.* **25** (3) (2012), 907–927.

[7] A. D. Sokal, Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions, *Combin. Probab. Comput.* **10** (1) (2001), 41–77.

[8] A. D. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In Surveys in Combinatorics 2005, volume 327 of London Math. Soc. Lecture Note Ser., pages 173–226. Cambridge Univ. Press, Cambridge, 2005.

In 2008, Dong and Koh [6][9] showed that $P(G,x) > 0$ holds for all real $x > 5.664\Delta$.

The problem is even open for the simplest case $\Delta = 3$.

Problem 6: Does $\tau(G,x)$ have real roots only for each simple graph G?

Background: This problem was first proposed by Brenti [2][10] in 1992.

For any simple graph G of order n, $\tau(G,x)$ is defined as

$$\tau(G,x) = \sum_{i=1}^{n} c_i x^i,$$

where c_i's are determined by the following expression:

$$P(G,x) = \sum_{i=1}^{n} (-1)^{p-i} c_i \langle x \rangle_i,$$

where $\langle x \rangle_i = x(x+1)\cdots(x+i-1)$.

For any integer $n > 0$, let

$$B_n(x) = \sum_{1\le i\le n} S(n,i)x^i,$$

where $S(n,i)$ is a *Stirling number of the second kind*, counting the number of partitions of $\{1,2,\cdots,p\}$ into i non-empty subsets. $B_n(x)$ is called a *Bell polynomial*.

Proposition 1 *Let N_n denote the null graph of order n. Then $\tau(N_n,x) = B_n(x)$.*

Proposition 2 *For any graph G with $e \in E(G)$,*

$$\tau(G,x) = \tau(G\backslash e,x) + \tau(G/e,x).$$

Proposition 3 *For any simple graph G of order n, if $P(G,x) = \sum_{k=1}^{n}(-1)^{n-k}a_k x^k$, then*

$$\tau(G,x) = \sum_{k=1}^{n} a_k B_k(x).$$

[9]Fengming Dong and K.M. Koh, Bounds for the real zeros of chromatic polynomials, *Combin. Probab. Comput.* **17** (2008), 749–759.

[10]F. Brenti, Expansion of chromatic polynomials and log-concavity, *Trans. Amer. Math. Soc.* **332** (1992), 729–755.

A graph G is said to be τ-*real* if $\tau(G, x)$ has real roots only.

In 1992, Brenti [2] asked if each simple graph is τ-real. No counter-example was found yet.

More details on τ-polynomials are provided in [2, 3, 5].

References

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[5] Fengming Dong, Polynomials related to chromatic polynomials, https://arxiv.org/abs/2007.05217, 2020.

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