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NUCLEI ON z-FRAMES

BY

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Abstract. This paper seeks for a further development of the theory of z-frames. More deep relations between nuclei, quotients and congruences on z-frames are revealed.

z-frame, which generalizes frame, σ-frame, preframe and k-frame, was introduced by the author in [5]. Nucleus is one of the several key structures in frame (or locale) theory. There have been lots of discussions on this topic for frames. The main reason for the importance of nuclei is that they can be regarded as subobjects in the category **Loc** of all locales. In [5] we have studied basic properties of nuclei on the general structure – z-frames. In this paper, we make a further investigation into the properties of nuclei and their connections with other structures. For a frame there are several different, but equivalent, ways to describe a nucleus. But for z-frames those ways are no longer equivalent. We will reveal more connections between these different characterizations. It is well known that the poset $\boldsymbol{N(A)}$ of all nuclei on a frame A is also a frame. So a natural problem is: Is $\boldsymbol{N(A)}$ a z-frame for any z-frame A? We give an example to show that $\boldsymbol{N(A)}$ is not necessarily a σ-frame for a σ-frame A. This answers an open problem. The next part of this paper is to study the complete lattices $\mathbf{Quo}(A)$ of all quotients of A and $\boldsymbol{zCong}(A)$ of all z-congruences relations on A which are closely related to nuclei. Although $\mathbf{Quo}(A)$ is generally not a z-frame, but it is a pseudo-Heyting algebra. This suggestes that we may need to take $\mathbf{Quo}(\boldsymbol{A})$, the set of all z-congruence relations on $\boldsymbol{A}$, as a replacement of $\boldsymbol{N(A)}$.

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1. z-frames

In the following, by a semilattice we will always mean a meet semilattice which has a top element 1. A semilattice homomorphism is a mapping between semilattice which preserves finite meets and the top element. **Slat** denotes the category of all semilattice and homomorphisms.

For a subset D of a semilattice S we write $\downarrow D = \{x \in S | \exists y \in D, x \leq y\}$. If $D = \{a\}$ is a singleton, we just write $\downarrow D$ as $\downarrow a$.

A system of sets on the category **Slat** is a function $\boldsymbol{Z}$ which assigns to each semilattice S a collection $\boldsymbol{Z}(S)$ of subsets of S such that the following conditions are satisfied:

(z1) Every $D \in \boldsymbol{Z}(S)$ is down closed, i.e. $D = \{x \in S | \exists y \in D, x \leq y\}$. And $\downarrow a \in \boldsymbol{Z}(S)$ for all $a \in S$;

(z2) If $f : S \to T$ is a mapping between semilattice which preserves binary meets, then $\downarrow f(D) \in \boldsymbol{Z}(T)$ for all $D \in \boldsymbol{Z}(S)$;

(z3) $D, C \in \boldsymbol{Z}(S)$ implies $D \cap C \in \boldsymbol{Z}(S)$;

(z4) $\Psi \in \boldsymbol{Z}(\boldsymbol{Z}(S))$ implies $\cup\Psi \in \boldsymbol{Z}(S)$.

Elements of $\boldsymbol{Z}(S)$ are called the z-ideals of S. A subset C of S is called a z-set if $\downarrow C \in \boldsymbol{Z}(S)$.

The idea of systems of sets is borrowed from Bandelt and Ernè who first introduced the set systems on the category of all posets to define z-continuous posets. For our purpose the condition (z3) has to be attached. It is easy to check that the condition (z3) is equivalent to that $\boldsymbol{Z}(S)$ is a semilattice.

A semilattice S is said to be z-complete if every z-ideal of S has a join in S. This is equivalent to that every z-set has a join in S.

Definition 1.1. A z-frame is a z-complete semilattice A such that

$$a \wedge \bigvee D = \bigvee \{a \wedge x | x \in D\}$$

holds for every $a \in A$ and z-ideal D (equivalently, for every z-set D).

Notice that the set $\{a \wedge x | x \in D\}$ is a z-set because the mapping $a \wedge - : S \to S$ preserves binary meets.

Examples 1.2.

(1) Let $\mathbf{Z}$ be the system of set which assigns to a semilattice S the set $\boldsymbol{Z}(S)$ of all down closed subsets of S. Then S is a z-frame if and only if it is a frame [1].

(2) For each semilattice S, assign $\boldsymbol{Z}(S)$ $=\mathbf{Idl}(S)$, the set of all ideals of S. Then S is a z-frame if and only if it is a preframe in the sense of [2]. Preframe are also called meet-continuous semilattice.

(3) Given a regular cardinal k, define $\boldsymbol{Z}(S) = \{\downarrow X | X \subset S, |X| < k\}$ for each semilattice S. Then z-frames are just the k-frames [3]. When $k = \aleph_1$, a k-frame is also called σ-frame.

A z-frame homomorphism $f : A \to B$ is a semilattice homomorphism which preserves joins of z-sets, that is

$$f(\vee D) = \vee f(D)$$

holds for every z-set D of A.

A subset B of a z-complete semilattice A is called a z-closed set if

(i) B is down closed, and

(ii) $D \in \boldsymbol{Z}(A)$ and $D \subset B$ imply $\vee D \in B$.

We use $C_z(A)$ to denote the poset of all z-closed subsets of A. It was proved in [5] that for any z-frame A, $C_z(A)$ is a frame and that there is a natural z-frame homomorphism

$$i_A : A \to C_z(A)$$

which sends $x \in A$ to $\downarrow x$. Moreover, for any z-frame homomorphism $f : A \to L$ to a frame L, there is a unique frame homomorphism $\overline{f} : C_z(A) \to L$ such that

$$f = \overline{f} \cdot i_A.$$

2. Nuclei and z-congruences

Nuclei were first introduced by Simmons for frames. Late on Isbell and Simmons also studied nuclei on complete preframes [4]. In this section we introduce nuclei on z-frames and discuss their basic properties. We also reveal some connections between nuclei and congruences on z-frames.

Definition 2.1. A nucleus on a semilattice S is a mapping $p : S \to S$ satisfying the following three conditions:

(i) $p(x) \geq x$, for all $x \in S$;

(ii) $p(x \wedge y) = p(x) \wedge p(y)$ for all $x, y \in S$;

(iii) $p^2 = p$.

Denote by $\boldsymbol{N}(S)$ the set of all nuclei on S. $\boldsymbol{N}(S)$ is a complete lattice with respect to the pointwise order. The bottom element is the identity mapping and the top element is the constant mapping with value 1_S.

The following proposition can be proved in a similar way as in the case for frames.

Proposition 2.2. *Let A be a z-frame, and $p \in \boldsymbol{N}(A)$, then*

(1) *as a subposet of A, $p(A)$ is a z-frame*;

(2) *the mapping $p^0 : A \to p(A)$ of the restriction of p to its codomain is a z-frame homomorphism, and the inclusion*

$$i_A : p(A) \to A$$

is right adjoint to p^0.

Definition 2.3. A quotient of a z-frame is a surjective mapping

$$q : A \to B,$$

where B is a semilattice and q is a semilattice homomorphism such that

$$q(\vee D) = \vee q(D)$$

for every z-set D of A.

Remarks.

(1) For a quotient $p : A \to B$ of a z-frame A, B is not necessarily a z-frame. Consider A which is the free semilattice generated by the set $X = \{a_n, d_n\}_{n \in N}$. Let $B = \{b_n, c_n\}_{n \in N} \cup \{1_B\}$ be the semilattice with the order given by

$$\cdots b_n < b_{n+1} < \cdots < c_{m+1} < c_m \cdots < 1_B.$$

Then A is a preframe. Consider the mapping $f : X \to B$ defined by $f(a_n) = b_n$, $f(d_n) = c_{n+1}$ for all $n \in \boldsymbol{N}$. Suppose $\overline{f} : A \to B$ is the extension of f over to the free semilattice A. A is clearly a preframe and $\overline{f} : A \to B$ is a quotient of this preframe A. But the chain $\{b_n\}_{n \in N}$ has no supremum in B, so B is not a preframe.

However it is easy to see that quotients of a k-frame are k-frames.

(2) From the proposition 2.2, for $j \in \boldsymbol{N}(A)$ there is a quotient of A

$$j^0 : A \to j(A).$$

(3) If A is a frame, then for any quotient $p : A \to B$, there is a unique nucleus $j \in \boldsymbol{N}(A)$ such that B is isomorphic to $j(A)$. This is no longer true for preframes. For instance, consider the quotient $\overline{f} : A \to B$ of the preframe A in remark (1). If there is $j \in \boldsymbol{N}(A)$ such that B is isomorphic to $j(A)$ then B must be a preframe by example 1.2.2 and proposition 2.2. But B is not a preframe.

Now we introduce z-congruence relations on a z-frame. Recall that a semilattice congruence relation τ on a semilattice S is an equivalence relation such that $a \sim b(\tau)$ and $c \sim d(\tau)$ imply $a \wedge c \sim b \wedge d(\tau)$ for all $a, b, c, d \in S$. Here $a \sim b(\tau)$ means a and b has the relation.

Definition 2.4. A z-congruence relation on a z-complete semilattice S is a semilattice congruence relation τ such that

(a) If $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are z-sets of S such that $a_i \sim b_i(\tau)$ for all $i \in I$, then

$$\bigvee_{i\in I} a_i \sim \bigvee_{i\in I} b_i(\tau).$$

Lemma 2.5. *Let τ be a semilattice congruence relation on a z-complete semilattice A satisfying*

(a') *for every z-set D of A and any $b \in A$, if $b \wedge x \sim x(\tau)$ for all $x \in D$, then*

$$b \wedge \vee D \sim \vee D(\tau).$$

Then τ is a z-congruence relation on A.

If A is a z-frame then a semilattice congruence relation on A is a z-congruence relation if and only if it satisfies (a').

Proof. Suppose that τ satisfies (a'). If $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are z-sets, and $a_i \sim b_i(\tau)$ for all $i \in I$, then it follows that for each $j \in I$, $a_j \wedge \vee_{i\in I} b_i \sim b_j \wedge \vee_{i\in I} b_i(\tau)$. $b_j \wedge \vee_{i\in I} b_i = b_j \sim a_j(\tau)$ so $a_j \wedge \vee_{i\in I} b_i \sim a_j(\tau)$. By (a'), $\vee_{i\in I} a_i \wedge \vee_{i\in I} b_i \sim \vee_{i\in I} a_i(\tau)$. Similarly, $\vee_{i\in I} a_i \wedge \vee_{i\in I} b_i \sim \vee_{i\in I} b_i(\tau)$. Hence $\vee_{i\in I} a_i \sim \vee_{i\in I} b_i(\tau)$. So τ is a z-congruence relation.

Now if A is a z-frame and τ is a z-congruence relation on A, we show that it satisfies the condition (a'). In fact, if D is a z-set such that for each $x \in D$, $a \wedge x \sim x(\tau)$, then by (a) and that $\{a \wedge x | x \in D\}$ is a z-set we have $\bigvee\{a \wedge x | x \in D\} \sim \vee D(\tau)$. Since A is a z-frame, $a \wedge \vee D = \bigvee\{a \wedge x | x \in D\}$, and so $a \wedge \vee D \sim \vee D(\tau)$.

For a z-frame A let $\mathbf{Quo}(A)$ denote the set of all quotients of A. Define an order $\leq$ on $\mathbf{Quo}(A)$ by $\boldsymbol{p}_1 \leq \boldsymbol{p}$ for two quotients $\boldsymbol{p} : A \to B$ and $\boldsymbol{p}_1 : A \to B'$ if there is a semilattice homomorphism $\boldsymbol{q} : B' \to B$ such that $\boldsymbol{p} = \boldsymbol{q} \circ \boldsymbol{p}_1$.

Two quotients $\boldsymbol{p}$ and $\boldsymbol{p}'$ of A is said to be equivalent if $\boldsymbol{p} \leq \boldsymbol{p}'$ and $\boldsymbol{p}' \leq \boldsymbol{p}$. We will also use $\mathbf{Quo}(A)$ to denote the poset of all equivalence classes of quotients of A.

As in the general case, from any z-congruence relation τ on A, we can construct a quotient of A

$$\boldsymbol{q} : A \to A/\tau,$$

where A/τ is the quotient semilattice corresponding to τ, $\boldsymbol{q}$ sends $x \in A$ to the equivalence class containing x.

Denote by $\mathbf{zCong}(A)$ the poset of all z-congruence relations on A. $\mathbf{zCong}$ (A) is clearly a complete lattice.

Proposition 2.6. $\mathbf{zCong}$ is isomorphic to $\mathbf{Quo}(A)$.

Proof. For each $\tau \in \mathrm{zCong}(A)$ let $\boldsymbol{F}(\tau)$ be the quotient $\boldsymbol{F}(\tau) = q : A \to A/\tau$. Then the mapping

$$\boldsymbol{F} : \mathbf{zCong}(A) \to \mathbf{Quo}(A)$$

obviously preserves order.

Conversely, for each quotient g of A we can define a relation $\boldsymbol{R}(g)$ on A by $x \sim y(\boldsymbol{R}(g))$ iff $g(x) = g(y)$. Then $\boldsymbol{R}(\boldsymbol{F}(\tau)) = \tau$ for all $\tau \in \mathbf{zCong}(A)$, and $\boldsymbol{F}(\boldsymbol{R}(g))$ is equivalent to g for all $g \in \mathbf{Quo}(A)$. Thus $\mathbf{zCong}(A)$ is isomorphic to $\mathbf{Quo}(A)$.

3. Structure of $N(A)$

It is well known that for every frame L, $\boldsymbol{N}(L)$ is a frame [1]. In this section we first construct a σ-frame A such that $\boldsymbol{N}(\boldsymbol{A})$ is not a σ-frame. This answers an open problem among folks. Then we discuss other properties of $\boldsymbol{N}(\boldsymbol{A})$.

Example 3.1. Let Ω be the set of all countable ordinals, and let L be the poset $\{(\alpha, \beta) \in \Omega \times \Omega : \alpha \leq \beta\}$ with a top element adjoined. L is clearly a σ-frame. Now define three elements $j_1, j_2, j_3 \in \boldsymbol{N}(\boldsymbol{L})$ as follows:

$j_1(\alpha, \beta) = (\alpha, \beta')$, where β' is the first limit ordinal satisfying $\beta' \geq \beta$;
$j_2(\alpha, \beta) = (\alpha, \beta'')$, where β'' is the first non-limit ordinal satisfying $\beta'' \geq \beta$;
$j_3(\alpha, \beta) = (\alpha \vee \beta, \beta)$.

Then $j_1 \vee j_2 = 1$. In fact, if $j = j_1 \vee j_2 \neq 1$, then $j(0,0) = (\alpha, \beta) \neq 1$. Suppose β is not a limit ordinal, then $j_1(\alpha, \beta) > (\alpha, \beta)$. This implies that $j(j(0,0)) \geq j_1(j(0,0)) > j(0,0)$. But this is impossible because $j^2 = j$. Similarly we can show that β must be a limit ordinal. This contradiction shows that $j = 1$. Observe that $j_2 \wedge j_3 = j_1 \wedge j_3 = 0$. As $j_3 \neq 0$, so $j_3 \wedge (j_1 \vee j_2) \neq (j_3 \wedge j_1) \vee (j_3 \wedge j_2)$. Hence $N(L)$ is not a distributive lattice let alone a σ-frame.

Proposition 3.2. *Let A be a z-frame and $\tau \in$ **zCong**(A). Then the following conditions are equivalent*:

(1) *every equivalent class of τ contains a maximal element, and if $x \wedge y \sim y(\tau)$ then there exists $x' \geq y$, $x' \sim x(\tau)$;*
(2) *there is a $j \in \boldsymbol{N}(\boldsymbol{A})$ such that $x \sim y(\tau)$ if and only if $j(x) = j(y)$;*
(3) *the quotient*

$$q : A \to A/\tau$$

has a right adjoint;
(4) *there is a quotient $g : A \to B$ of A which has a right adjoint and for all $x, y \in A$, $x \sim y(\tau)$ iff $g(x) = g(y)$.*

The proof of this proposition is direct, we leave it to the reader.

A z-congruence relation τ is said to be induced by a nucleus if it satisfies the condition (2) in the above proposition. The following example shows that, the condition that every equivalence class contains a maximal element is not enough for τ to be induced by a nucleus.

Example 3.3. Let $\boldsymbol{K} = \{(a, b, i) \in \boldsymbol{N} \times \boldsymbol{N} \times \{0, 1\} : a \leq b\}$ ordered by

$$(a, b, i) \leq (a', b', i')$$

iff $a \leq a'$, $b \geq b'$ and $i = i'$. Let L be the preframe obtained by adjoining a top element and a bottom element to K. Define a z-congruence relation τ on L by stating that

$$x \sim y(\tau) \quad \text{if} \quad x = y \quad \text{or if} \quad x, y \in K \quad \text{and} \quad \pi_1(x) = \pi_1(y), \pi_3(x) = \pi_3(y),$$

where π's are projections. Then for each equivalence class $[(a, b, i)]_\tau$, $[(a, b, i)]_\tau = \{(a, c, i) : c \in \boldsymbol{N}, c \geq a\}$ of which (a, a, i) is the maximal element. If τ is induced by a nucleus j on L, then it follows that $j((a, b, i)) = (a, a, i)$. Now

$j((0,0,0) \wedge (1,1,0)) = j((0,1,0)) = (0,0,0) \neq (0,0,0) \wedge (1,1,0) = j((0,0,0) \wedge j((1,1,0))$. This contradicts to that j preserves binary meets.

For a k-frame A if every equivalence class of a congruence relation τ contains a maximal element then τ is induced by some nucleus. As a matter of fact, suppose that $x \wedge y \sim y(\tau)$ and x' is the maximal element of $[x]_\tau$, then from $x' = x' \vee (x \wedge y) \sim x' \vee y(\tau)$ it follows that $x' \geq y$ and also $x' \sim x(\tau)$. By proposition 3.2(2), τ is induced by a nucleus.

Given a z-frame A we have a mapping

$$\phi : \boldsymbol{N(A)} \to \mathbf{zCong}(A),$$

which sends $j \in \boldsymbol{N(A)}$ to the z-congruence relation induced by j. ϕ is order preserving and injective. We are interested in the cases when ϕ is surjective.

Recall that a Heyting algebra A is a semilattice and for all $a \in A$ and $b \in A$ there is an element $a \to b \in A$ such that $x \wedge a \leq b$ iff $x \leq a \to b$.

Corollary. *If $\phi : \boldsymbol{N(A)} \to \mathbf{zCong}(A)$ is surjective then A is a Heyting algebra.*

Proof. For any $a \in A$, we have a quotient

$$q_a : A \to \downarrow a,$$

defined by $q_a(x) = x \wedge a$. By proposition 3.2, q_a has a right adjoint, say $f : \downarrow a \to A$. Now for each $b \in A$ we claim that $f(a \wedge b) = a \to b$. First as q_a is left adjoint to f, $f(a \wedge b) \wedge a = q_a(f(a \wedge b)) \leq a \wedge b \leq b$. If $x \wedge a \leq b$ then $q_a(x) = x \wedge a \leq a \wedge b$, so $x \leq f(a \wedge b)$.

Lemma 3.4. *Let A be a z-frame such that every equivalence class of z-congruence relation has a maximal element, then A is a complete lattice provided it contains a bottom element.*

Proof. Let X be any non-empty subset of A, and τ be the minimal z-congruence relation on A for which all the elements of X are equivalent to each other. We can assume that X is closed under meets, i.e. $x \in X$, $y \in X$ implies $x \wedge y \in X$. We give a transfinite construction defining τ.

First define a relation τ_1 on A by stating that $a \sim b(\tau_1)$ if either $a = b$ or $a = u \wedge x$ and $b = u \wedge y$ for some $u \in A$ and $x, y \in X$. Suppose now that β is an ordinal and τ_α has been defined for every ordinal $\alpha < \beta$.

(i) If β is a limit ordinal, we define

$$a \sim b(\tau_\beta) \quad \text{iff} \quad a \sim b(\tau_\alpha) \quad \text{for some} \quad \alpha < \beta.$$

(ii) If $\beta = \alpha + n$, where α is a limit ordinal and $n > 1$ is a even integer, we define $a \sim b(\tau_\beta)$ iff there exist z-sets C and D of A such that $a = \vee C$, $b = \vee D$, $u \sim b \wedge u(\tau_{\beta-1})$ for all $u \in C$, and $v \sim a \wedge v(\tau_{\beta-1})$ for all $v \in D$.

(iii) If $\beta = \alpha + n$ with α a limit ordinal and $n > 1$ is an odd integer, we define $a \sim b(\tau_\beta)$ iff there are $c_1, c_2, \ldots, c_m \in A$ such that $a = c_1$, $b = c_m$ and $c_i \sim c_{i+1}(\tau_{\beta-1})$ for each $i = 1, 2, \ldots, m-1$.

For the reson of cardinality there is some ordinal β, $\tau_{\beta+1} = \tau_\beta$. This then implies that τ_β is a z-congruence relation and $\tau = \tau_\beta$.

Now let m be the maximal element of the equivalence class containing all the elements of X. We show that $m = \bigvee X$. m is obviously an up bound of X. Let w be an arbitrary up bound of X, we claim that for any $a \in \downarrow w$, $a \sim b(\tau)$ implies $b \leq w$. It is clear that τ_1 has this property. Suppose that τ_α has this property for all $\alpha < \beta$, it is easy to show that τ_β also has this property. Hence τ has this property.

For each $a \in X$, $a \in \downarrow w$ and $a \sim m(\tau)$ so $m \leq w$. Hence $m = \bigvee X$.

Example 3.5. Let $\mathcal{Z}$ be the set of all integers with the natural order of numbers. Let A be the preframe obtained from $\mathcal{Z}$ by adding a top element. It is easy to see that every equivalence class of z-congruence on the preframe A has a maximal element. However A is not complete because it has no bottom element.

Theorem 3.6. *If the mapping $\phi : \boldsymbol{N(A)} \to \mathbf{zCong}(A)$ is surjective then A is a frame provided that it has a bottom element.*

Lemma 3.7. *Let A be a z-frame such that $\phi : \boldsymbol{N(A)} \to \mathbf{zCong}(A)$ is surjective. Then for any non-empty set $X \subset A$, $\bigvee X = \bigvee Y$ where Y is the smallest z-closed set containing X.*

Proof. Let m, τ and τ_β be defined as in lemma 3.4. It is easy to check, by an inductive argument, that for every $y \in Y$ and every $u \in A$, $u \sim y(\tau)$ implies $u \in Y$. Since $m \sim x(\tau)$ for all $x \in X$ and $m \in Y$. By the construction of Y, m is an up bound of Y, so $m = \bigvee Y$.

In the rest of this section we assume that $\boldsymbol{Z}(S)$ contains empty set as an element for each semilattice S. Then every z-complete semilattice has a bottom element.

Corollary 3.8. *If A is a z-frame such that $\phi : \boldsymbol{N(A)} \to \mathbf{zCong}(A)$ is surjective, then A is isomorphic to $\boldsymbol{C}_z(A)$ via the mapping $i_A : A \to C_z(A)$.*

Proof. By lemma 3.7 for every z-closed set Y of A, Y has a maximal element, say a, then $Y = \downarrow a$. Hence i_A is onto, and so is an isomorphism.

Lemma 3.9. *Suppose that A is a z-frame such that A is isomorphic to $C_z(A)$ via i_A, then $\phi : \boldsymbol{N(A)} \to \mathbf{zCong}(A)$ is surjective.*

Proof. Let $p : A \to B$ be any quotient of A. Then there is a unique frame homomorphism

$$f : C_z(A) = A \to C_z(B)$$

such that $i_B \cdot p = f$, where $i_B : B \to C_z(B)$. Let $f_* : C_z(B) \to A$ be the right adjoint of f, and $p_* = f_* \cdot i_B$. Then p_* is right adjoint to p. From proposition 3.2 and proposition 2.6 it follows that every z-congruence relation on A is induced by some nucleus. So ϕ is surjective.

Theorem 3.10. *For a z-frame A the following conditions are equivalent to each other*:

(1) *$\boldsymbol{N(A)}$ is isomorphic to $\mathbf{zCong}(A)$ via ϕ;*

(2) *A is isomorphic to $C_z(A)$ via i;*

(3) *Every quotient* $p : A \to B$ *has a right adjoint*;

(4) *Every equivalence class of all* z*-congruence relation has a maximal element, and for each* $\tau \in \mathbf{zCong}(A)$ *if* $x \wedge y \sim y(\tau)$ *then there is a* $x' \geq y$ *such that* $x' \sim x(\tau)$.

Notice that each of the equivalent conditions in the above theorem implies that A is a frame. But a frame need not satisfies those conditions.

4. Pseudo-Heyting algebra and $\mathbf{zCong}(A)$

For a z-frame A, the complete lattice $\mathbf{zCong}(A)$ is generally not a frame (see example 4.3). However $\mathbf{zCong}(A)$ has several properties very similar to those of frames. A complete lattice is a frame if and only if it is a Heyting algebra. Although $\mathbf{zCong}(A)$ is generally not a Heyting algebra, it is a pseudo-Heyting algebra. Let us first define and discuss pseudo-Heyting algebra.

Definition 4.1. A lattice S is called a pseudo-Heyting algebra if for any two elements a and b of S, there is an element $a \hookrightarrow b$ such that for any $x \in S$, $x \geq b$, $x \wedge a \leq b$ iff $x \leq a \hookrightarrow b$.

So $a \hookrightarrow b$ is the greatest element among all those $x \in S$ that are greater than or equale to b and $x \wedge a \leq b$. Thus generally, $a \hookrightarrow b \leq a \to b$.

Lemma 4.2. *A pseudo-Heyting algebra is a Heyting algebra if and only if it is distributive.*

Proof. Since every Heyting algebra is distributive, we only need to prove the sufficiency. Let A be a distributive pseudo-Heyting algebra and let a, b be two elements of A. We show that $a \hookrightarrow b = a \to b$. For any $x \in A$, if $x \wedge a \leq b$, then $a \wedge (x \vee (a \hookrightarrow b)) = (a \wedge x) \vee (a \wedge (a \hookrightarrow b)) \leq b$. Since $x \vee (a \hookrightarrow b) \geq a \hookrightarrow b \geq b$, by the definition of $a \hookrightarrow b$ it follows that $x \vee (a \hookrightarrow b) \leq a \hookrightarrow b$, so $x \leq a \hookrightarrow b$. This shows that $a \hookrightarrow b = a \to b$. Hence A is a Heyting algebra.

Lemma 4.3. *A pseudo-Heyting algebra is distributive if it is modular.*

Proof. Suppose A is a pseudo-Heyting algebra that is modular. If A is not distributive, then A has a five element modular but non-distributive sublattice, say $S = \{a, b, c, d, e\}$, where $a < b < e$, $a < c < e$, $a < d < e$. Then $b \hookrightarrow a$ does not exist. Otherwise as $c > a$, $c \wedge b = a$, $d > a$, $d \wedge b = a$ so $c \leq b \hookrightarrow a$, $d \leq b \hookrightarrow a$, and $e = c \vee d \leq b \hookrightarrow a$. This implies that $(b \hookrightarrow a) \wedge b \geq e \wedge b = b > a$. But according to the definition of $b \hookrightarrow a$ we should have $(b \hookrightarrow a) \wedge b \leq a$. This contradiction shows that A is distributive.

Theorem 4.4. *For any z-frame A,* $\mathbf{zCong}(A)$ *is a pseudo-Heyting algebra.*

Proof. Let γ and λ be any elements of $\mathbf{zCong}(A)$. Define τ by $x \sim y(\tau)$ iff whenever $a \sim b(\lambda)$, then $x \wedge a \sim x \wedge b(\gamma)$ if and only if $y \wedge a \sim y \wedge b(\gamma)$. It is easy to verify that τ is really a semilattice congruence relation. Now we show that it satisfies the condition (a') of lemma 2.5. Let X be a z-set, $c \in A$ and $c \wedge x \sim x(\tau)$ for all $x \in X$. If $a \sim b(\lambda)$ and $(\bigvee X) \wedge a \sim (\bigvee X) \wedge b(\gamma)$, then for all $x \in X$, $x \wedge a = x \wedge \bigvee X \wedge a \sim x \wedge (\bigvee X) \wedge b = x \wedge b(\gamma)$. So $c \wedge x \wedge a \sim c \wedge x \wedge b(\gamma)$ for all $x \in X$. Hence $(c \wedge \bigvee X) \wedge a \sim (c \wedge \bigvee X) \wedge b(\gamma)$. Conversely, if $(c \wedge \bigvee X) \wedge a \sim (c \wedge \bigvee X) \wedge b(\gamma)$, then $c \wedge x \wedge a \sim c \wedge x \wedge b(\gamma)$ for all $x \in X$. From $c \wedge x \sim x(\tau)$ it follows that $x \wedge a \sim x \wedge b(\gamma)$ for all $x \in X$. Thus $\bigvee X \wedge a \sim \bigvee X \wedge b(\gamma)$. So $c \wedge \bigvee X \sim \bigvee X(\tau)$. Obviously $\gamma \leq \tau$.

(i) Suppose that $x \sim y(\tau \wedge \lambda)$, then $y \sim x \wedge y(\lambda)$. Since $x \wedge y \sim x \wedge (x \wedge y)(\gamma)$ we have $y = y \wedge y \sim y \wedge (x \wedge y) = x \wedge y(\gamma)$. Similary, $x \sim x \wedge y(\gamma)$, so $x \sim y(\gamma)$. This shows that $\lambda \wedge \tau \leq \gamma$.

(ii) Suppose $\tau' \in$$\mathbf{zCong}(A)$, $\tau' \geq \gamma$ and $\tau' \wedge \lambda \leq \gamma$. Let $x \sim y(\tau')$, we want to show that $x \sim y(\tau)$. For this, assume that $a \sim b(\lambda)$ and $x \wedge a \sim x \wedge b(\gamma)$. Then $x \wedge a \sim x \wedge b(\tau')$, and $y \wedge a \sim x \wedge a \sim x \wedge b \sim y \wedge b(\tau')$, $y \wedge a \sim y \wedge b(\lambda)$. Hence $y \wedge a \sim y \wedge b(\lambda \wedge \tau')$, so $y \wedge a \sim y \wedge b(\gamma)$. Similarly, $y \wedge a \sim y \wedge b(\gamma)$ implies that $x \wedge a \sim x \wedge b(\gamma)$. So $x \sim y(\gamma)$, and $\tau' \leq \tau$.

Hence $\tau = \lambda \hookrightarrow \gamma$, and $\mathbf{zCong}(A)$ is a pseudo-Heyting algebra.

Corollary. *Every $\tau \in$ $\mathbf{zCong}(A)$ has a pseudo complement τ^*, which is the maximal element of* $\mathbf{zCong}(A)$ *satisfying $\tau \wedge \tau^* = 0$.*

Corollary. *If* $\mathbf{zCong}(A)$ *is distributive then it is a frame.*

Corollary. *If* $\mathbf{zCong}(A)$ *is modular, it is distributive.*

Example 4.5. $\mathbf{zCong}(A)$ could be non-modular. Take $\mathbf{Z}$ as the system of sets which assigns $\mathbf{Id}|(S)$ to every semilattices S. Consider the five- point non-modular lattice $A = \{0, a, b, c, 1\}$, where $0 < a < b < 1$, $0 < c < 1$. A is obviously a preframe. $\mathbf{zCong}(A)$ contains a sublattice

$$B = \{0, \tau_{a-1,c-0}, \tau_{a-b-0,c-1}, \tau_{a-b-0}, 1\},$$

where $\tau_{a-b-0,c-1}$ is the smallest z-congruence relation containing $\{(a, b), (a, 0), (b, 0), (c, 1)\}$. $\tau_{a-1,c-0}$ and τ_{a-b-0} are defined similarly. Then $0 < \tau_{a-b-0} < \tau_{a-b-0,c-1} < 1$, $0 < \tau_{a-1,c-0} < 1$. So B is not a modular lattice, and hence neither is $\mathbf{zCong}(A)$.

This example also shows that a pseudo-Heyting algebra is not necessarily distributive.

For any $a \in A$ we have a $\tau_a \in \mathbf{zCong}(A)$ which is determined by the quotient

$$a \wedge - : A \longrightarrow \downarrow a,$$

that is $x \sim y(\tau_a)$ iff $a \wedge x = a \wedge y$.

Proposition 4.6. *Let A be a z-frame, then:*

(1) *for any $a \in A$,*

$$\tau_a^{**} = \tau_a.$$

(2) *For any z-set X of A, $\bigwedge_{x \in X} \tau_x = \tau_{\vee X}$.*

Proof. (1) Suppose that $x \sim y(\tau^{**})$ we want to show that $x \sim y(\tau)$. From the proof of theorem 4.1, $x' \sim y'(\tau^*)$ if and only if, whenever $c \sim d(\tau_a)$ (i.e. $a \wedge c = a \wedge d$) then $x' \wedge c = x' \wedge d$ iff $y' \wedge c = y' \wedge d$. Now if $c \sim d(\tau_a)$ then $a \wedge x \wedge c = a \wedge x \wedge d$ and $a \wedge y \wedge c = a \wedge y \wedge d$ hold simultaneously, so $x \wedge a \sim y \wedge a(\tau^*)$. But $x \wedge a \sim y \wedge a(\tau^{**})$ because $x \sim y(\tau^{**})$. Hence

$x \wedge a \sim y \wedge a(\tau^{**} \wedge \tau^*)$. As $\tau^{**} \wedge \tau^* = 0$, so $x \wedge a = y \wedge a$. This shows that $\tau^{**} \leq \tau$, and so $\tau^{**} = \tau$.

(2) It is clear that $\tau_{\vee X} \leq \bigwedge_{x \in X} \tau_x$. Suppose that $a \sim b(\bigwedge_{x \in X} \tau_x)$, that is for each $x \in X$, $a \wedge x = b \wedge x$. Then by the properties of z-congruence relation we have $a \wedge \bigvee X = b \wedge \bigvee X$. This means that $a \sim b(\tau_{\vee X})$. So $\bigwedge_{x \in X} \tau_x \leq \tau_{\vee X}$. So it is proved that $\tau_{\vee X} = \bigwedge_{x \in X} \tau_x$.

It is well known that for any frame A, $\boldsymbol{B} = (\boldsymbol{N}(\boldsymbol{A}))_{\neg\neg} = \{x \in \boldsymbol{N}(\boldsymbol{A}) | \neg\neg x = x\}$ is a Boolean algebra, and A is isomorphic to a subframe of this Boolean algebra. For a z-frame A we let $\mathcal{B}^*(A)$ be the dual poset of $\{\tau \in \mathbf{Quo}(A) | \tau^{**} = \tau\}$.

The above lemma shows that there is an embedding from A into $\mathcal{B}^*(A)$ that preserves joins of z-sets. But to have a similar result for z-frames, we still have to answer the following questions, which are open problems:

1. Is $\mathcal{B}^*(A)$ a Boolean algebra, or some weaker form of boolean algebra?
2. Is the embedding from A into $\mathcal{B}^*(A)$ preserves finite meets?

We also expect more deep discussions on pseudo-Heyting algebras. In another paper we will discuss projective z-frames that are closely connected to z-continuous posets.

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