Poset models of topological spaces

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Abstract. We consider poset models of topological spaces and show that every T_1-space has an bounded complete algebraic poset model, thus give a positive answer to a question asked in a recent paper by Waszkiewicz. It is also proved that every T_1-space is homeomorphic to the maximal point space of a d-space.

In the classic general topology, people are mainly interested in the spaces which satisfy at least T_1 separation axiom. One possible reason for this phenomena is that in the early time, people considered only those spaces which are subspaces of Euclidean n-spaces. Another reason is that there is no meaningful natural examples of non T_1 spaces, most of the existing non T_1 spaces are artificially constructed. It is, probably, only until the appear of domain theory which has a deep root in computer science, people begin to be interested in T_0 spaces. The most important examples of T_0 spaces are the Scott spaces, defined originally for complete lattices by Dana Scott. For every complete lattice L, Scott introduced a topology, $\sigma(L)$ on L, which is always T_0 but not T_1. Later this topology was defined for directed complete posets (dcpos), and more recently, for arbitrary posets. The Scott topology on a poset is not T_1 unless the poset has the discrete order.

On the first look, it seems that Scott spaces are too weak in separation to be interested by classical topologists. However, to certain extend, two results on Scott spaces proved in the past decades have change one's views on the theoretical importance of such spaces. The first one was proved by Dana Scott in [**10**], which characterizes the injective T_0 spaces–these as exactly the continuous lattices with their Scott topology. The second significant result was proved in [**1**]: every complete metric space is homeomorphic to

2000 *Mathematics Subject Classification.* 54E30, 06A06.

Key words and phrases. Bounded complete poset; algebraic poset; poset models of spaces .

the subspace of maximal points of a continuous dcpo equipped with the relative Scott topology.

The second result reveals that a large class of traditional spaces can be represented as subspaces of Scott spaces in a "normal " way. In [**6**], Martin proved that if a space is homeomorphic to the maximal point space of a continuous dcpo, then the space must be Choquet complete (every Choquet complete space is Baire). Thus not every space can be represented as the maximal point space of some continuous dcpo. It is therefore natural to consider the maximal point space of more general posets.

In this paper we consider the more general question: which spaces are homeomorphic to the maximal point space of (certain class of) posets ? The main result we shall prove is that every T_1 space is homeomorphic to the maximal point space of a bounded complete algebraic poset. This gives a positive answer to a question raised in [**11**]. This result will allows us to define a functor from the category of T_1 spaces to the category of bounded algebraic posets and Scott continuous functions.

1. Bounded complete algebraic posets

A nonempty subset D of a poset P is directed if for any $x, y \in D$ there is $z \in D$ such that $x \leq z, y \leq z$. A directed complete poset, called dcpo for short, is a poset whose every directed subset has a supremum.

For two elements x, y in P, x is way-below y, denoted by $x << y$, if for any directed set $D \subseteq P$, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$ whenever $\bigvee D$ exists in P.

A base B of a poset P is a subset of P such that for any $x \in P, \Downarrow x \cap B = \{y \in B : y \ll x\}$ is a directed set and its supremum is x.

A poset P is called continuous if it has a base.

If $x << x$, x is called a compact element. The set of all compact elements of P is denoted by $K(P)$. A poset P is called an algebraic poset if $K(P)$ is a basis of P.

A poset P is bounded complete if for any $D \subseteq P$ with an upper bound in P, $sup\, D$ exists in P. This is equivalent to that every nonempty subset of P has an infimum.

EXAMPLE 1. *(a) Let X be an infinite set and $\mathcal{P}_0(X)$ be the set of all finite subsets of X. Then $(\mathcal{P}_0(X), \subseteq)$ is a bounded complete algebraic poset which is not a dcpo.*

(b) Let X be a topological space and $C_0(X)$ be the set of all nonempty closed subsets of X. Then $(C_0(X), \leq)$ is a bounded complete poset, where for any $A, B \in C_0(X), A \leq B$ if and only if $B \subseteq A$.

(c) The set $P = [0, \infty)$ of all nonnegative real numbers is a bounded complete continuous poset under the ordinary order of numbers.

If P is a poset, $x, y \in K(P)$ and $x \vee y$ exists in P, then $x \vee y \in K(P)$. This property was proved for dcpos in [**2**] (See Remark I-4.4. of [**2**]), but it is easy to show that the conclusion is also valid for any poset.

A subset F of a poset P is a filter if $F = \uparrow F = \{y \in P : y \geq x \text{ for some } x \in F\}$ and for any $x, y \in F$ there is $z \in F$ with $z \leq x, y$.

A nonempty subset S of a complete lattice L is called an m-set of L if $0_L \notin S$ and for any $x, y \in S, x \wedge y \neq 0_L$ implies $x \wedge y \in S$.

Lemma 1. *For any m-set S of a complete lattice L, the set $Filt^l(S)$ of all filters F of S satisfying $\bigwedge F \neq 0_L$ is a bounded complete algebraic poset with respect to the inclusion order.*

Proof. (i) If $\mathcal{F} = \{F_i : i \in I\} \subseteq Filt^l(S)$ has an upper bound, say F, then $F \supseteq \bigcup \mathcal{F}$. The filter G of S generated by $\bigcup \mathcal{F}$ is still contained in F, thus $\bigwedge G \geq \bigwedge F$ and so $\bigwedge G \neq 0_L$. Obviously G is the supremum of $\mathcal{F}$ in $Filt^L(S)$. Thus $Filt^l(S)$ is bounded complete.

(ii) If $\mathcal{F} = \{F_i : i \in I\}$ is a directed subset of $Filt^l(S)$ and $sup\mathcal{F}$ exists in $Filt^l(S)$, then it follows easily that $sup\mathcal{F} = \bigcup \mathcal{F}$. For each $x \in S, \uparrow x \in Filt^l(S)$ and since the existing supremum of a directed subset of $Filt^l(S)$ is the union of the subsets, we see that $F \in Filt^l(S)$ is a compact element of $Filt^l(S)$ iff $F = \uparrow x$ for some $x \in S$.

Then for each $F \in Filt^l(S)$, the set of compact elements of $Filt^l(S)$ below F equals $\{\uparrow x : x \in F\}$ which is obviously directed because F is a filter. Also $F = sup\{\uparrow x : x \in F\}$. Hence $Filt^l(S)$ is algebraic. □

Example 2. *(1) Let $Int(R)$ be the set of all non empty closed intervals of R including R. Then $Int(R)$ is a subset of $\mathcal{P}(R)$ and for any $I, J \in Int(R)$, $I \cap J \neq \emptyset$ implies $I \cap J \in Int(R)$. Thus $Int(R)$ is an m-set of $\mathcal{P}(R)$, so $Filt^l(Int(R))$ is a bounded complete algebraic poset. In addition, due to the compactness of closed intervals, every filter of $Int(R)$ has a nonempty intersection, thus $Filt^l(Int(R)) = Filt(Int(R))$, and $Filt(Int(R))$ is a directed complete poset.*

In general, if X is a Hausdorff space and S is the set of all nonempty compact sets of X. Then $Filt^l(S) = Filt(S)$ is a bounded complete algebraic dcpo.

(2) Let X be a topological space and $\mathcal{O}^*(X) = \mathcal{O}(X) - \{\emptyset\}$ be the set of all nonempty open sets of X. Then $(Filt^l(\mathcal{O}^*(X)), \subseteq)$ is a bounded complete algebraic poset. Note that $Filt^l(\mathcal{O}^*(X))$ need not be directed complete.

If X is T_0, then a filter $\mathcal{F}$ is an maximal element of $Filt^l(\mathcal{O}^*(X))$ if and only if the intersection $\bigcap \mathcal{F}$ is a singleton.

It is well known that if P is an algebraic dcpo, then P is isomorphic to the poset $Idl(K(P))$ of all ideals of $K(P)$. Thus if P and Q are two algebraic dcpos and $K(P)$ is isomorphic to $K(Q)$ as posets, then P and Q

are isomorphic. This fact is not true for arbitrary algebraic posets. For instance, let $P = \{1 - \frac{1}{n} : n \in N\}$ and $Q = P \cup \{1\}$. Then both P and Q are algebraic posets with respect to the ordinary order of numbers. Also $K(P) = K(Q) = P$. But P and Q are obviously not isomorphic.

For a poset P, we use $max(P)$ to denote the set of all maximal elements of P.

PROPOSITION 1. *A bounded complete algebraic poset P is isomorphic to $Filt^l(\mathcal{O}^*(X))$ for some topological space X iff the following conditions are satisfied:*

(i) for any nonempty subset $A \subseteq K(P)$, it holds that $inf_P A \in K(P)$;

(ii) for any $x \in max(P)$ and nonempty $A \subseteq K(P)$, $inf_P A \leq x$ implies $a \leq x$ for some $a \in A$;

(iii) every element of $K(P)$ is the meet of some elements in $max(P)$.

(iv) $\uparrow a \cap max(P) \neq \emptyset$ for all $a \in P$.

PROOF. If $P = Filt^l(\mathcal{O}^*(X))$, then it follows easily that $F \in P$ is a maximal element iff there is a point $x \in X$ such that $F = N(x)$ – the filter of open neighbourhoods of x. And $F \in K(P)$ iff $F = \uparrow U$ for some open set U in X. Then it is straightforward to verify that all the four conditions (i)-(iv) are satisfied.

Conversely, suppose that the four conditions are satisfied. Put $X = max(P)$ and $\tau = \{X \cap \uparrow a : a \in K(P)\} \cup \{\emptyset\}$.

Then $X = X \cap \uparrow u \in \tau$, where $u = inf_P K(P)$. Let $\{X \cap \uparrow x_i : i \in I\}$ be any subset of τ. Put $x = \bigwedge_{i \in I} x_i$. Then $x \in K(P)$ by (i), and $\bigcup\{X \cap x_i : i \in I\} = X \cap \uparrow x$ by (ii). So τ is closed under arbitrary unions. For any $a, b \in K(P)$, if $\uparrow a \cap \uparrow b \neq \emptyset$, then $a \vee b$ exists in P, so $a \vee b \in K(P)$. Thus $\uparrow a \cap \uparrow b = \uparrow (a \vee b)$, it follows that τ is closed under finite intersection, and hence is a topology on X.

Now we define the mapping $F : P \longrightarrow Filt^l(\mathcal{O} * (X))$ by $F(u) = \{X \cap \uparrow x : x \leq u, x \in K(P)\}$. For each $u \in P$, $\{x \in K(P) : x \leq u\}$ is a directed set, so $F(u)$ is a filter base. Now suppose that $X \cap \uparrow v \in \tau$ and $X \cap \uparrow v \supseteq X \cap \uparrow x$ for some $x \in K(u)$ and $x \leq u$, then by condition (iii), $v = sup(X \cap \uparrow v) \leq sup(X \cap \uparrow x) = x \leq u$, thus $X \cap \uparrow v \in F(u)$. Hence $F(u)$ is a filter. By (iv), there is $a \in max(P)$ with $a \geq u$, then a belongs to the intersection of members of $F(u)$. Thus $F(U) \in Filt^l(\mathcal{O}^*(X))$. Since P is algebraic, F is an order embedding by condition (iii). Now it remains to show that F is surjective.

Let $\mathcal{A} = \{X \cap \uparrow x_i : x_i \in K(P), i \in I\}$ be a filter of nonempty open sets of X such $\bigcap\{X \cap \uparrow x_i : i \in I\}$ is nonempty. Then by condition (iii), $\{x_i : i \in I\}$ is a directed subset of $K(P)$ which is up bounded. Let $u = sup\{x_i : i \in I\}$. Then $F(u) = \mathcal{A}$. The proof is completed. $\square$

In general, the poset $Filt^l(\mathcal{O}^*(X))$ is not a dcpo. As a matter of fact, one can easily show that for a T_1 space X, $Filt^l(\mathcal{O}^*(X))$ is a dcpo iff every filter of nonempty open sets has nonempty intersection.

2. Models of topological spaces

A subset U of a poset P is Scott open if U is an upper set($U = \uparrow U = \{x \in P : y \leq x \text{ for some } y \in U\}$) and for any directed subset D of P, $supD \in U$ implies $D \cap U \neq \emptyset$ whenever $supD$ exists. All Scott open sets form a topology on P, denoted by $\sigma(P)$, called the Scott topology on P. If P is continuous, then the sets $W(x) = \{y \in P : x << y\}, x \in P$, form a base of $\sigma(P)$. If P is algebraic, then the sets $\uparrow a, a \in K(P)$, form a base of $\sigma(P)$. See [**2**] for more about Scott topology.

For a poset P, if the set $max(P)$ of maximal points is equipped with the relative Scott topology on P (that is $V \subseteq max(P)$ is open in $max(P)$ iff $V = U \cap max(P)$ for some Scott open set U of P), then $max(P)$ is called the maximal point space of P. Following [**7**], a poset model of a topological space X is a poset P together with a homeomorphism $\phi : X \longrightarrow max(P)$. We shall use (P, ϕ) to denote a poset model of X.

Note that for any poset P and any $x \in P$, $\downarrow x$ is the closure of $\{x\}$ with respect to the Scott topology on P. Thus if x and y are two different maximal points of P, then $x \notin \downarrow y = cl\{y\}$ and $y \notin cl\{x\}$. Hence $max(P)$ is always a T_1 space, it follows that only T_1- spaces may have a poset model.

If P is a continuous poset (domain, algebraic poset), then the model (P, ϕ) of X is called a continuous (domain, algebraic) model of X.

A poset model (P, ϕ) is said to satisfy the Lawson condition if for any $x \in P$, $\uparrow x \cap max(P)$ is closed in $max(P)$ (for the relative Scott topology).

EXAMPLE 3. *(1) Every Hausdorff locally compact space has a domain model. If X is locally compact, take $P = K(X)$ to be the set of all nonempty compact subsets of X. With the reverse inclusion order, P is a domain and X is isomorphic to $Max(P)$(see Example V-6.3 of* [**2**]*).*

(2) The set $\mathbf{P}I = \{[a, b] : a \leq b, a, b \in R\}$ with the inverse inclusion order is a domain model of the real line R. This model satisfies the Lawson condition.

(3) For a complete metric space (X, d), let $\mathbf{B}X = X \times [0, +\infty)$ be equipped with the order defined by

$$(x, r) \leq (y, s) \text{ iff } d(x, y) \leq r - s.$$

Then $\mathbf{B}X$ is a domain and it is a model of X(see [**1**]*).*

(3) In [**4**]*, Lawson proved that a space has an ω-continuous dcpo (a continuous dcpo that has a countable base) model satisfying Lawson condition iff it is a Polish space.*

(4) Liang and Keimel proved in [**5**] *that a space has a continuous poset model satisfying the Lawson condition iff the space is Tychonoff.*

(5) In [**3**], *it is proved that a space has a bounded complete algebraic dcpo model with a countable base iff the space has a clopen countable base $\mathcal{B}$ closed under consistent finite intersections (if $U \subseteq U_1, \cdots, U_n$ with $U \in \mathcal{B}, U_i \in \mathcal{B}(i = 1, 2, \cdots, n)$ then $\bigcap_{i=1}^{n} U_i \in \mathcal{B}$) and every filtered subset of the base has a nonempty intersection. It is also noted that every bounded complete algebraic dcpo model satisfies the Lawson condition.*

In [**11**], Waszkiewizc asks if every T_1 space has a continuous poset model. The following is an answer to his problem.

Theorem 1. *Every T_1 space has a bounded complete algebraic poset model.*

Proof. Let X be a T_1 space. Take P to be the set of all filters of nonempty open sets of X that has a nonempty intersection. Then by Lemma 1, P is a bounded complete algebraic poset. The compact elements of P are of the form $L(U) = \{V \in \mathcal{O}(X) : U \subseteq V\}$, $U \in \mathcal{O}(X)$. And $Max(P) = \{N(x) : x \in X\}$, where $N(x) = \{U \in \mathcal{O}(X) : x \in U\}$ is the open neighbourhood filter of $x \in X$.

Define $\phi : X \longrightarrow P$ by

$$\phi(x) = N(x), x \in X.$$

Then f is a bijection. Note that as P is algebraic, the subsets of P of the form $\uparrow L(U)$ $(U \in \mathcal{O}(X))$ form a basis of the Scott topology on P. Now for any $U \in \mathcal{O}(X)$, $\phi^{-1}(\uparrow L(U)) = \{x : L(U) \subseteq \phi(x)\} = \{x : U \in \phi(x)\} = U$. So ϕ is continuous. For any open set U of X, $f(U) \cap max(P) = \{\phi(x) : x \in U\} = \uparrow L(U) \cap max(P)$, which is open in $max(P)$. Hence f is also an open mapping, therefore it is a homeomorphism.

□

If $\mathcal{B}$ is a base of the T_1 space X such that $U, V \in \mathcal{B}, U \cap V \neq \emptyset$ implies $U \cap V \in \mathcal{B}$, then we can also show that the set of all filters of $\mathcal{B}$ that have a nonempty intersection is a bounded complete algebraic model for X.

A given topological space my have two very different models. For instance, $\mathbf{P}I = \{[a, b] : a \leq b, a, b \in [0, 1]\}$ is a continuous dcpo model of $I = [0, 1]$, which is very different from the model for $I = [0, 1]$ constructed in the proof of Theorem 1.

A topological space is second countable if it has a countable base.

Theorem 2. *For a topological space X, the followings are equivalent:*
(a) X is a second countable T_1 space.
(b) X has a countably based continuous poset model.
(c) X has a countably based bounded complete algebraic model.

Proof. Obviously (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) Now suppose that X is T_1 and is second countable. Let $\mathcal{B}$ be a countable base of X with $\emptyset \notin \mathcal{B}$. Take P be the set of all filters of $\mathcal{B}$ that have a nonempty intersection. Then as in Theorem 1, we can verify that P is a bounded complete algebraic poset and $max(P)$ is a model for X. Since $K(P) = \{\uparrow L(U) : U \in \mathcal{B}\}$, where $L(U) = \{V \in \mathcal{B} : U \subseteq V\}$, is a countable set, P is countably based. So (c) is satisfied. $\square$

By [**7**], a poset P is called ideal algebraic if every element of P is either a compact or a maximal element.

Proposition 2. *If X is a first countable T_1 topological space, then X has an ideal algebraic poset model.*

Proof. For each $x \in X$ choose a countable neighbourhood base at x, $N(x) = \{U(x, n) : n \in N\}$ such that $U(x, n) \subseteq U(x, n + 1)$ for all n. Put $P = \{U(x, n) : x \in X, n \in N\} \bigcup \{\{x\} : x \in X\}$. Define the order on P by $\{x\} \leq U(y, m)$ iff $x \in U(y, m)$ and $U(x, n) < U(y, m)$ iff $U(y, m) \subseteq U(x, n)$ and $n < m$. Then

(i) each $U(x, n)$ is a compact element of P;

(ii) for any $x \in X, \{x\}$ is the supremum of the directed set $\{U(x, n) : n \in \omega\}$;

(iii) $max(P) = \{\{x\} : x \in X\}$.

Thus P is an ideal algebraic poset.

The mapping $\eta : X \longrightarrow max(P)$ is an homeomorphism from X to $max(P)$ with the relative Scott topology, where $\eta(x) = \{x\}, x \in X$.

$\square$

It is still not known whether every T_1 space have a dcpo(not necessary continuous) model. Recall that a T_0 space X is called a d-space(or monotone convergence space) if X is a dcpo with respect to the specialization order ($x \leq y$ iff $x \in cl\{y\}$) and every open set of X is Scott open with respect to the specialization order.

Proposition 3. *For any T_1 space X there is a monotone convergence space Y such that X is homeomorphic to the subspace $max(Y)$ of Y and Y is a meet semilattice with respect to the specialization order.*

Proof. Let $K(X)$ be the set of all nonempty closed compact subsets of X. With respect to the inverse inclusion, $K(X)$ is a dcpo and a meet semilattice. For each open set U of X, let $L(U) = \{A \in Y : A \subseteq U\}$. Then $\{L(U) : U \in \mathcal{O}(X)\}$ is a base for a topology τ on $K(X)$.

(i) If $A, B \in K(X)$ and $A \neq B$, then $A \not\subseteq B$ or $B \not\subseteq A$. Assume $A \not\subseteq B$, then there is $x \in A - B$. Now $L(U)$, where $U = X - \{x\}$ is an open set that contains B but not A. So $K(X)$ is T_0.

(ii) From the proof of (i) it also follows that $A \subseteq B$ iff for any $L(U)$ in the base, $B \in L(U)$ implies $A \in L(U)$. Thus the specialization order on $K(X)$ is the inverse inclusion order.

(iii) Clearly every open set $L(U)$ in the base of τ is a Scott open set of $(K(X), \supseteq)$, thus every member of τ is Scott open. Hence $(K(X), \tau)$ is a d-space.

Now $max(K(X)) = \{\{x\} : x \in X\}$ and the mapping $\eta : X \longrightarrow max(K(X))$, sending $x \in X$ to $\{x\}$ is clearly a homeomorphism. □

It is yet to know whether every T_1-space is homeomorphic to the maximal point space of a d-space that is continuous with respect to the specialization order.

In [**12**], a dcpo $E(P)$ is constructed for each poset P, called the dcpo-completion, which is the smallest dcpo extension of P in certain sense. It is natural to wonder how $max(P)$ and $max(E(P)$ are related. In particular, is every maximal point of P maximal when regarded as a point of $max(E(P))$. The following example shows that this is not always the case.

EXAMPLE 4. *Let* $P = \{a_n\}_{n=1}^{\infty} \cup \{a\} \cup \{b_n\}_{n=1}^{\infty}$ *with the order generated by* $a_n < a_{n+1} < a, a_n < b_n, \forall n$. *Then by the construction of* $D(P)$*, which is the smallest subset of* $\sigma^{op}(P)$ *containing all* $\{\downarrow x : x \in P\}$ *and closed under directed joins. The point* a *is in* $max(P)$. *Also* $\downarrow a$ *is below the join of* $\{\downarrow b_n : n = 1, 2, \cdots\}$ *in* $E(P)$*, so it is not a maximal point in* $D(P)$.

In [**5**], it is proved that a space has a continuous poset model satisfying Lawson condition if and only if it is Tychonoff. We now consider the following questions: which spaces have a (bounded complete) algebraic poset model satisfying Lawson condition?

THEOREM 3. *A* T_1 *space has a bounded complete algebraic poset model that satisfies Lawson condition iff* X *is zero-dimensional.*

PROOF. Sufficiency: Assume that X is zero-dimensional and let $\mathcal{S}$ be a base of X consisting of nonempty clopen sets. Then, by the remark after Theorem 1, $Filt^l(\mathcal{S})$ is a model of X. Now we only need to show that for each $\mathcal{F} \in Filt^l(\mathcal{S})$, $\uparrow\mathcal{F} \cap max\, Filt^l(\mathcal{S})$ is closed. Suppose that $x \in X$ with $\phi(x) \notin\uparrow\mathcal{F}$, then $\mathcal{F} \not\subseteq \phi(X)$, that is there is $V \in \mathcal{F}, x \notin V$. Then, as $\mathcal{S}$ is a base of X consisting of clopen sets, there is $U \in \mathcal{S}, x \in U, V \cap U = \emptyset$. Note that $\uparrow U = \{W \in \mathcal{S} : U \subseteq W\}$ is a compact element of $Filt^l(\mathcal{S})$, so $\uparrow(\uparrow U) \cap max(Filt^l(\mathcal{S})$ is open in $max(Filt^l(\mathcal{S}))$, $\phi(x) \in\uparrow(\uparrow U)$ and $\uparrow(\uparrow U) \subseteq\uparrow \mathcal{F}$. This shows that $\uparrow\mathcal{F} \cap max(Filt^l(S))$ is closed. Therefore $max(Filt^l(\mathcal{S}))$ is a bounded complete algebraic poset model of X, and it satisfies Lawson condition.

Next, assume that X has a bounded complete algebraic model (P, ϕ) which satisfies Lawson condition. Since P is algebraic, $\{\uparrow e : e \in K(P)\}$ is a base of the Scott topology $\sigma(P)$. Also as (P, ϕ) satisfies Lawson condition, each $\uparrow e (e \in K(P))$ is clopen. It follows that $\{\uparrow e \cap max(P), e \in K(P)\}$ is a base of $max(P)$ consisting of clopensets. Thus $max(P)$, and therefore X is zero-dimensional.

□

3. A functor from the category of T_1 spaces to a category of algebraic posets

In section 2, we defined a bounded complete algebraic poset for each T_1 space. In this section we show that this construction can be extended to a functor from the category TOP_1 of T_1-spaces to a category of bounded complete algebraic posets.

A mapping $f : P \longrightarrow Q$ between two posets is Scott continuous iff it is continuous with respect to the Scott topologies on P and Q, this is equivalent to that for any directed set $D \subseteq P$, $f(sup\, D) = sup\, f(D)$ whenever $supD$ exists in P. A local dcpo is a poset such that every subposet $\downarrow a$ $(a \in P)$ is a dcpo, iff every upper bounded directed subset has a supremum.

Lemma 2. *Let Q be a local dcpos and $f : P \longrightarrow Q$ be a monotone mapping from a continuous poset P to Q. Then there is a largest Scott continuous mapping $F : P \longrightarrow Q$ satisfying $F \leq f$ (that is, $F(x) \leq f(x)$ for all $x \in P$).*

Proof. For each $x \in P$, let $\Downarrow x = \{y \in P : y \ll x\}$. For any $x \in P$, define $F(x) = sup\, f(\Downarrow x)$. By the monotonicity of f and the continuity of P, the set $f(\Downarrow x)$ is directed and bounded by $f(x)$, so $sup\, f(\Downarrow x)$ exists in Q. For any directed set $D, \Downarrow sup\, D = \bigcup\{\Downarrow x : x \in D\}$ whenever $sup\, D$ exists, thus F is Scott continuous. Obviously, $F(x) \leq f(x)$ for all $x \in P$ and is the largest such mapping.

□

The following result is a slight generalization of Proposition 3.8 in [**9**].

Lemma 3. *Let P be a continuous poset and Q be a bounded complete continuous poset. If $P_1 \subseteq P$ with $\downarrow P_1 = P$ and $f : P_1 \longrightarrow Q$ is a continuous mapping with respect to the relative Scott topology on P_1, then f has a (Scott) continuous extension over P.*

Proof. For any $x \in P$, let $g(x) = inf\{f(y) : y \in P_1, x \leq y\}$. Then g is a well defined monotone mapping from P to Q because Q is bounded complete and for each $x \in P, \uparrow x \cap P_1 \neq \emptyset$. Also $g(x) = f(x)$ for all $x \in P_1$. Now for each $x \in P$, define $F(x) = \sup g(\Downarrow x), x \in P$. By Lemma 2, F is Scott continuous.

To show that F is an extension of f, let $a \in P_1$ and $z << f(a)$. There is w such that $z << w << f(a)$ because Q is continuous. First note that $g(a) = f(a)$. The set $V = \{y \in Q : w << y\}$ is open in Q, so $f^{-1}(V)$ is open in P_1 and $a \in f^{-1}(V)$. As P is continuous, there is $r \in P, r << a$ such that $\{u \in P : r << u\} \cap P_1 \subseteq f^{-1}(V)$. Choose $r', r << r' << a$, then Obviously $g(r') \geq w$, so $g(r') \geq z$. Thus $F(a) \geq g(r') \geq z$ for each $z << f(a)$, hence $F(a) \geq f(a)$. In addition, $F(a) \leq g(a) = f(a)$, therefore $F(a) = f(a)$. □

Note that for a subset P_1 of a poset P, the relative Scott topology on P_1 need not be the same as $\sigma(P_1)$.

THEOREM 4. *The assignment X to $Filt^l(\mathcal{O}^*(X))$ extends to a functor from the category TOP_1 of T_1-spaces to the category $BCALG$ of bounded complete algebraic posets and Scott continuous mappings.*

PROOF. For any continuous map $f : X \longrightarrow Y$ between T_1-spaces, the corresponding map sending $N(x)$ to $N(f(x))$, denoted by $\hat{f}$, can be viewed as a continuous mapping from $max(Filt^l(\mathcal{O}^*(X)))$ to $max(Filt^l(\mathcal{O}^*(Y)))$. Now note that every member of
$Filt^l(\mathcal{O}^*(X))$ is below a member of $max(Filt^l(\mathcal{O}^*(X)))$, applying Lemma 3, we obtain a continuous extension of $\hat{f}$, denoted by $H(f)$. Obviously, in this way we obtain a functor from TOP_1 to $BCALG$. □

The following result follows directly from Corollary 3.10 of [**9**].

PROPOSITION 4. *If $f : X \longrightarrow X$ is a continuous mapping on a T_1 space which has a fixed point, then $H(f) : Filt^l(\mathcal{O}^*(X)) \longrightarrow Filt^l(\mathcal{O}^*(X))$ has a smallest fixed point.*

References

[1] A. Edalat and R. Heckmann, *A computational model for metric spaces* , Theoretical computer science, **193** (1998), 53-73.
[2] G. Gierz, K. H. Hoffmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. S. Scott, Continuous Lattices and Domains, Cambridge University Press, 2003.
[3] T. Kamimura and A. Tang, *Total objects of domains* , Theoretical computer science, **34** (1984), 275-288.
[4] J. Lawson, *Spaces of maximal points*, Math. Structures comput. Sci., **7**(1997), 5: 543-555.
[5] J. Liang and K. Klause, *Order environment of topological spaces* , Acta Math. sinica, **20**(2004), 5: 943-948.
[6] K. Martin, Topological games in domain theory, Topology Appl. **129** (2003), 2: 177–186.

[7] K. Martin, *Ideal models of spaces* , Theoretical computer science, **305** (2003), 277-297.

[8] K. Martin, *The regular spaces with countably based models*, Theoret. Comput. Sci. **305** (2003), 1-3: 299–310.

[9] M. W. Mislove, *Local DCPOs, Local CPOs and Local completions*, Electronic Notes in Theoretical Computer Science **20**(1999).

[10] D. Scott, *Continuous lattices,* In F.W. Lawvere, editor, Toposes, Algebraic Geometry and Logic, Dalhousie University, Halifax, Nova Scotia, January 16-19, 1971, volume 274 of Lecture Notes in Mathematics, pg 97-136. Springer-Verlag, 1972.

[11] P. Waszkiewicz, *How do domains model topologies* , Electronic Notes in Theoretical Computer Science, **83** (2004),1-18.

[12] D. Zhao and T. Fan, *Dcpo-completion of posets* , (Preprint) 2007.

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