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DIRECTED COMPLETE POSETS DETERMINED BY SCOTT CLOSED SET LATTICES AND RELATED WORK

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Abstract. Let F be an assignment that associates each object C of a category $\mathcal{C}$ with an object $F(C)$ in a category $\mathcal{D}$, such that $F(C_1)$ is isomorphic to $F(C_2)$ whenever C_1 is isomorphic to C_2. An object C of $\mathcal{C}$ is called F-determined if for any object A of $\mathcal{C}$, $F(C) \cong F(A)$ if and only if $C \cong A$. Such objects have been studied in various different categories and for various assignments F, such as the category of all topological spaces and the assignment that sends each space to the ring of all continuous real valued functions; the category of all topological spaces and the assignment that sends each space to its closed set lattice; the category of all topological spaces and the assignment that sends each space to the lattice of all lower semicontinuous functions. Recently, such objects in the category of directed complete posets for the assignment that sends a directed complete poset to its Scott open set lattice have also been studied. In this paper, we shall try to present a panoramic survey on such work in various mathematics disciplines, especially those about directed complete posets. Some problems will be posed and elaborated for further work.

Contents

For any T_0 space X, let $\Gamma(X)$ be the lattice of all closed sets of X. Using a result by Drake and Thron in [6], one deduces the following result: A topological space X has the property that $\Gamma(X)$ isomorphic to $\Gamma(Y)$ implies X is homeomorphic to Y iff X is sober and T_D (every derived set $d(\{x\}) = cl(\{x\}) - \{x\}$ of point $x \in X$ is closed)(see also [23], line 11-13, page 504). Some similar results have been obtained

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in other fields of mathematics. For example, the countable infinite cyclic group C_∞ has the property that for any group G, C_∞ is isomorphic to G if and only if the lattices $\mathrm{Sub}(C_\infty)$ and $\mathrm{Sub}(G)$ are isomorphic, where $\mathrm{Sub}(G)$ denotes the lattice of all subgroups of G [1].

In this paper, we first present a survey on such results in some classical categories. Then we focus on the category of all directed complete posets, review the recent work and list some open problems for further work. Hope this survey will provide the reader with a relatively complete picture on such work carried out independently in various different mathematics disciplines.

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1. General F-determined objects

Two objects C and D in a category are *isomorphic*, denoted by $C \cong D$, if there are morphisms $p : C \to D, q : D \to C$ such that $1_C = q \circ p$ and $1_D = p \circ q$.

Definition 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Assume that F is an assignment that associates each object C of $\mathcal{C}$ with an object $F(C)$ of $\mathcal{D}$ such that $F(C) \cong F(C')$ whenever $C \cong C'$. Such an assignment F will be called an "isomorphism preserving assignment" from category $\mathcal{C}$ to category $\mathcal{D}$.*

(1) *An object A of $\mathcal{C}$ is called* F-determined *if for any object B in $\mathcal{C}$,*

$$F(A) \cong F(B) \text{ implies } A \cong B.$$

(2) *A class $\mathcal{A}$ of objects of $\mathcal{C}$ is called* F-faithful *if for any two objects A_1, A_2 in $\mathcal{A}$,*

$$F(A_1) \cong F(A_2) \text{ implies } A_1 \cong A_2.$$

Remark 1. *Let F be an isomorphsm preserving assignment from a category $\mathcal{C}$ to a category $\mathcal{D}$.*

Two objects C_1 and C_2 of $\mathcal{C}$ are said to be F-equivalent, *written $C_1 \equiv_F C_2$, if $F(C_1) \cong F(C_2)$.*

A property p of objects in $\mathcal{C}$ is F-equivalent invariant *if for any two F-equivalent objects C_1 and C_2, C_1 has p iff C_2 has p.*

A subcategory $\mathcal{C}_1$ of $\mathcal{C}$ is called F-equivalent closed if an object C is in $\mathcal{C}_1$ whenever it is F-equivalent to an object in $\mathcal{C}_1$.

For any subcategory $\mathcal{C}_1$ of $\mathcal{C}$, there is a smallest F-equivalent closed subcategory containing $\mathcal{C}_1$ (consisting of all those objects that are F-equivalent to some objects in $\mathcal{C}_1$).

The following are some general problems on a given F.

(1) Which classes of objects are F-faithful?
(2) Which objects are F-determined?
(3) Which properties are F-equivalent invariant?

Clearly if C is F-determined and $C \cong C'$, then C' is also F-determined.

In the following, all subcategories considered will be assumed to be isomorphic closed: if $C \cong C'$ and C is in the subcategory, then so is C'.

Lemma 1. *If $\mathcal{C}_1$ is a subcategory of a category $\mathcal{C}$ such that for any object C in $\mathcal{C}$, there is an object C_1 in $\mathcal{C}_1$ that is F-equivalent to C, then all F-determined objects of $\mathcal{C}$ belong to $\mathcal{C}_1$.*

Proof. Let A be an F-determined object of $\mathcal{C}$. There is an object A_1 in $\mathcal{C}_1$ such that $F(A) \cong F(A_1)$. Since A is F-determined, we have $A \cong A_1$, thus A is in $\mathcal{C}_1$. □

2. Groups determined by subgroup lattices

Let GRP be the category of all groups and group homomorphisms, and LAT be the category of lattices and lattice homomorphisms.

For each group G, let $\mathrm{Sub}(G)$ be the set of all subgroups of G. As the intersection of any collection of subgroups of G is a subgroup of G, $Sub(G)$ is a (complete) lattice with respect to the inclusion order $\subseteq$. The assignment $G \mapsto \mathrm{Sub}(G)$ is isomorphism preserving. The book [20] provides a complete information on $\mathrm{Sub}(G)$ for various different types of groups G. The following are some results related to Sub-determined objects.

Note that all infinite cyclic groups are isomorphic to the additive group $(\mathbb{Z}, +)$ of integers.

Theorem 1 ([1])**.** *The infinite cyclic group is Sub-determined.*

Note that the additive group $(\mathbb{Q}, +)$ of all rational numbers is not cyclic.

Theorem 2 ([9])**.** *The additive group $\mathbb{Q}$ of all rational numbers is Sub-determined.*

The alternating group A_4 is the group of all even permutations on four elements.

Theorem 3 ([20])**.** *The alternating group A_4 is Sub-determined.*

Let $M = [m_{ij}]_{1 \le i,j \le n}$ be a symmetric $n \times n$ matrix with entries from $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \le n$ and $m_{ij} > 1$ whenever $i \ne j$. The finite Coxeter group of type M is the group given by the following representation:

$$W(M) = \langle s_1, \cdots, s_n | (s_i s_j)^{m_{ij}} = 1, 1 \le i, j \le n, m_{ij} < \infty \rangle.$$

Theorem 4 ([24])**.** *Every finite Coxeter group of rank not less than 3 is Sub-determined.*

Recall that a *simple group* is a group whose only normal subgroups are the trivial subgroups.

Theorem 5 ([17])**.** *Every finite simple group of Lie type of rank greater than or equal 2 is Sub-determined.*

The following are some Sub-equivalent invariant properties for groups.

Recall that a *locally cyclic* group is a group in which every finitely generated subgroup is cyclic (equivalently, the subgroup generated by any two elements of the group is cyclic). A group is locally cyclic if and only if it is isomorphic to a subquotient (i.e., a quotient group of a subgroup) of the group of rational numbers.

Theorem 6 ([20]). *A group G is locally cyclic if and only if $Sub(G)$ is a distributive lattice.*

Theorem 7 ([20]). *A group G is cyclic iff $Sub(G)$ is distributive and satisfies the ascending chain condition.*

Up to date, there is still no a complete characterization of Sub-determined groups. We even do not know any necessary condition for such groups.

Problem 1. Which groups are exactly the Sub-determined groups?

3. Topological spaces determined by the rings of continuous functions

Topology and algebra are two of the fundamental structures in mathematics. For each topological space, the set $C(X)$ ($C^*(X)$) of all real valued (bounded) continuous functions is an algebra in several senses. The basic algebraic structure on $C(X)$ are the ordinary addition and product of functions and $C(X)$ is a commutative ring with respect to such operations. The assignment of $C(X)$ to X establishes a strong bridge between algebra and topology. As pointed out in the Prospectus of the book [10], "One of the main problems will be to specify conditions under which X is determined as a topological space by the algebraic structure of $C(X)$ ($C^*(X)$). In other words, what restrictions on X and Y, if any are needed at all, will allow us to conclude that X is homeomorphic with Y, when we are given that $C(X)$ is isomorphic to $C(Y)$?"

In this section, we shall retract some results in [10] concerning the C-determined spaces. The two categories we are concerned are the category TOP of all topological spaces and continuous mappings and the category Ring of all commutative rings and ring homomorphisms.

Theorem 8 ([10]). *For any topological space X, there is a completely regular space Y such that $C(X) \cong C(Y)$ (as rings).*

By Lemma 1 and the above theorem we deduce the following.

Corollary 1. *If a space X is C-determined, then X is completely regular.*

Since there exists non-completely regular spaces, we have the following.

Corollary 2. *The class TOP of all topological spaces is not C-faithful.*

As such, in the study of the rings of continuous functions, it is enough to consider only completely regular spaces.

The following is the famous Gelfand - Kolmogorov Theorem (Theorem 4.9, [10]).

Theorem 9. *Two compact spaces X and Y are homeomorphic iff $C(X)$ and $C(Y)$ are isomorphic.*

Corollary 3. *The class of all compact spaces is C-faithful.*

Hewitt introduced the notion of realcompact space, and showed that to a very large extent these spaces play the same role in the theory of $C(X)$ that the compact spaces do in the theory of $C^*(X)$(the ring of all bounded real continuous functions on X).

Definition 2. *A topological space X is realcompact if it is homeomorphic to a closed subspace of a product space of $\mathbb{R}'s$. Here $\mathbb{R}$ is the set of real numbers with the usual topology.*

Remark 2.
(1) *A space is Lindelöf if its every open cover has a countable subcover. Every Lindelöf space is realcompact. In particular, every subspace of $\mathbb{R}^n$ is realcompact.*
(2) *A Hausdorff space is compact if and only if it is realcompact and pseudo-compact ($C(X) = C^*(X)$)* [7].
(3) *Every realcompact space is completely regular.*

The following theorem is clearly a generalization of the Gelfand-Kolmogorov theorem.

Theorem 10 ([10])**.** *For any two realcompact spaces X and Y, X is homeomorphic to Y iff $C(X)$ is isomorphic to $C(Y)$.*

Corollary 4. *The class of all realcompact spaces is C-faithful.*

Theorem 11 (Realcompactification)**.** *Let X be a completely regular space.*
(1) *There is a realcompact space νX such that X can be embedded in it as a dense subset.*
(2) $C(X) \cong C(\nu X)$.

The space νX is called a Hewitt realcompactification of X. Since there exists a completely regular space that is not realcompect, thus we have the following conclusions.

Corollary 5. *The class of all completely regular spaces is not C-faithful.*

Corollary 6. *All realcompact spaces form a maximal C-faithful class of topological spaces.*

Again, by Theorem 11 and Lemma 1 we deduce the following necessary condition for C-determined spaces.

Corollary 7. *Every C-determined space is realcompact.*

Unfortunately, we still haven't seen a necessary and sufficient conditions for C-determined space.

Problem 2. Which spaces are exactly the C-determined spaces?

Problem 3. Is every compact Hausdorff space C-determined?

Recently, Deb Ray et al. considered the ring $B_1(X)$ of all Baire one functions on a space X (a function $f : X \longrightarrow \mathbb{R}$ is Baire one if it is the pointwise limit of a sequence of continuous functions) and obtained some similar results[3, 4, 5].

4. Topological spaces determined by closed set lattices

For each topological space X, let $\Gamma(X)$ be the set of all closed sets of X. With the inclusion order, $\Gamma(X)$ is a complete lattice. Thus Γ assigns a complete lattice to each topological space, which is clearly isomorphism preserving.

Note that for any two spaces X and Y, $\Gamma(X) \cong \Gamma(Y)$ if and only if $\mathcal{O}(X) \cong \mathcal{O}(Y)$, where $\mathcal{O}(X)$ is the lattice of all open sets of X.

A non-empty subset A of a topological space X is irreducible if for any two closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$.

A non-empty closed set A of X is *strongly irreducible* if for any collection $\{B_i : i \in I\}$ of closed sets $B_i's$, $A = cl(\bigcup\{B_i : i \in I\})$ implies $A = B_i$ for some $i \in I$. This is clearly equivalent to that if $A = cl(B)$ for some set B, then $A = cl(\{b\})$ for some $b \in B$.

A T_0 space is *sober* if for every irreducible closed set F, there is a (unique) point x such that $F = cl(\{x\})$.

Let ${}^sX = \{F : F \text{ is a closed irreducible set of } X\}$. The sets $\diamond U = \{F \in {}^sX : F \cap U \neq \emptyset\}$ (U is an open set in X) form a topology on ${}^sX$. The set ${}^sX$ with this topology is sober, usually called the *sobrification* of X, and $\Gamma(X) \cong \Gamma({}^sX)$. Thus we have the following classic result.

Theorem 12. *For any T_0 space X, there is a sober space Y such that $\Gamma(X) \cong \Gamma(Y)$ (as lattices).*

Since there are non sober T_0 spaces, we have the following corollary.

Corollary 8. *The class of all T_0 spaces is not Γ-faithful.*

The following result shows that the sober spaces play the similar roles in the theory of Γ-determined objects as realcompact spaces in that of C-determined objects (see Theorem 11 and Theorem 9).

Theorem 13. *If X and Y are sober spaces, then X is homeomorphic to Y if and only if $\Gamma(X) \cong \Gamma(Y)$.*

Corollary 9. *The class of all sober spaces form a maximal Γ-faithful class.*

A nice thing about the Γ-objects is that they have a complete characterization.

Definition 3. *A topological space X is called a T_D space if for each $x \in X$, $cl(\{x\}) - \{x\}$ is a closed set.*

Every T_1 space is a T_D space and there are T_D spaces which are not T_1.

The following result can be proved directly.

Lemma 2. *A T_0 space X is T_D if and only if every point closure $cl(\{x\})$ is strongly irreducible.*

If X is a T_D space, then X is homeomorphic to the subspace $Irr_s(X)$ of ${}^sX$ consisting of all strongly closed irreducible sets of X. Hence one can deduce the following result.

Theorem 14. *If X and Y are T_D spaces, then X and Y are homeomorphic if and only if $\Gamma(X)$ and $\Gamma(Y)$ are isomorphic.*

Corollary 10. *For any two T_1 spaces X and Y, X is homeomorphic to Y if and only if $\Gamma(X)$ is isomorphic to $\Gamma(Y)$.*

Corollary 11. *The class of all T_D spaces is Γ-faithful.*

The following characterization of Γ-determined spaces was obtained by Drake and Thron [6] (they stated this in a different and equivalent form).

Theorem 15. *A topological space X is Γ-determined if and only if its every irreducible closed set is strongly irreducible.*

Lemma 3. (1) *A T_0 space X is T_D if and only if every point closure $cl(\{x\})$ is strongly irreducible.*

(2) *A T_0 space is both sober and T_D if and only if every irreducible closed set is strongly irreducible.*

See Appendix of [28] for a direct and detailed proof of (2).

The combination of the above results leads to the following nice characterization of Γ-determined spaces.

Theorem 16. *A topological space X is Γ-determined if and only if it is both sober and T_D.*

5. Topological spaces determined by the lattices of lower semicontinuous functions

The set $S = \{0, 1\}$ with the topology $\{\emptyset, S, \{1\}\}$ is called the *Sierpinski space*. For any topological space X, the set $C(X, S)$ of all continuous mappings $f : X \longrightarrow S$ with the pointwise order is isomorphic to the lattice $\Gamma(X)$ under the mapping $F : C(X, S) \longrightarrow \Gamma(X)$ sending f to $f^{-1}(\{0\})$. Now consider the partially ordered set $\mathbb{R}$ of all reals with the *lower topology* $\{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. This is actually the Scott topology $\sigma(\mathbb{R})$ on the poset $(\mathbb{R}, \leq)$ (see the definition of Scott topology in

Section 7). The Sierpinski space is clearly a subspace of the space $(\mathbb{R}, \sigma(\mathbb{R}))$. The set of all continuous functions from a space X to $(\mathbb{R}, \sigma(\mathbb{R}))$ coincides with the set $\mathrm{LS}(X)$ of all lower semicontinuous functions on X, which is a lattice with respect to the pointwise order.

Now we have another assignment that associates the lattice $\mathrm{LS}(X)$ to each space X. In [23], Thornton studied the LS-determined spaces and obtained a complete characterization of such spaces.

A family $\mathcal{G}$ of open sets of a space is an *open filter* if it is closed under finite intersections and $G \in \mathcal{G}, G \subseteq U \in \mathcal{O}(X)$ imply $U \in \mathcal{G}$.

Definition 4. *A subset A of a space X has the filter countable intersection property, or FCI-property, if every open filter $\{G_\lambda : \lambda \in \Lambda\}$ in X with $G_\lambda \cap A \neq \emptyset$ for each λ has nonempty countable intersections (every countable subfamily has a nonempty intersection).*

Remark 3. *By Lemma 4.1 of* [19]*, $A \subseteq X$ has the FCI-property iff every lower semicontinuous function $f : X \longrightarrow \mathbb{R}$ is upper bounded on A.*

For any point $x \in X$, $cl(\{x\})$ has the FCI-property.

Definition 5. *A T_0 space X is called an fc space if every irreducible closed set with the FCI-property is the closure of a point.*

For a T_0 space X, let $\mathrm{Irr}_s(X)$, $cl(X)$, $\mathrm{Irr}_f(X)$ and $\mathrm{Irr}(X)$ denote the set of all strongly irreducible closed sets, all point closures, all closed irreducible sets with the FCI-property and all irreducible closed sets, respectively. Then we have the following inclusions:

$$\mathrm{Irr}_s(X) \subseteq cl(X) \subseteq \mathrm{Irr}_f(X) \subseteq \mathrm{Irr}(X).$$

Hence every sober space is an fc space.

The following is the Theorem 4.3 in [19].

Theorem 17. *The subcategory consisting of all fc spaces is epireflecitive in the category of all T_0 spaces.*

Remark 4. *By the above theorem, one deduces that for any T_0 space X, there is an fc space ϕX and a continuous map $\phi_X : X \longrightarrow \phi X$ such that for any continuous map $f : X \longrightarrow Y$ with Y an fc space there is a unique continuous map $\hat{f} : \phi X \longrightarrow Y$ such that $f = \hat{f} \circ \phi_X$. In addition, as pointed out in* [19]*, ϕX can be identified with a subspace of the sobrification ${}^sX$ of X.*

Recall that for any T_0 space X, there is a sober space ${}^sX$ such that $\Gamma(X)$ is isomorphic to $\Gamma({}^sX)$. Also, if X and Y are sober spaces then X is homeomorphic to Y if and only if $\Gamma(X)$ is isomorphic to $\Gamma(Y)$. For fc property, there exist similar results.

The result below is the Theorem 4.5 in [19].

Theorem 18. *For any T_0 space X, the lattices $LS(X)$ and $LS(\phi X)$ are isomorphic.*

If X and Y are fc spaces, then X and Y are homeomorphic iff $LS(X)$ and $LS(Y)$ are isomorphic.

Definition 6. *A T_0 space X is called a T_P space if for any point x, $\{x\}$ is a G_δ-set or $cl(\{x\}) - \{x\}$ is closed.*

Clearly, every T_D space is T_P.

The following result is the Theorem 16 in [23].

Theorem 19. *For any two T_P spaces X and Y, X is homeomorphic to Y if and only if $LS(X)$ is isomorphic to $LS(Y)$.*

Corollary 12. *The class of all T_P spaces is LS-faithful.*

The following is the remarkable characterization of LS-determined spaces (Theorem 15 in [23]), similar to that of Γ-determined spaces.

Theorem 20. *A T_0 space X is LS-determined iff it is both fc and T_P.*

Example 1. The set $\mathbb{N}$ of all positive integers with the co-finite topology τ_{cof} ($U \in \tau_{cof}$ iff either $U = \emptyset$ or $\mathbb{N} - U$ is a finite set) is both T_P and fc. Hence it is SL-determined.

Note that $(\mathbb{N}, \tau_{cof})$ is not sober.

The set $\mathbb{R}$ of reals with the co-countable topology is neither fc nor T_P.

For any T_0 space X, the b-topology on X (introduced by Skula in [21]) has $\{U \cap V^c : U, V \in \mathcal{O}(X)\}$ as a base. Clearly, $\{U \cap cl(\{x\}) : x \in X, U \in \mathcal{O}(X)\}$ is also a base of the b-topology. It is easy to verify that a space is T_D if and only if its b-topology is discrete.

The b-topology is called the *front topology* in [19]. A subset B of a space X is b-closed if it is closed with respect to the b-topology.

Remark 5. *The following are some more interesting properties of sober spaces (also called pc spaces) and fc spaces in terms of b-topology proved in* [19].

(1) *A b-closed subspace of a sober space is sober.*
(2) *A sober subspace of a T_0 space Y is b-closed.*
(3) *A T_0 space is sober if and only if it is homeomorphic to a b-closed subspace of some cub S^I, where S is the Sierpinski space.*
(4) *A T_0 space is fc if and only if it is homeomorphic to a b-closed subspace of some cub $\mathbb{R}^I$, where $\mathbb{R}$ is the real line with the Scott topology $\{\mathbb{R}, \emptyset\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$.*

6. Directed complete posets determined by Scott closed set lattices

Given a poset P, one may define various different intrinsic topologies on P. The most important such topologies in domain theory is the Scott topology. In this section, we consider the C_σ-determined posets, where $C_\sigma(P)$ is the set of all Scott closed sets of a poset P.

A nonempty subset D of a poset $(P, \leq)$ is directed if any two elements in D has an upper bound in D (for any $d_1, d_2 \in D$, there is a $d_3 \in D$ such that $d_1 \leq d_3, d_2 \leq d_3$). A subset U of a poset $(P, \leq)$ is *Scott open* if i) U is an upper set (that is, $U = \uparrow U = \{x \in P : y \leq x \text{ for some } y \in U\}$), and ii) for any directed subset $D \subseteq P$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the *Scott topology* on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the *Scott space* of P. We shall use $C_\sigma(P)$ to denote the set of all Scott closed sets of P (it is a complete lattice with respect to the set inclusion order).

One can easily show that two posets are isomorphic if and only if their Scott spaces are homeomorphic.

A poset is called a *directed complete poset* (*dcpo*, for short) if every directed subset of the poset has a supremum. A mapping $f : P \longrightarrow Q$ between posets is Scott continuous if it is continuous with respect to the Scott topologies on P and Q. It is well known that $f : P \longrightarrow Q$ is Scott continuous if and only if it preserves existing suprema of directed sets: for any directed set $D \subseteq P$ with $\bigvee D$ existing, it holds that $f(\bigvee D) = \bigvee f(D)$.

For more about the Scott topology and dcpos, see [8] and [11].

Let POS_d be the category of all posets and Scott continuous mappings and DCPO_d be the full subcategory of POS_d consisting of all directed complete posets.

Then $P \mapsto C_\sigma(P)$ defines an isomorphism preserving assignment from POS_d (DCPO_d, resp.) to the category LAT of lattices and lattice homomorphisms.

The following result shows that every poset is C_σ-equivalent to a directed complete poset.

Theorem 21 ([27])**.** *For any poset P, there is a directed complete poset $E(P)$ such that $C_\sigma(P)$ is isomorphic to $C_\sigma(E(P))$.*

The dcpo $E(P)$ in the above theorem is called the dcpo-completion of P which has the following property: there is a Scott continuous mapping $\eta_P : P \longrightarrow E(P)$ such that for any Scott continuous $f : P \longrightarrow Q$ to a dcpo Q, there is a unique Scott continuous mapping $\hat{f} : E(D) \longrightarrow Q$ satisfying $f = \hat{f} \circ \eta_P$.

Corollary 13. *The class of all posets is not C_σ-faithful.*

Corollary 14. *Every C_σ-determined poset is directed complete.*

A poset P is called *bounded complete* if every upper bounded subset has a supremum. In particular, every complete lattice is bounded complete.

Theorem 22 ([13])**.** *Two bounded complete lattices L and M are isomorphic if and only if $C_\sigma(L)$ and $C_\sigma(Q)$ are isomorphic.*

Thus the class of all bounded complete lattices is C_σ-faithful.

It is then natural to ask whether the class of all dcpos is C_σ-faithful.

Example 2. In [12], the authors constructed a dcpo P whose Scott space is not sober, yet the soberification of ΣP is the Scott space of another dcpo Q. Hence $C_\sigma(P)$ and $C_\sigma(Q)$ are isomorphic but P and Q are not.

This example also shows the existence of non C_σ-determined dcpo.

A dcpo P is called sober if its Scott space is sober. Thus by the above counterexample, a sober dcpo need not be C_σ-determined.

Corollary 15. *The class of all dcpos is not C_σ-faithful.*

For two elements x and y in a poset P, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P with $\bigvee D$ existing, $y \leq \bigvee D$ implies $D \cap \uparrow x \neq \emptyset$. Let $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$. A continuous dcpo is also called a *domain*.

Continuous dcpos (domains) are the most important order structures in domain theory.

Remark 6. *The following are some well-known results about domains* (*see* [8] *and* [11]).

(1) *For any domain P, the Scott space ΣP is sober.*

(2) *A dcpo P is continuous if and only if the lattice $C_\sigma(P)$ is a completely distributive lattice.*

A complete lattice L is completely distributive if it satisfies the most general distributivity: for any family $\{A_i : i \in I\}$ of subsets of L, it holds that

$$\bigwedge_{i \in I} \bigvee A_i = \bigvee_{f \in \Pi_{i \in I} A_i} \bigwedge \{f(i) : i \in I\}.$$

In the following, to simplify statements, we shall regard C_σ as an assignment from the category DCPO_d to the category of complete lattices. Thus a dcpo P is C_σ-determined if and only if for any dcpo Q, $C_\sigma(P)$ is isomorphic to $C_\sigma(Q)$ if and only if P is isomorphic to Q.

Theorem 23. *Let P be a domain. Then for any dcpo Q, $C_\sigma(P)$ is isomorphic to $C_\sigma(Q)$ if and only if P is isomorphic to Q.*

Proof. Assume that $C_\sigma(P)$ is isomorphic to $C_\sigma(Q)$, then $C_\sigma(P)$ is a completely distributive lattice, hence so is $C_\sigma(Q)$. Thus Q is a domain, therefore ΣQ is sober. Now $C_\sigma(P)$ is isomorphic to $C_\sigma(Q)$, thus the open set lattices $\sigma(P)$ and $\sigma(Q)$ are isomorphic, hence by Theorem 13, ΣP is homeomorphic to ΣQ. Thus P is isomorphic to Q. The other direction of implication is trivial. $\square$

Corollary 16. *Every domain is C_σ-determined.*

In general, assume that r is a property of dcpos and s is a property of complete lattices such that (i) if a dcpo P has property r then ΣP is sober, (ii) P has property r if and only if $C_\sigma(P)$ has property s. Then every dcpo with the property r is C_σ-determined.

Another such type of dcpos are the quasicontinuous dcpos.

A finite subset F of a dcpo P is way-below an element $a \in P$, denoted by $F \ll a$, if for any directed subset $D \subseteq P$, $a \leq \bigvee D$ implies $D \cap \uparrow F \neq \emptyset$. A dcpo P is quasicontinuous if for any $x \in P$, the family

$$fin(x) = \{F : F \text{ is finite and } F \ll x\}$$

is a directed family (for any $F_1, F_2 \in fin(x)$ there is $F \in fin(x)$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$) and for any $x \not\leq y$ there is $F \in fin(x)$ satisfying $y \notin \uparrow F$ (see Definition III-3.2 of [8]).

Qusicontinuous dcpos have the following properties:

(i) Every domain is quasicontinuous;
(ii) every quasicontinuous dcpo is sober;
(iii) a dcpo P is quasicontinuous iff the Scott open set lattice $\sigma(P)$ of P is hypercontinuous (Theorem VII-3.9 of [8]).

Thus by the remarks after the Corollary 16 (just note that the Scott open set lattice $\sigma(P)$ is dual to $C_\sigma(P)$, hence $\sigma(P) \cong \sigma(Q)$ if and only if $C_\sigma(P) \cong C_\sigma(Q)$), we obtain the following more general result.

Theorem 24. *Every quasicontinuous dcpo is C_σ-determined.*

Recall that every Γ-determined topological space is sober. Thus a natural question arising is:

Is every C_σ-determined dcpo sober?

The answer is no.

An element a of a dcpo P is quasicontinuous if the subdcpo $\downarrow a = \{x \in P : x \leq a\}$ is quasicontinuous. Trivially, every element of a quasicontinuous dcpo is a quasicontinuous element.

A T_0 space X is called bounded sober if every upper bounded closed irreducible subset F (i.e. there is an $x \in X$ such that $F \subseteq cl(\{x\})$) is the closure of a point. Every sober space is bounded sober. The set of all positive integers with the co-finite topology is bounded sober but not sober.

The following is the Theorem 3.9 of [28], where the C_σ-determined dcpos are called C_σ-unique dcpos.

Theorem 25. *A dcpo P is C_σ-determined if it satisfies the following conditions:*

(a) *the Scott space ΣP is bounded sober;*
(b) *for every element $a \in P$, there is a directed set $D \subseteq P$ consisting of quasicontinuous elements such that $a = \bigvee D$.*

Example 3. In [15], Peter Johnstone constructed the first non sober dcpo as $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with the partial order $\leq$ defined by

$$(m, n) \leq (m', n') \Leftrightarrow \text{either } m = m' \text{ and } n \leq n' \text{or } n' = \infty \text{ and } n \leq m'.$$

(i) $(\mathbb{J}, \leq)$ is a dcpo.
(ii) The largest Scott closed set $\mathbb{J}$ is irreducible and $\mathbb{J} \not\subseteq cl(\{x\})$ for any $x \in \mathbb{J}$ (i.e. it is not upper bounded).
(iii) If F is a proper irreducible Scott closed set of $\mathbb{J}$, then $F = \downarrow(m, n) = cl(\{(m, n)\})$ for some $(m, n) \in X$. Hence $\Sigma\mathbb{J}$ is bounded sober.
(iv) If $n \neq \infty$, the subdcpo $\downarrow(m, n)$ is a finite chain, hence continuous (thus also quasicontinuous), implying that (m, n) is a quasicontinuous element.
(v) If $n = \infty$, then (m, n) is the supremum of the chain $\{(m, k) : k \neq \infty\}$ whose members are quasicontinuous elements.

Hence by Theorem 25, we deduce that the dcpo $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ is C_σ-determined.

Thus a C_σ-determined dcpo need not be sober. We can also verify that the Scott space $\Sigma\mathbb{J}$ is not a T_D space.

One can easily verify that the Scott space ΣP of a dcpo P is a T_D space if and only if P is Noetherian (it does not contain an infinite ascending chain).

Thus, by Theorem 16 we have the following result (note that if the Scott space ΣP of a dcpo P is Γ-determined, then P is C_σ-determined).

Proposition 1. *If P is a Noetherian and sober dcpo, then P is Γ-determined and thus C_σ-determined.*

To answer the question whether every well-filtered Scott space is sober, Kou constructed another non sober dcpo [16] whose Scott space is well-filtered but not sober. One can check that Kou's dcpo also satisfies the conditions (a) and (b) in Theorem 25, thus it is C_σ-determined.

A dcpo P is called *locally quasicontinuous* if every subdcpo $\downarrow x$ ($x \in P$) is quasicontinuous (i.e. every element is quasicontinuous). It is easy to see that every locally quasicontinuous dcpo is bounded sober. By Theorem 25 we deduce the following result [25].

Corollary 17. *Every locally quasicontinuous dcpo is C_σ-determined.*

Note that Johnstone's dcpo $\mathbb{J}$ is locally quasicontinuous.

Any finite product of locally quasicontinuous dcpos is locally quasicontinuous. Thus for any n, the product dcpo $\mathbb{J}^n$ is C_σ-determined.

See [25] for more recent development on the study of C_σ-determined dcpos.

7. Some problems on C_σ-determined dcpos

We now list and elaborate some problems on C_σ-determined dcpos.

Problem 4. Which dcpos are exactly the C_σ-determined dcpos?

At the moment we have obtained some sufficient conditions for a dcpo to be C_σ-determined. However, we still do not have any necessary condition for such dcpos, not to mention a complete characterization. It would be desirable if an order characterization of C_σ-determined dcpos can be find.

Problem 5. Is every complete lattice C_σ-determined?

In [12], two non isomorphic dcpos are constructed such that they have isomorphic Scott closed set lattices. But no one of these dcpos is a complete lattice. Although by Theorem 22 if both L and M are complete lattices, they are isomorphic if $C_\sigma(L)$ is isomorphic to $C_\sigma(M)$, we do not know if the conclusion still hold if only one of them is a complete lattice. In [14], Isbell constructed a non sober complete lattice. One may like to check whether this complete lattice is C_σ-determined first.

The following relevant problem may have a positive answer.

Problem 6. Is every sober countable complete lattice C_σ-determined?

Problem 7. What does a maximal C_σ-faithful class consist of?

A maximal C_σ-faithful class $\mathcal{M}$ of dcpos has the following properties:

(i) for any dcpo P, there is a dcpo $\overline{P} \in \mathcal{M}$ such that $C_\sigma(P)$ is isomorphic to $C_\sigma(\overline{P})$;

(ii) $\mathcal{M}$ is C_σ-faithful.

By the definition, a maximal C_σ-faithful class contains all C_σ-determined dcpos (such as all quasicontinuous dcpos and $\mathbb{J}$).

Note that such maximal classes have been explicitly described for the assignment $\Gamma(X)$ by Corollary 9 as well as for the assignment $C(X)$ by Theorem 6.

Problem 8. Is the product of two C_σ-determined dcpos also C_σ-determined?

It is well know that the product of two quasicontinuous dcpos is quasicontinuous [11]. The product of any two of the C_σ-determined dcpos, we have identified so far, are C_σ-determined. Thus it is natural to consider the above problem.

Problem 9. Is the product $\mathbb{J}^N$ of countable copies of $\mathbb{J}$ C_σ-determined?

Since the dcpo $\mathbb{J}$ is locally quasicontinuous, any product of finite copies of $\mathbb{J}$ is locally quasicontinuous, thus C_σ-determined. But we do not know whether the product $\mathbb{J}^N$ of countable copies of $\mathbb{J}$ is C_σ-determined. If the answer is yes, we would have a much simpler example of non C_σ-determined dcpo.

Problem 10. Is the product of two sober dcpos also a sober dcpo?

In order to determine whether the product of two dcpos satisfying the conditions (a) and (b) in Theorem 25 also satisfies these conditions (hence the product is C_σ-determined), we need to know whether the product of two bounded sober dcpos is bounded sober. But this is equivalent to whether the product of two sober dcpos is a sober dcpo. This problem was also posed in [26].

It is well known that the product of any collection of sober topological spaces is sober. Given two sober dcpos P and Q, the set $P \times Q$ with the product topology (generated by $U \times V$, $U \in \sigma(P), V \in \sigma(Q)$) is sober. However the product topology on $P \times Q$ need not equal the Scott topology on $P \times Q$, that is, $\sigma(P \times Q) = \mathcal{O}(\Sigma P \times \Sigma Q)$ need not hold. Hence the product of two sober dcpos may not be sober, but we do not have a counterexample yet. By Theorem II-4.13 [8], if the lattice $\sigma(P)$ is a continuous lattice, then for any dcpo Q, $\sigma(P \times Q) = \mathcal{O}(\Sigma P \times \Sigma Q)$ holds. Hence it follows that if $\sigma(P)$ is continuous (in particular, if P is a domain) and P is sober, then $P \times Q$ is sober for any sober dcpo Q.

Problem 11. Which dcpos P have the property that $P \times Q$ is sober for any sober dcpo Q?

It is well known that a sober space X is locally compact if and only if its open set lattice $(\mathcal{O}(X), \subseteq)$ is a continuous poset (such a space is called core-compact). By the remarks following Problem 10, it follows that if P is a dcpo whose Scott space ΣP is both sober and locally compact, then $P \times Q$ is sober for any sober dcpo Q. It is natural to wonder whether the converse implication also holds.

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