Introduction to general topology

Zhao Dongsheng

Aug 2012

Chapter one : Metric spaces

Outline:

- Definition of metric spaces, examples
- Continuous functions
- Open sets and closed sets

1.1 Example Let **R** be the set of all real numbers. For any x, y in **R**, define $d(x, y)=|x-y|$. Then

i) $d(x, y)\geq 0$;

ii) $d(y, x)=0$ if and only if ____________;

iii) $d(x, y)$ _____ $d(y, x)$;

iv) $d(x, y)$ $d(x, z)+ d(z, y)$.

1.2 Definition A metric space is an ordered pair (M, ρ) consisting a set M together with a function ρ: $M\times M\rightarrow R$ such that for any $x, y, z \in M$:

M-a) $\rho(x, y) \geq 0$;

M-b) $\rho(y, x)=0$ if and only if x y ;

M-c) $\rho(x, y)$ $\rho(y, x)$; (Symmetric)

M-d) $\rho(x, y)$ $\rho(x, z)+ \rho(z, y)$. (Triangle Inequality)

If all conditions except M-b are satisfied, the function ρ is called a pseudometric on M, and (M, ρ) is called a pseudometric space.

Remark: We may use different symbols for the function ρ. For instance, $d(x, y)$, $\lambda(x, y)$ etc.

1.3 Examples

a) (**R**, d) is a metric space, where **R** is the set of all real numbers and $d(x, y)=|x-y|$.

b) Let $\mathbf{R}^n=\{ (x_1, x_2, \ldots, x_n): x_i's$ are real numbers $\}$. Define

$$\rho((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n))=\sqrt{\sum_{k=1}^{n}(x_k - y_k)^2} .$$

Then $(\mathbf{R}^n, \rho)$ is a metric space and this function ρ is called the usual metric on $\mathbf{R}^n$.

c) For $\mathbf{R}^2$, the function $\rho_1((x_1, y_1), (x_2, y_2))=|x_1-x_2|+|y_1-y_2|$ is a metric .

d) Let (M, ρ) be a metric space and A be a subset of M . Then (A, ρ) is also a metric space, called the subspace of M.

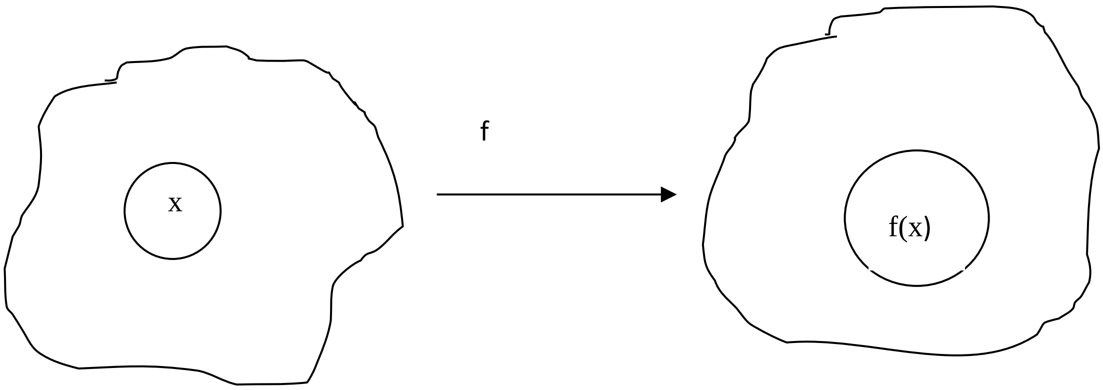
e) The discrete metric : Let X be a set. Define $\rho(x, x)=0$ and $\rho(x, y)=1$ for $x\neq y$. Then ρ is a metric on X, called the **discrete metric**.

1.4 Definition A function f: M→N from a metric space (M, ρ) to a metric space (N, σ) is ***continuous*** at appoint $x \in$ M if for any number $\varepsilon>0$,

there is a positive number $\delta>0$ such that

$$\sigma(f(x), f(y)) < \varepsilon \quad \text{whenever} \quad \rho(x, y)<\delta.$$

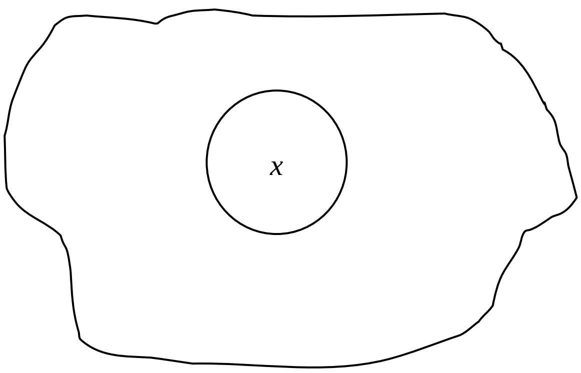
The function is called a continuous function if it is continuous at **every point** of X.



1.5 Definition Let (M, ρ) be a metric space and $x \in$ M . For each number $\varepsilon>0$, let

$$U(x, \varepsilon)=\{ y\in M: \rho(x, y)< \varepsilon \},$$

called the **ε-disk** (or ε-open ball) about x.



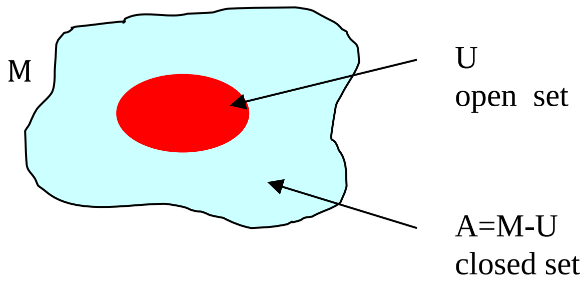
Exercise

1) In **R**, determine the set U(0, 4).

2) In the Example 1.3 e), find U(x, 1) and U(x, 2).

1.6 Definition A subset V of a metric space (M, ρ) is an **open set** if for each *x* in U, there is an ε>0 such that U(x, ε) ⊆ V.

A subset A is called a **closed set** if it's complement A^c=M - A is open.



For example, in **R**, the set U=(0, 1)∪(4, 5) is open.

The set [0, 1] is not open in **R**.

1.7 Theorem (properties of open sets) In any metric space (M, ρ) we have:

1) Any union of open sets is open.

2) Any finite intersection of open sets is open.

3) The empty set and M are open.

Proof

1.8 Examples

a) In **R**, a subset A is open if and only if it is the disjoint union of open intervals, i.e.

$$A=\bigcup_{k=1}^{\infty}(a_k, b_k), \quad \text{where } (a_k,\ b_k) \text{ are disjoint.}$$

b) Every disk U(x, ε) is open. (Exercise)

c) If (X, d) is a **discrete metric space**, then every set is open. In fact, for any subset A and

for any x in A, $U(x, 1)=\{x\}\subseteq A$.

d) Every finite set is closed.

1.9 Theorem A function f: M → N from a metric space (M, ρ) to a metric space (N, σ) is ***continuous*** at x_0 if and only if for any open set W of N containing $f(x_0)$, there is an open set U containing x_0 such that $f(U)\subseteq W$.

Proof

Recall that if $f: X \to Y$ is a function, then for any subset $B \subseteq Y$,

$$f^{-1}(B)=\{ x\in X: f(x) \in B \}$$, called the inverse image of B under f.

1.10 Corollary function $f: M\to N$ from a metric space (M, ρ) to a metric space (N, σ) is continuous if and only if for any open set W of N,

$$f^{-1}(W)=\{ x\in M: f(x) \in W \}$$

is open.

Hands-On- Exercise

Let $U=\{ (x, y): x>0, y>0 \}$. Show that U is an open set of $\mathbf{R}^2$ with the usual metric.

Summary

- A metric space is an ordered pair (M, ρ) consisting of a set M and a function

  $\rho : M\times M\to R$ satisfying the four conditions.

- A function $f: M\to N$ from a metric space (M, ρ) to a metric space (N, σ) is *continuous* at x_0 if ____

- A subset A is an open set if ______________________________________

- Every disk is an open set

- The union of any open sets is ______________________

- Any finite intersection of open sets is open

- A function f: M→N from a metric space (M, ρ) to a metric space (N, σ) is *continuous* if and only if ______________________________

Exercise 1

1. Verify that the following function ρ is a metric on $\mathbf{R}^n$

$$\rho(x, y)=\max\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_n-y_n|\}.$$

2. Let C([0, 1]) be the set of all of all continuous functions on the interval [0, 1].

(i) Verify that the following function σ is a metric on C([0,1]).

$$\sigma(f,g)=\int_0^1 |f(x)-g(x)|\,dx .$$

(ii) Verify that the following function η is a pseudo metric on C([0,1])

$$\eta(f,g)=|f(1/2)-g(1/2)| .$$

3. Show that every disk U(x, ε) in a metric space is an open set.

[Hint: For any $y \in$ U(x, ε), U(y, ε') is contained in U(x, ε), where ε'= ε- d(x, y)]

4. A mapping f from a metric space (M, ρ) to metric space (N, σ) is an **isometry** if f is a bijection and $\rho(x, y)=\sigma(f(x), f(y))$ for all x, y in M. Two spaces M and N are isometric if there is an isometry between them.

Prove

(i) Every isometry f and its inverse f^{-1} are continuous.

(ii) The subspaces [0, 1] and [a, b] (a< b) of **R** are isomertric .

5. Let (M, ρ) be a metric space. Show that a subset A is closed iff whenever every disk about x meets A then x is in A.

6. Let ρ be a metric on M. Show that the following functions ρ_1 and ρ_2 are also metrics on M.

(i) $\rho_1(x, y)=2\,\rho(x, y)$.

(ii) $\rho_2(x, y)=\min\{1, \rho(x, y)\}$.

7*(Optional)

Let Q be the set of all rational numbers and p be a prime number. For each x in Q, define

$|x|_p=0$ if $x=0$ and $|x|_p=p^{-k}$ if $x=p^k\frac{m}{n}$, where m and n are integers not divisible by p.

Define $\rho(x, y)=|x-y|_p$ for any x, y in Q.

(a) Find $|15/9|_5$ and $|2.6|_7$.

(b) Show that $|xy|_p=|x|_p|y|_p$.

(c) Show that $|x+y|_p\le\max\{|x|_p, |y|_p\}$.

[hint: Assume $|x|_p=\max\{|x|_p, |y|_p\}$]

(d) Show that $\rho(x, y)=|x-y|_p$ defines a metric on Q. This called the p-adic metric on Q.

Chapter two: Topological Spaces

Outline:

- Definition of topological spaces, examples
- Closure operator
- Interior operator
- Neighbourhoods,
- Bases and subbases

2.1 Topological spaces

2.1.1 Definition A topology on a set X is a collection τ of subsets of X such that the following conditions are satisfied:

T-1) Any union of members of τ is a member of τ; (closed under arbitrary unions)

T-2) any finite intersection of members of τ is a member of τ;

(closed under finite intersections)

T-3) $\varnothing$ and X are members of τ.

If τ is a topology on X, the members of τ are called **open sets** of X.

The pair (X, τ) (or just X) is called a topological space.

If $\tau_1 \subseteq \tau_2$ are topologies, then τ_2 is said to be finer than τ_1 .

2.1.2 Example

a) Let (M, ρ) be a metric space. The set of all open sets of M form a topology, called the metric topology and denoted by τ_ρ .

If (X, τ) is a topological space such that $\tau = \tau_\rho$ for some metric ρ, then (X, τ) is called **metrizable.**

b) The metric topology generated by the usual metric on any subset of R^n is called the usual topology. Hereafter, when a topology is used on a subset of $\mathbf{R}^n$ without mention it is assumed to be the usual topology.

c) Let X be any set. The power set **P**(X) (all subsets of X) is a topology on X, called the **discrete topology**. Discrete topology is the finest topology on X.

d) For any set X, $\tau = \{ \emptyset , X \}$ is a topology, called the **indiscrete topology**. It is the coarsest topology on X.

e) Sierpinski topology .

Let X={ 0, 1} and let $\tau=\{ \emptyset, \{1\}, X \}$. Then τ is a topology. The space (X, τ) is called the Sierpinski space.

2.1.3 Definition If (X, τ) is a topological space and A⊆ X, then A is a closed set if its complement X - A is open.

2.1.4 Examples

a) Every closed interval [a, b] is closed in R.

b) In the discrete topological space, every subset is closed.

c) In the Sierpinski space, the closed sets are ∅, {0} and X

2.1.5 Theorem

C-1) Any intersection of closed sets is closed;

C-2) any finite intersection of closed set is closed;

C-3) the empty set and X are closed.

2.1.6 Definition The <u>closure</u> of a subset A of a topological space (X, τ) is defined to be

$$\overline{A} = cl(A) = \bigcap\{K \subseteq X \mid K \text{ is closed and } A \subseteq K \}.$$

Since any intersection of closed sets is closed, the closure of a subset is closed and is the smallest closed set containing the set.

Remark.

1) A⊆B implies cl(A) ⊆cl(B).

2) cl(X)=X.

3) In the discrete space, the closure of any set A is A.

4) In the indiscrete space X, cl(A)=X for any nonempty set A.

2.1.7 Theorem Let A, B and E be subsets of a topological space X. Then

K-1) E ⊆ cl(E);

K-2) cl(cl(E))=cl(E);

K-3) cl(A∪ B)=cl(A) ∪ cl(B);

K-4) cl(∅)=∅;

K-5) E is closed iff cl(E)=E.

Proof.

2.1.8 Definition Let A be a subset of a topological space X. The **interior** of A in X is the set

$$\text{int}(A)=A^{o}=\cup\{\, U\subseteq A:\ U \text{ is open }\}.$$

Remark

1) int(A) is the largest open set contained in A.

2) A⊆B implies int(A) ⊆ int(B).

3) int(A)=X-cl(X-A), cl(A)=X-int(X-A). (Exercise)

2.1.9 Theorem Let A, B and E be subsets of a topological space X. Then

I-1) int(E) ⊆ E ;

I-2) int(int(E))=int(E);

K-3) int(A∩ B)=int(A) ∩ int(B);

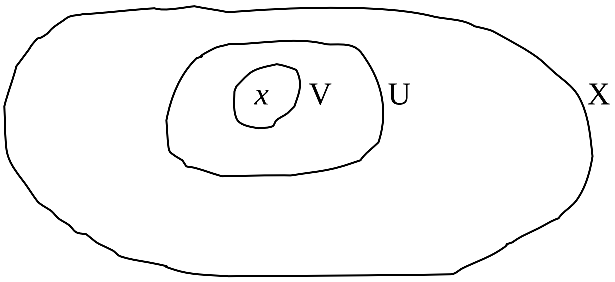
K-4) int(X)=X;

K-5) E is open iff int(E)=E.

Proof (Exercise)

2.2 Neighbourhoods

2.2.1 Definition A neighbourhood U of a point x in a topological space is a subset that contains an open V containing x.



A neighbourhood that is open is called an open neighbourhood. The set of all neighbourhoods of x is denoted by $\mathbf{N}_x$, called the neighbourhood system at x.

2.2.2 Theorem Let x be any point of a topological space X. then

N- 1) for any U in $\mathbf{N}_x$, x is in U;

N-2) if $U, V \in \mathbf{N}_x$, then $U \cap V \in \mathbf{N}_x$;

N-3) if $U \in \mathbf{N}_x$ and $U \subseteq V$ then $V \in \mathbf{N}_x$;

N-4) $G \subseteq X$ is open if and only if G is a neighbourhood of every point in G.

Proof

2.2.3 Definition A neighbourhood <u>base</u> at x in a topological space X is a collection $\mathbf{B}_x$ of neighbourhoods of x such that for any U in $\mathbf{N}_x$, there is V in $\mathbf{B}_x$ such that $V \subseteq U$.

2.2.4 Examples

1) For any point x in a topological space X, all the open neighbourhoods of x form a neighbourhood base at x.

2) In any metrizable space X, all disks about x form a neighbourhood base at x.

Also, all disks $U(x, r)$ with r a positive rational number form a neighbourhood base at x.

Thus every x has a countable neighbourhood base.

3) If X is a discrete space, then for each x, $B_x=\{\{x\}\}$ is a neighbourhood base at x.

Exercise Show that for any x in R, $\{(x-1/n, x+1/n): n \text{ in } N\}$ is a nbhd base at x.

2.2.5 Definition (Accumulation points) An **accumulation point** (cluster point) of a set A in a topological space X is a point x of X such that every neighbourhood of x contains a point of A, other than x. The set A' of all cluster points of A is called the derived set of A.

For example, the accumulation points of (0, 1) in R form the set [0, 1].

2.2.6 Theorem For any set A in a topological space,

$$cl(A)= A \cup A'.$$

Proof

2.3 Bases and subbases

Sometimes using a subfamily to define a topology is easier than directly describing the topology.

2.3.1 Definition Let (X, τ) be a topological space. A base **B f**or X is a collection $B \subseteq \tau$, such that every member U of τ is a union of some members of **B**,

that is for each U in τ, there exist $\{ V_i : i \in I\} \subseteq \mathbf{B}$, such that $U=\bigcup\{V_i : i \in I\}$.

2.3.2 Examples

a) In R, all the open intervals form a bases of the usual topology.

In any metrizable space, all the open disks form a bases.

b) $\{\{x\}: x\in X\}$ is a bases of the discrete space.

c) In R, all the intervals (s, r) with s, r rational numbers form a bases of the usual topology.

2.3.3 Theorem A collection B of subsets of X is the base of a topology on X if and only if

1) $X=\bigcup\{V: V\in \mathbf{B}\}$;

2) if $V_1, V_2\in \mathbf{B}$ and $x\in V_1\cap V_2$, then there is V in B such that

$$x\in V \subseteq V_1\cap V_2.$$

Proof

2.3.4 Example The family B={ [a, b): a< b, a, b are in R } satisfies the conditions of Theorem 3, so it is the base of a topology. The set R with this topology is called the Sorgenfrey line. The topology of the Sogenfrey line is strictly finer than the usual topology.

2.3.5 Definition A subbase **C** for a topology τ on a set X is a collection of subsets of X such that all the <u>finite intersections</u> of members of **C** form a base of τ .

2.3.6 Example

$\mathbf{C}=\{ (a, +\infty): a\in R \}\cup\{ (-\infty, b): b\in R \}$ is a subbase of the usual topology on R.

Summary

- A topology τ on a set X is a collection of subsets of X which contains ___ and X and is closed under taking arbitrary _______ and finite _____________.

  Members of τ are called ______ sets

- A subset A of (X, τ) is a closed set if ____________
- The closure cl(A) of set A is ______________
- The interior int(A) of set A is _______________
- A cluster point of A, the derive of A
- A base, subbase of a topology
- Neighbourhoods of a point

Exercise 2

1. Let N be the set of all natural numbers. Let τ be the set of all subset U of N such that N-U is finite.

(i) Show that τ is a topology on N. This is called the co-finite topology.

(ii) Find cl({1, 2, 3}) and cl(E), where E is the set of all even numbers.

(iii) Let D be the set of all odd numbers . Is $cl(D\cap E) = cl(D)\cap cl(E)$ true?

2. Let τ and σ be two topologies on set X. Show that $\tau \cap \sigma$ is also a topology on X.

3. Prove for any subset A of a topological space X,

$$int(A)=X-cl(X-A) \quad \text{and} \quad cl(A)=X-int(X-A) \text{ hold.}$$

4. Use the results in exercise 3 to prove Theorem 2.1.9 .

5. Show that x is in cl(A) if and only if every neighbourhood of x intersects A.

6. Let A be a fixed subset of a set X.

a) Show that $\tau =\{ U\subseteq X:\ A\subseteq U \}$ is a topology on X.

b) Describe the closure of a subset B with respect to the topology τ in a).

7. Call a subset U of R^2 radially open if it contains an open line segment in each direction about each of its point.

a) Show that all the radially open set of R^2 form a topology on R^2. The plane with this topology is called the radial plane.

b) Compare this topology with the usual topology.

8.

a) Show that for any open set U in a topological space X,

cl(int(cl(U))=cl(U). (or $U^{-o-}=U^{-}$)

b) Using a) to show that for any set A , there are at most 14 different sets in the following sequences

$$A,\ A^{c},\ A^{c-},\ A^{c-c},\ldots$$

$$A,\ A^{-c},\ A^{-c-},\ A^{-c-c},\ldots$$

where $A^{c}=X-A$, B^{-} is the closure of B.

9. Let P(X) denote the power set of X and σ: $P(X)\rightarrow P(X)$ be a mapping such that for any A, B in P(X),

1) $\sigma(\varnothing)=\varnothing$;
2) $A\subseteq\sigma(A)$ for all subset A;
3) $\sigma(\sigma(A))=\sigma(A)$;
4) $\sigma(A\cup B)=\sigma(A)\cup\sigma(B)$.

Define $\tau=\{U: \sigma(X-U)=X-U\}$.

Show that τ is a topology on X and for any subset A, $cl(A)=\sigma(A)$ holds.

*A mapping σ: $P(X)\rightarrow P(X)$ satisfying 1) - 4) is called a closure operator on X.

Chapter three: Continuous functions

Outline:

- Definition
- Example
- Equivalent conditions
- Homeomorphisms
- Subspaces

3.1 Continuous functions

We now define continuous function from a topological space to another space.
Recall by 1.10 Corollary that a function f: $(M, d) \to (N, \sigma)$ between two metric space is continuous if and only if for any open set W of N, $f^{-1}(W)=\{ x\in M: f(x) \in W \}$ is open.
Now we define continuous functions between topological spaces.

3.1.1 Definition
A function $f: X\to Y$ from a topological space (X, τ) to a topological space (Y, σ) is continuous at a point x_0 in X if for any nbhd(abbreviation for neighbourhood) V of $f(x_0)$ there is a nbhd U of x_0 such that $f(U) \subseteq V$.
If f is continuous at every point in X, it is said to be continuous(everywhere).

3.1.2 Theorem Let $f: X\to Y$ be a function from a topological space X to a topological space Y. Then the followings are equivalent:

a) f is continuous;
b) for each open set V of Y, $f^{-1}(V)$ is open in X;
c) for each closed set K of Y, $f^{-1}(K)$ is closed in X;
d) for each subset A of X, $f(cl_X(A)) \subseteq cl_Y(f(A))$.
e) for each subset B of Y, $cl_X(f^{-1}(B))) \subseteq f^{-1}(cl_Y(B))$.

Proof: We prove the theorem by showing a) $\Rightarrow$b) $\Rightarrow$ c) $\Rightarrow$d)$\Rightarrow$e) $\Rightarrow$a).

3.1.3 Example
If X is a discrete space, then every function from X to another topological space is continuous.
Any function from a topological space to an indiscrete space is continuous.

3.1.4 Proposition Let f: $X \rightarrow Y$ be a function from a topological space X to a topological space Y and let **B** be a **base** of Y. Then f is continuous iff for any open set V in **B**, $f^{-1}(V)$ is open.

Proof

3.1.5 Example
The function f: $R \to R$, defined by $f(x)=|x|$, is continuous with respect to the usual topology.

3.1.6 Theorem If f: $X \to Y$ and g:$Y \to Z$ are continuous functions between topological spaces, then the composition $g \circ f$: $X \to Z$ is continuous.
Proof (exercise)

3.1.7 Definition Let f: $X \to Y$ be a function from a topological space X to a topological space Y **.**
a) f is called an open function if for any open set U of X, f(U) is an open set of Y.
b) f is called a closed function if for any closed K of X, f(K) is a closed set of Y.
c) f is called a clopen function if it is both open and closed.

3.1.8 Theorem A function f: $X \to Y$ between topological spaces is open if and only if for any open set U in a base **B** of X, f(U) is open.

Proof

3.1.9 Examples

a) The function f: $R \to R$ defined by f(x)=x+1 is both open and closed.

b) The function g: $R \to R$ defined by $g(x)=x^2$ is continuous that is not open. To see this, let U=(-1, 1). Then U is open, but g((-1, 1))=[0, 1) which is not open.

c) Every function from a space to a discrete space is both open and closed. Thus an open (closed) function need not be continuous.

Exercise

Consider the continuous function f: $R \to R$ given by f(x)=|x|. Is f open ?
Can you

3.2 Homeomorphisms

3.2.1 Definition A function f: $X \to Y$ between two topological spaces is called an homeomorphism if it is a bijection and both f and f^{-1} are continuous.
Two spaces X and Y are homeomorphic if there is a homeomorhism between them.

3.2.2 Remark

a) Given two topological spaces X and Y, one often wants to know whether the two spaces are homeomorphic. For example,
i) are R and R^2 homeomorphic?
ii) are Q homeomorphic to R?
In order to prove two spaces are homeomorphic, we need to define a homeomorphism between them.

b) A property p of topological spaces is called a topological property if a space X has property p, then every space homeomorphic to X also has property p. One method of proving two spaces X and Y are <u>not homeomorphic</u> is to find a topological property satisfied by X but not satisfied by Y.

3.2.3 Example

The space X=(0, 1) is homeomorphic to any space Y=(a, b) with

a<b, where both X and Y have the metric topology induced by the usual metric.
To show this, we define f: X $\to$ Y by f(x)=a+x(b-a) for each x in X. Then f is bijective. The inverse f^{-1}: Y $\to$ X sends y in (a, b) to (y-a)/(b-a) for y in Y. Thus both f and its inverse are continuous.

3.2.4 Example The space Q of rational numbers (with the metric topology induced by the usual metric) is not homeomorphic to R. This is because Q is a countable set and R is not countable.

3.2.5 Proposition If X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z.
Proof

3.2.6 Theorem Let f: X $\to$ Y be a bijection between two topological spaces. Then the following statements are equivalent:
a) f is an homeomorphism;
b) for any G $\subseteq$ X, f(G) is open in Y iff G is open in X;
c) for any F $\subseteq$ X, f(F) is closed in Y iff F is closed in X;
d) for any E $\subseteq$ X, $f(cl_X(E))=cl_Y(f(E))$.
e) f is clopen.
Proof

3.3 Subspaces

3.3.1 Definition Let (X, τ) be a topological space and A be a subset of X. The collection τ'={ U $\cap$ A : U $\in$ τ } is a topology on A, called the relative topology for A. A subset of X equipped with the relative topology is called a **subspace** of (X, τ).

3.3.2 Examples
a) The real line R with the usual topology is a subspace of R^2.
b) Let Z be the set of all integers. As a subspace of R, Z inherits the discrete topology.
c) Any subspace of a discrete space is discrete, and any subspace of a indiscrete space is indiscrete.

Exercise
Show that the function i_A: A$\to$X from a subspace A of X into X defined by $i_A(x)=x$ is continuous.

3.3.3 Theorem Let A be a subspace of a topological space X. Then

a) $H\subseteq A$ is open in A iff $H=A\cap U$ for some open set U of X;
b) $F\subseteq A$ is closed in A iff $H=A\cap K$ for some closed set K of X;
c) for any $E\subseteq A$, $cl_A(E)=A\cap cl_X(E)$;
d) if $\mathbf{B_x}$ is a nbhd base for x in X, then $\{U\cap A: U\in \mathbf{B_x}\}$ is a nbhd base for x in A;
e) if B is a base of X, then $\{U\cap A: U\in \mathbf{B}\}$ is a base for A.

Proof:

Remark Note that $int_A(E)=A\cap int_X(E)$ need not be true for all subsets E of A.
For example, let $X=R^2$, A=E= the ox-axis. Then $int_A(E)=E$, while

$$int_X(E)\cap A=\varnothing\cap A=\varnothing.$$

If f: X$\to$Y is a function and A is a subset of X, then f restricts to a function from A to denoted by $f|_A$: A$\to$Y such that $f|_A(x)=f(x)$ for any x in A.

3.3.4 Theorem Let f: X$\to$Y be a continuous functions between two topological spaces. Then for any subspace A of X, the restriction function $f|_A$: A $\to$Y is continuous.

Exercise Prove Theorem 3.3.4.

Summary

- A function f: X$\to$Y between two topological spaces is continuous at a point x_0 in X if ____
- A function f: X$\to$Y is called a continuous function if it is continuous ________________
- f: X$\to$Y is continuous iff for any open set V of Y, ___________ is an open set of X
- f: X$\to$Y is continuous iff for any closed set V of Y, ___________ is a closed set of X

- $f: X \to Y$ is continuous iff for any subset A of X,

  $cl_Y(f(A)$ ________ $f(cl_X(A)$

- $f: X \to Y$ is continuous iff for any subset B of Y,

  $cl_X(f^{-1}(B))$ ______ $f^{-1}(cl_Y(B))$
- a function $f: X \to Y$ between two topological spaces is a homeomorphism
  if f is a bijection such that ____________________________________
- A subset A of a topological space X equipped with the _________ topology is called a subspace of X.

Exercise 3

1. Prove Theorem 3.1.6.

2. Show that f: $X \to Y$ is continuous iff for any subset B of Y,

$$f^{-1}(\text{int}_Y(B)) \subseteq \text{int}_X f^{-1}(B).$$

[hint: use Theorem 3.1.2 and note that $\text{int}_Y(B)=B$ if B is open]

3. Let A be a subspace of a topological space X and g: $Z \to A$ be a function from a space Z to A. Show that g is continuous if and only if the composition $i_A \circ g$: $Z \to X$ is continuous.

4. Prove:
(a) the composition of two open functions is an open function;
(b) the composition of two closed functions is a closed function.

5.. Give a bijective continuous function f: $X \to Y$ such that it's inverse f^{-1}: $Y \to X$ is not continuous.

6. Let f: $X \to Y$ be a homeomorphism.

(a) Show that if X is a discrete space then Y is discrete.

(b) Show that if X is indiscrete then Y is indiscrete.

Chapter four:
Cartesian product and quotient space

Outline:

- (finite)Cartesian product
- Example
- Quotient spaces

In this lesson, we study more methods of constructing new topological spaces from given ones.

4.1 Cartesian product(finite)

Let $X_1, X_2, \ldots, X_n$ be sets. The Cartesian product of X_i's is defined as

$$\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n = \{(x_1, x_2, \ldots, x_n): x_i \in X_i, i=1,\ldots, n\}.$$

For any subsets $U_i \subseteq X_i (i=1,\ldots,n)$, define

$$U_1 \times U_2 \times \ldots \times U_n = \{(x_1, x_2, \ldots, x_n): x_i \in U_i, i=1,\ldots, n\}.$$

Note: We can define the product of any collection of sets.

4.1.1 Examples

a) $\{1, 2\} \times \{a, b\} = \{ (1, a), (1, b), (2, a), (2, b)\}$.

b) $R \times R = \{ (x, y): x, y \in R \} = R^2$.

If $X_1, X_2, \ldots, X_n$ are topological spaces, we can define a topology on their Cartesian product set so that it becomes a topological space.

4.1.2 Lemma

Let $(X_1, \tau_1), (X_2, \tau_2), \ldots, (X_n, \tau_n)$ be topological spaces. The family

$$\mathcal{B} = \{ U_1 \times U_2 \times \ldots \times U_n: U_i \in \tau_i, i=1, 2,\ldots, n\}$$

of subsets of $X = X_1 \times X_2 \times \ldots \times X_n$ is the base of a topology on X.

Proof

4.1.3 Definition Let { (X_i, τ_i): i=1, 2, …, n} be topological spaces.
The topology τ on $X= X_1\times X_2\times \ldots\times X_n$ generated by the base **B**={ $U_1\times U_2\times \ldots\times U_n$: $U_i\in \tau_i$, i=1, 2,…, n} is called the **product topology** and the space (X, τ) is called the Cartesian product of (X_i, τ_i)'s.

4.1.4 Example If (X, τ) and (Y, υ) are topological spaces. Then { $U\times V$: $U\in \tau$, $V\in \upsilon$} is a base of the product topology on $X\times Y$.
Thus a subset M of $X\times Y$ is open if and only if for each (x,y) in M, there are open sets U and V in X and Y respectively such that $x\in U$, $y\in V$ and $U\times V\subseteq M$.

4.1.5 Theorem Let (X_1, τ_1), (X_2, τ_2), …, (X_n, τ_n) be topological spaces. Assume that for each i, $\mathcal{B}_i$ is a base of X_i (i=1, …, n). Then

$$\mathcal{B}'=\{ V_1\times V_2\times \ldots\times V_n: V_i\in \mathcal{B}_i, i=1, 2,\ldots, n\}$$

is a base of the product topology.
Proof:

4.1.6 Example The real line R has a base consisting of open intervals**, so** the plane $R^2=R\times R$ has a base consists of product of open intervals $(a_1, b_1)\times(a_2, b_2)$.

4.1.7 Example
A product of discrete spaces is discrete and a product of indiscrete spaces is indiscrete.

Let $X_1, X_2, \ldots, X_n$ be sets. For each i (i=1, 2,…, n) the *projection* from $X=X_1\times X_2\times \ldots\times X_n$ to X_i, denoted by

$$p_i: X_1\times X_2\times \ldots\times X_n \to X_i$$

is defined by

$$p_i(x_1, x_2, \ldots, x_n)=x_i.$$

For example, $p_1: X_1\times X_2\times \ldots\times X_n \to X_1$
$P_1(x_1, x_2, \ldots, x_n)=x_1$.

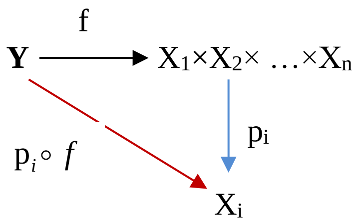
4.1.8 Example

$p_1: R\times R \to R$ sends (x, y) to __________ , and $p_2: R\times R \to R$ sends (x,y) to ________

Exercise Consider $X_1\times X_2\times \ldots\times X_n$.
Let U be a subset of X_i. Show that $p_i^{-1}(U)= X_1\times\ldots\times X_{i-1}\times U\times X_{i+1}\times\ldots\times X_n$.

4.1.9 Theorem Let $(X_1, \tau_1), (X_2, \tau_2), \ldots, (X_n, \tau_n)$ be topological spaces.
For each *i*, the projection from the product space $X_1\times X_2\times \ldots\times X_n$ to X_i is continuous.
Proof (Exercise)

4.1.8 Theorem Let $(X_1, \tau_1), (X_2, \tau_2), \ldots, (X_n, \tau_n)$ be topological spaces and $f: Y\to X$ a function from a space Y to the product space of X_i's. Then f is continuous if and only if for each i, the composition function $p_i\circ f: Y\to X_i$ is continuous.



Proof

Note: The above theorem shows that the product space has the initial topology with respect to the projection functions.

4.2 Quotient spaces

4.2.1 Definition

Let (X, τ) be a topological space and $f: X\to Y$ be an onto function from X to a set Y. Then $\tau_f=\{ V\subseteq Y: f^{-1}(V) \in \tau \}$ is a topology on Y, called the quotient topology induced on Y by f. In this case the space Y is called a quotient space of X and f is called the quotient function.

Exercise

Verify that $\tau_f=\{ V\subseteq Y: f^{-1}(V) \in \tau \}$ is a topology.

4.2.2 Remark
Every quotient function is a continuous function.

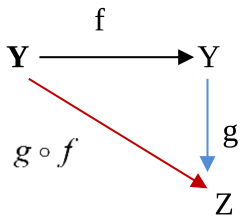
4.2.3 Theorem If X and Y are topological spaces and f: X$\rightarrow$Y is a continuous onto function. Then the topology on Y is the quotient topology τ_f if f is either open or closed.
Proof:

4.2.4 Example Let X=[0, 2π] with the usual topology, and

$$Y=\{(x,y) \in R^2: x^2+y^2=1\}$$

with its usual subspace topology. Define f: X$\rightarrow$Y by f(x)=(cos x, sin y). Then f is continuous, closed and onto. So Y is a quotient space of X.

4.2.5 Theorem Let Y have the quotient topology induced by a function f from X onto Y. Then a function g: Y $\rightarrow$ Z is continuous if and only if the composition g$\circ$f: X$\rightarrow$Z is continuous.



Proof.

Summary

- A base of the product topology
- Each projection function from the product space is continuous
- Quotient topology induced by a onto function
- Properties of quotient space

Exercise 4

1. (a) Show that each projection function from a product space is an open function.
   (b) Let p_1: $R^2 \to R$ be the projection to the ox-axis. Determine if p_1 is a closed function.

2. Show that if Y is a quotient space of X, and Z is a quotient space of Y, then Z is a quotient space of X.

3. Let A and B be subsets of spaces X and Y, respectively.
 (a) Show that cl(A×B)=cl(A) ×cl(B).
 (b) Show that A×B is a closed set of the product space X×Y iff A and B are closed sets of X and Y.

4. Let A and B be subsets of spaces X and Y, respectively.
 Show that int(A×B)=int(A) ×int(B).

5. Show that the function f: X×Y $\to$Y×X is an homeomorphism, where f(x,y)=(y, x) for each (x, y) in X×Y.

6. Let X and Y be disjoint topological spaces and Z=X $\cup$ Y.
Let υ={ U⊆Z: U∩X is open in X and U∩Y is open Y}. Show that υ is a topology on Z.
[The space Z is called the sum of X and Y]

7. Spaces of closed sets.
For any topological space X, let Γ(X) be the set of all non-empty closed subsets of X.
For any open sets $U_1, U_2, \ldots, U_n$ of X, let

$$\mathbf{V}(U_1, U_2, \ldots, U_n) = \{B \in \Gamma(X) : B \subseteq \bigcup_{i=1}^{i=n} U_i \text{ and } B \cap U_i \neq \emptyset \text{ for each } i\}.$$

Show that all $\mathbf{V}(U_1, U_2, \ldots, U_n)$ form a base of a topology on Γ(X); this topology is called the Vietoris topology on Γ(X).

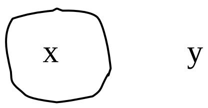
Chapter five: Axioms of separation

Outline:

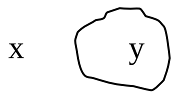
- T_0, T_1 and T_2 spaces
- Convergence in topological space
- Regular spaces
- Normal spaces

5.1 T_0, T_1 and T_2 spaces

5.1.1 Definition(T_0 space)
A topological space X is a T_0 space if for any **two distinct points** x and y in X, there exists an open set containing one and not another.



or



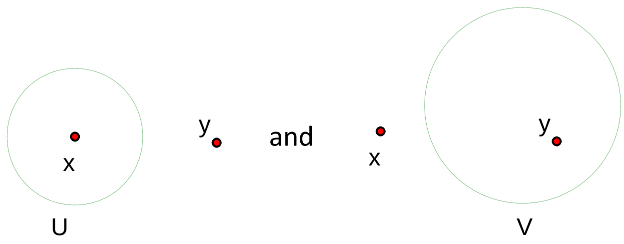
5.1.2 Example

a) Every discrete space is T_0. A indiscrete space containing more than one point is NOT T_0.

b) The Sierpinski space X={0,1} is a T_0 space.

c) The real line R is a T_0 space.
For any two different points a and b(assume $a < b$), the open set U=(a-1, b) contains a but not b.

5.1.3 Definition(T_1 space)
A topological space X is a T_1 space if for any two distinct points x and y in X, there is an open set U containing x but not y **and** an open set V containing y but not x .



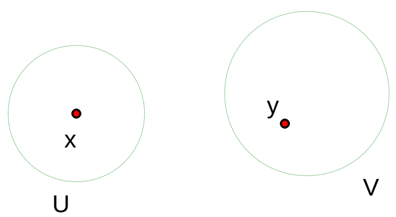
5.1.4 Example

a) Every T_1 space is T_0.
b) The Sierpinski space X={0, 1} is a T_0 space but not T_1.
There is no open set U containing 0 but not 1.

Exercise Prove that if X is a T_1 space and A is a subspace of X, then A is T_1.

5.1.5 Theorem(Properties of T_1 spaces)
a) A space X is T_1 iff cl({x})={x} for any point x in X.
b) Every subspace of a T_1 space is T_1.
c) The product space of two T_1 spaces is T_1.
Proof:

5.1.6 Definition(T_2 spaces)
A space X is a T_2 space (or Hausdorff space) if for any two distinct points x and y in X, there exist **disjoint open sets** U and V such that $x \in$ U and $y \in$ V.



Exercise Show that every T_2 space is T_1.

5.1.7 Example

a) The real line R is T_2.

If $a < b$, then the open sets $U=(a-1, (a+b)/2)$ and $V=((a+b)/2, b+1)$ satisfy the requirement.

b) If (X, d) is a metric space, then for any two distinct points x and y, $U=B(x, a)$, $V=B(y, a)$ are disjoint open sets containing x and y respectively, where $a=1/2d(x, y)$. Thus every metric space is T_2.

5.1.8 Example

Let X=N with the finite complement topology. Then X is T_1 but not T_2. For example, if U is an open set containing x=1 and V be an open set containing y=2. Then X-U and X-V are finite sets, so $X-(U\cap V)=(X-U)\cup(X-V)\neq X$, hence $U\cap V\neq\emptyset$.

5.1.9 Theorem If $f: X\to Y$ is a continuous function and Y is Hausdorff, then

$$\{ (x, y): f(x)=f(y) \}$$

is a closed subset of $X\times Y$.

5.2 Convergence in topological spaces

5.2.1 Example

Let N be the set of all natural numbers and let $\leq$ be the ordinary order of numbers. Then the relation $\leq$ is

i) reflexive (for any n, $n\leq n$),

ii) transitive ($n\leq m$, $m\leq k$ imply $n\leq k$), and

iii) directed (for any two members m and n in N, there is k such that $n, m\leq k$).

b) Let X be a set and D is the set of all finite subsets of X. Then $(D, \subseteq)$ is a directed set.

Let D be a set. We say the set D is directed by relation $\leq$ (or $(D, \leq)$ is a directed set)
If the following conditions are satisfied:

i) $x\leq y\leq z$ imply $x\leq z$; (transitive)

ii) for any x in D, $x\leq x$; (reflexive)

iii) for any x, y in D, there is z in D such that $x\leq z$ and $y\leq z$. (directed)

5.2.2 Example

a) Let X be a set and D be the set of all finite subsets of X. Then (D, $\subseteq$) is a directed set.

b) Let x be a point of a topological space X. The neighbourhood system N(x) of x is a directed set with respect to the inverse inclusion relation $\supseteq$.

c) The set of all partitions of [0, 1] is a directed set, where $D_1 \leq D_2$ for two partitions iff D_2 is finer than D_1 (D_2 has more partition points).

d) Let X={1} and define 1≤1. Then (X, ≤) is a directed set.

5.2.3 Definition(Net and sequence)

A **net** in a topological space X is a function from a directed set Σ into X. We shall use $S=\{x_\sigma : \sigma \in \Sigma\}$ (or $\{x_\sigma\}$) to denote a net in X), where Σ is called the index set of the net. If Σ=N, then the net is called a **sequence**.

5.2.4 Definition(Convergence of nets)

A net $S=\{x_\sigma : \sigma \in \Sigma\}$ in a space X is said to **converge** to a point x in X (or x is a limit of S) if for each neighbourhood U of x, there is a $\sigma_0 \in \Sigma$, such that $x_\sigma \in U$ holds for all $\sigma \geq \sigma_0$. We write $x_\sigma \to x$ (or $S \to x$) to denote the net S converges to x.

The set of all limits of S is denoted by lim S.

A point x is called a cluster point of a net $S=\{x_\sigma : \sigma \in \Sigma\}$, if for each neighbourhood U of x and each $\sigma_0 \in \Sigma$, there exists $\sigma \geq \sigma_0$, such that $x_{\sigma 0} \in U$.

5.2.5 Example

a) Let $x_n = 1 - \frac{1}{n}$, for each n in N. Then $x_n \to 1$ in R.

b) Let X={0, 1} be the Sierpinski space. The net $\{x_1 : 1 \in \{1\}\}$ converges to both point 0 and 1.

So the limits of a net **need not be unique**.

Exercise

Show that a point x is in cl(A) iff for any neighbourhood U of x, U∩A is non-empty.

5.2.3 Theorem A point x is in cl(A) iff there is a net in A that converges to x.

Proof:

5.2.4 Theorem(Net characterization of Continuous functions)
A function $f: X \rightarrow Y$ between two topological spaces is continuous iff for any net $S=\{x_\sigma : \sigma \in \Sigma\}$ in X, $S \rightarrow x$ in X implies $f(S) \rightarrow f(x)$ in Y, where

$$f(S)=\{f(x_\sigma) : \sigma \in \Sigma\} .$$

Proof:

5.2.5 Theorem (Property of hausdorff spaces)
A topological space X is a Hausdorff space if and only if every net in X converges to **at most one point**.

Summary

- A topological space X is a T_0 space if for any two points x and y, ____
- A topological space X is a T_1 space if for any two points x and y, ________________
- A topological space X is a T_2 space if for any two points x and y, ____

- The product of T_0 (T_1, T_2) spaces is T_0 (T_1, T_2) . The converses are also true.

- A function f: X $\to$Y between two topological spaces is continuous iff for any net S in X, S $\to$x in X implies f(S) $\to$f(x) in Y.
- A space X is T_2 if and only if for any net S in X, S$\to$x and S$\to$y imply _____________________

Exercise 5

1. Show that the product X×Y of spaces X and Y is T_2 if and only if both X and Y are T_2.

2. Show that a space X is Hausdorff iff the set diagonal

$$\Delta=\{(x, x): x\in X\}$$

is a closed set of the Cartesian product X×X.

3. The Zariski topology

For a polynomial P in n variables, let

$$K(P)=\{ (x_1, \ldots, x_n) \in R^n : P(x_1, \ldots, x_n)\neq 0\}.$$

a) Show that {K(P): P is a polynomial in n variables } is a base of a topology on R^n. The corresponding topology is called the Zariski topology.

b) Show that the Zariski topology is T_1.

c) Describe the Zariski topology on R. Is it T_2?

4. Let X=R and $\tau=\{ (a, +\infty): a\in R \text{ or } a = -\infty \}$.

a) Show that τ is a topology.

b) Which separation axioms does (X, τ) satisfy?

c) Find a sequence in X that converges to infinite different points.

5. Show that a subspace of a T_2 space is T_2.

6.

a) Let *f*, *g*: X $\to$ Y be continuous functions and Y be a T_2 space, then

$$\{x \mid f(x)=g(x)\}$$

is a closed set of X.

b) A subset A of space X is a **dense set** if cl(A)=X (or A is dense in X).
Use a) to deduce that if f, g: X $\to$ Y are continuous functions and Y is a T_2 space such that f(x)=g(x) for all x in a dense subset A of X, then f=g.

7. Let X be a T_0 space. Define $x\leq y$ for *x*, *y* in X if $x\in cl(\{y\})$. Prove each of the following statements:

i) $x \leq x$ for all x in X (reflexive);
ii) $x \leq y \leq z$ imply $x \leq z$ (transitive);
iii) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetric).

* A binary relation $\leq$ on a set X satisfying the above three conditions is called a partial order X. The partial order proved above is called the **specialization order** on space X.

5.3 Regularity and complete regularity

5.3.1 Definition(Regular space)
A topological space X is a **regular space** if for any closed set A and point x with $x \notin A$, there are disjoint open sets U and V such that $x \in U$ and $A \subseteq V$.
A T_1 regular space is called a T_3 space.

Remark
1) A regular space need not be T_1. For example, every indiscrete space is regular.

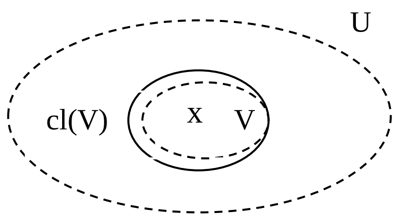
2) Every T_3 space is T_2. This is because for each point y in a T_1 space, A=cl({y})={y}.

Exercise
Let X be a regular space. Show that if A is a closed set which is disjoint from $B = \{b_1, b_2, \ldots, b_n\}$, then there are disjoint open sets U and V containing A and B respectively.

5.3.2 Theorem The followings are equivalent for a topological space X.

a) X is regular.

b) If U is an open set with $x \in U$, then there is an open set V such that
$$x \in V \subseteq cl(V) \subseteq U.$$



Proof

5.3.3 Theorem

a) Every subspace of a regular space is regular.

b) The product X×Y of two spaces is regular if and only if both X and Y are regular.

Proof

Let I=[0, 1] denote the closed unit interval of real numbers with its usual topology.

5.3.4 Definition(Completely regular space)

A topological space X is **completely regular** iff for any closed set A and $b \notin A$, there is a continuous function f: X→I=[0,1] such that f(b)=0 and f(A)={1}.

A T_1 completely regular space is called a **Tychonoff** space (or $T3\frac{1}{2}$ space).

Remark In the definition of complete regular spaces, we can change the condition into:
There is a continuous f: X→R such that f(A)={a}, f(x)=b and $a \neq b$.

Exercise
Show that every complete regular space is regular.

5.3.5 Example
Let (X, d) be a metric space, A be a closed set and $b \notin$ A.
Define f: X→R by

f(y)=d(y, A) for each y in X (see Exercise 6.3).

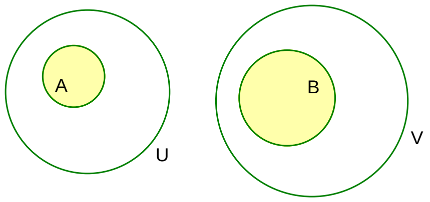
Then f(A)={0} and f(x)≠0. Thus every metric space is completely regular.

In particular, **R**, $\mathbf{R}^2$ and $\mathbf{R}^n$ are completely regular.

5.3.6 Theorem
a) Every subspace of a completely regular space is completely regular.
b) The product of two topological spaces is completely regular iff each factor space is completely regular.

5.4 Normal spaces

5.4.1 Definition(Normal space)
A topological space X is **normal** if for any *two disjoint closed* sets A and B in X, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.
A normal T_1 –space is called a T_4 space.



5.4.2 Examples

(a) Every discrete space is normal.

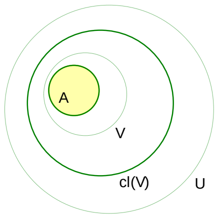
(b) Let A and B be disjoint closed sets in a metric space (X, d). For each x in A and y in B choose δ_x and δ_y with $U(x, \delta_x) \subseteq X - B$ and $U(y, \delta_y) \subseteq X - A$.

Let $U = \bigcup \{U(x, \frac{\delta_x}{3}) : x \in A\}$ and $V = \bigcup \{U(y, \frac{\delta_y}{3}) : y \in B\}$.

Then U and V are disjoint open sets with $A \subseteq U$ and $B \subseteq V$. (Exercise)
Thus *every metric space is normal*.

In particular, **R, R**^{2}, and **R**^{n} are all normal spaces.

5.4.3 Remark A topological space X is normal if for any closed sets A and open set U containing A, there is an open set V, $A \subseteq V \subseteq cl(V) \subseteq U$.



5.4.4 Urysohn's Lemma

A space X is normal iff for any two disjoint closed sets A and B in X, there is a continuous function f: X→[0, 1] such that f(A)={0} and f(B)={1}.

Proof:

5.4.5 Tietze's extension theorem

A space X is normal iff for any closed set and continuous f: A→R, there is an *extension* of f on X; that is there is a continuous F: X→R such that F(x)=f(x) for all x in A.

5.4.6 Remarks

(a) A subspace of a normal space need not be normal.

(b) A product of two normal spaces need not be normal.

Summary

- Regular space, T_3 space
- Completely regular space, Tychonoff space
- Normal space, T_4 space
- Uryson's lemma
- Tiez's extension theorem

Exercise 6

1.

a) Show that the real line **R** with the usual topology is regular.

b) Show that every metric topology is regular.

2. Let X be a regular space. Show that for each closed set A, A is the intersection of all open sets containing A.

Is the converse conclusion true?

3. Verify the Example 5.4.2 (b).

4. Show that every T_4 space is T_3.

5. Prove Remark 5.4.3.

6. Show that every closed subspace of a normal space is normal.

7. Show that if X is regular, then for any point *x* and closed set A that does not contain *x*, there are disjoint open sets U and V containing *x* and A respectively and

$$cl(U)\cap cl(V)=\emptyset.$$

[Hint: Use Theorem 5.3.2 b)]

8.*(Optional) A topological space X is called **completely Hausdorff** if for any two distinct points *x* and y in X, there is a continuous function f: X→I=[0, 1] such that f(*x*)=0 and f(*y*)=1.

(a) Show that every completely Hausdorff space is Hausdorff.

(b) Is every subspace of a completely Hausdorff space a completely Hausdorff space?

Chapter six:
Countability properties

Outline

- First countable spaces
- Second countable spaces
- Separable spaces
- Lindelöff spaces

In this chapter we study some topological properties which are defined by means of countable families of sets.

6.1 First countable spaces

Recall that for a point *x* in a space X, N(*x*) denotes the set of all neighbourhoods of *x*.
A neighbourhood base of *x* is a subset **B** of N(*x*) such that for each U in N(*x*) there is V in **B** so that V is contained in U.

6.1.1 Definition (First Countablility) A space X is called first countable (C1 space) if every point in X has a countable neighbourhood base.

6.1.2 Examples

(a) The real line **R** with the ordinary topology is first countable.

(b) Every metric space is first countable.

6.1. 3 Example
Let X be a non-countable set. Then X with the **finite complement** topology is not first countable.

6.1.4 Proposition Every *subspace* of a first countable space is first countable.

A base **B f**or a topological space (X, τ) is a collection **B** of open sets, such that every member U of τ is a union of some members of **B**.

6.1.5 Definition A space X is second countable (C2 space) if it has a countable base (i.e. there is a base **B**={ U_i: $i \in N$ } consisting of countable number of members).

6.1.6 Example

(a) The real line **R** is second countable. The set **B**={ (r, s): r < s are rational numbers } is a countable base of R.

(b) Let X be a non-countable set and X have the discrete topology, then X is not second countable.

> If there is an surjective (open) continuous mapping f: X→Y from the space X onto space Y, then Y is called a continuous (open) image of X.

[Optional]

6. 1.7 Theorem

(1) A continuous open image of a second countable space is second countable.
(2) Every subspace of a second countable space is second countable.
(3) The product of two second countable spaces X and Y is second countable.

6.2 Separable spaces

6.2.1 Definition
A subset A of a topological space X is called a dense set if cl(A)=X.

6.2.2 Proposition
A subset A is dense in X iff for any nonempty open set U of X, $A \cap \mathbf{U} \neq \emptyset$**.**

6.2.3 Example

(a) The set Q of all rational numbers is dense in the real line **R**.

(b) The set **R - Q** (of all irrational numbers) is also dense in R.

6.2.4 Definition
A topological space X is separable iff X has a **countable dense subset**.

6.2.5 Example
(1) The real line **R** is separable.

(2) If X is a non-countable set and X has the discrete topology, then X is not separable.

6.2.6 Theorem

a) The continuous image of a separable space is separable.

b) An open subspace of a separable space is separable.

6.3 Lindelöff spaces

A collection **U**={ U_j: $j \in J$} of open sets of a space X is called an <u>open cover</u> if the union of all U_j's equals X, i.e. if

$$X = \bigcup \{U_j : j \in J\}.$$

If **U** contains countable U_i, U is called a countable cover.

A subcover **U`** of **U** is a subcollection of U which is also a cover of X.

6.3.1 Definition A space X is called a Lindelöff space if every open cover of X has a *<u>countable subcover</u>*.

6.3.2 Proposition Every closed subspace of a Lindelöff space is Lindelöff.
[Exercise]

6.3.3 Theorem A regular, Lindelöff space is a normal space.

Summary

- A topological space is first countable if every point has a ________
- For example, __________________ are first countable.
- A topological space is second countable if it has a _________________
- Every second countable space is ___________________
- The product of ________ spaces is ___________
- A subspace of a __________ space is ____________
- A space X is a Lindelöff space if _______________________________________

- A _________ and _____________ is normal.

Exercise 7

1. Show that the product $X \times Y$ of spaces X and Y is first countable iff both X and Y are first countable.

2. Prove that every second countable space is first countable.
[Hint: Let **B** be a base of X. For each x in X, consider $\mathbf{B}_x=\{U \in B: x \in U\}$]

3. Show that a subset A of X is dense in X iff for any nonempty open set U in a base **B** of X, $A \cap \mathbf{U} \neq \varnothing$.

4. Show that a discrete space X is separable iff X is a countable set.

5. Let X be second countable and $\mathbf{B}=\{U_i: i \in N\}$ be a countable base of X. Show that X is separable.
[Hint: Choose a point b_i from each U_i, then consider the subset $A=\{b_i: i \in N\}$]

6. Show that the product $X \times Y$ of two separable spaces is separable.
Is the converse also true?
[Hint: Let A and B be countable dense subsets of X and Y. Show $A \times B$ is dense in the product space]

7. Show that if X is second countable, then it is Lindelöff.
[Hint: Let B be a countable base for X. Suppose **U** is any open cover of X. For each U in **U** and x in U, choose some $B_{x,U}$ in **B** such that $x \in B_{x,U} \subseteq U$. Then $B'=\{B_{x,U} : x \in U, U \in \mathbf{U}\}$ is countable because it is a subset of **B**. Assume $B'=\{B_{x1,U1}, B_{x2,U2}, \ldots\}$. Show {U1, U2,… } is a subcover of U]

Chapter 7 : Compactness

Outline:

- Definition and examples
- Tychonoff Theorem
- Continuous functions on compact spaces

7.1 Definition (Compact space)

A topological space X is **compact** if every open cover of X has a finite subcover.

7.2 Example

1) The family **U**={ (n, $+\infty$): n=0, -1, -2, … } is an open cover of real line **R**, but it has no finite subcover. Thus **R** is not compact.

2) The subspace **I**=[0,1] of R is compact. In fact, if $\mathscr{U}$ is an open cover of **I**. Let K be the set of all points c such that a finite subcover of $\mathscr{U}$ covers [0, c]. Then 0 is in K and if *d<c* and c is in K then d is in K. Thus K is an interval. If K=[0, c], then c must equal 1. In fact , assume c<1 we can choose a member U of $\mathscr{U}$ that contains c, then there is $\varepsilon > 0$ such that $c \in (c-\varepsilon, c+\varepsilon) \subseteq U$. Since [0, c] is covered by finite number members of $\mathscr{U}$, $[0, c+\frac{1}{2}\varepsilon]$ is also covered by finite number of members of **U,** so $c+\frac{1}{2}\varepsilon$ is also in K, which contradicts that K=[0, c]. On the other hand, if K=[0, c), let c be contained in a member U_c of $\mathscr{U}$ and $c \in (c-\varepsilon, c+\varepsilon) \subseteq U_c$, then $[0, c-\frac{1}{2}\varepsilon]$ i1s covered by finite numbers of members of $\mathscr{U}$(as $c-\frac{1}{2}\varepsilon$ $c-\frac{1}{2}\varepsilon$ is in K), so [0, c] is also covered by finite numbers of members of $\mathscr{U}$**,** which implies c is in K , a contradiction. All these show that K=[0,1]=I, that is I is covered by a finite number of members of $\mathscr{U}$**.** So I is compact.

3) The subspace E = (0, 1) of I is not compact. The open cover

$$\mathscr{U} = \{ (\tfrac{1}{n}, 1-\tfrac{1}{n}) : n=1,2,3 \ldots\}$$

of E does not have a finite subcover.

Thus a subspace of a compact space need not be compact.

4) Every indiscrete space is compact. A discrete space X is compact iff X is a finite set.

7.3 Definition A family **E** of subsets of X has the <u>finite intersection property</u> if the intersection of any finite numbers of members of **E** is nonempty.

7.4 Example

(1) The family $\{(r, \infty): r \in \mathbf{R}\}$ has the finite intersection property.

(2) The family { A: A is a subset of N and N-A is finite} has the finite intersection property.

(3) The family { (r, s): r< s and r and s are rational numbers } does not have the finite intersection property.

Recall that a **net** $S=\{x_\sigma : \sigma \in \Sigma\}$ in a topological space X is a function from a directed set Σ into X. A point x is called a cluster point of a net $S=\{x_\sigma : \sigma \in \Sigma\}$, if for each neighbourhood U of x and each $\sigma_0 \in \Sigma$, there exists $\sigma \geq \sigma_0$, such that $x_{\sigma 0} \in U$.

7.5 Theorem For a topological space X, the following statements are equivalent.

(1) X is compact.

(2) Every family **E** of closed subsets of X with the finite intersection property has a nonempty intersection.

(3) Every net in X has a cluster point.

A subset A of a topological space is called a <u>compact subset</u> of X if the subspace A is compact. A subset A of X is compact iff any open cover of A has a finite subcover.

7.6 Theorem

(1) Every closed subset of a compact space is compact.

(2) A compact subset of a Hausdorff space is a closed set.

Proof:

7.7 Corollary A subset B of the real line **R** is compact iff B is a bounded (i.e. $B \subseteq [-n, n]$ for some positive number n) closed subset.

Recall that if there is an onto continuous function f: X $\to$Y, then Y is called a continuous image of X.

7.8 Theorem The continuous image of a compact space is compact.

Proof:

7.9 Theorem (Tychonoff) The product of topological spaces is compact iff each factor space is compact.

Let X be a Hausdorff space and A, B be closed subsets of X. Then both A and B are compact subsets of X by Theorem 6.6.

By Exercise 8.2, there are disjoint open sets containing A and B respectively. So we have

7.10 Theorem Every compact Hausdorff space is normal.

Recall that in calculus we learned that every continuous function f: [0, 1] $\rightarrow$R is bounded and f achieves its maximal and minimal values at some points. The following is a more general result.

7.11 Theorem Every continuous real function defined on a compact space is bounded.

Summary

- A topological space X is compact if every open cover of X has a ___________
- X is compact iff every net in X has a ___________
- X is compact iff every family of closed with the ______________ property has _____________ intersection
- Closed subsets of compact space are ________
- Ever compact subset of a Hausdorff space is _________

- The product of spaces is compact iff each ____________
- Every Hausdorff compact space is _____________
- Every continuous real function on a compact space is _________

Exercise 8

1. Let X be a Hausdorff space. Prove that for any compact subset A of X and a point *x* not in A, there are disjoint open sets U and V such that U contains x and V contains A.

2. Let A and B be two disjoint compact subsets of a Hausdorff space X. Show that there are disjoint open sets U and V containing A and B respectively.

3. Let $A \times B$ be a compact subset of $X \times Y$ contained in an open set W of $X \times Y$. Show that there are open sets U of X and open sets V of Y such that $A \times B \subseteq U \times V \subseteq W$.

4. Show that a subset of R^2 is compact iff it is closed bounded.

5. Prove Theorem 6.11.

[Hint: Let f: $X \rightarrow R$. Consider the open cover { $f^{-1}(-n, n)$: $n \in N$ }]

6. Let A and B be two compact subsets of a Hausdorff space X.

(a) Show that $A \cup B$ is compact.

(b) Show that $A \cap B$ is compact.

Chapter 8:
Connectedness of topological spaces

Outline

- Definition, examples and basic properties
- More properties
- Some applications

8.1 Connected spaces

Consider the subspaces of real line R:

$X=[0, 1]$, $Y=[0, 1/2) \cup (1/2, 1]$

Are the subspaces X and Y of R homeomorphism ?

That is, is there a bijection $f : X \to Y$ such that both f and $f^{-1} : Y \to X$ are continuous?

The space Y can be expressed the union of two **disjoint, non-empty open subsets** (closed sets).
But X cannot be expressed as the union of two disjoint, non-empty open subsets.

8.1.1 Definition

A topological space X is called connected if there are no closed subsets F and E such that

(i) $X = F \cup E$;
(ii) $F \cap E = \varnothing$;
(iii) F and E are non-empty.

8.1.2 Example

(1) The real line **R** is connected. (See Appendix 1 for the proof .)
(2) The subspace $I=[0, 1]$ of **R** is connected.
(3) Every indiscrete space is connected, as it has only one non-empty closed set.

8.1.3 Example

(1) The subspace $Y=[0, 1] \cup [3, 4]$ of **R** is not connected.
This is because $F=[0, 1] = Y \cap [-1, 2]$ and $E=[3, 4]= Y \cap [2, 5]$ are non-empty, disjoint closed sets of Y and $Y= F \cup E$.

(2) The subspace **Q** of **R** consisting of all rational numbers is not connected.

Exercise:
Express Q as the union of two disjoint, non-empty closed sets.

8.1.4 Remark
A subspace X of the real line **R** is connected if and only if it is an interval (finite or infinite)

8.1.5 Lemma Let X be a topological space. Then the following statements are equivalent:
1) X is not connected.
2) X is the union of two disjoint, non-empty open sets.
3) There is a non-empty, **proper** subset that is both closed and open.

8.1.6 Definition
A subset A of a topological space X is called a connected subset of X, if A is connected with respect to the subspace topology.

8.1.7 Example
(a) Q is not a connected subset of R.
(b) Every closed interval $[a, b]$ is a connected subset of R.
(c) The square $[0, 1] \times [0, 1]=\{ (x, y): 0 \leq x, y \leq 1 \}$ is a connected subset of $\mathbf{R}^2$

8.2 More properties

8.2.1 Proposition

If f: X→ Y is a continuous function and X is connected, then f(X) is a connected subset of Y.

Proof We prove **by contradiction.**
Assume that f(X) is not connected.
There are open sets U, V of Y such that

$f(X)= (U\cap f(X)) \cup (V\cap f(X)$,

$U' = U\cap f(X)$ and $V' = V\cap f(X)$ are non-empty and disjoint.

Now $X= f^{-1} (U' \cup V')$

$= f^{-1}(U') \cup f^{-1} (V')$

$= f^{-1} (U\cap f(X)) \cup f^{-1} (V\cap f(X))$

$= [f^{-1} (U)\cap f^{-1} (f (X))] \cup [f^{-1} (V) \cap f^{-1} (f(X))]$

$= f^{-1}U)\cap X \cup f^{-1} (V) \cap X$

$= f^{-1} (U) \cup f^{-1} (V)$.

Note that

$f^{-1} (f(X))=X$

f^{-1} (U) and f^{-1} (V) are open sets as f is continuous,
they are non-empty and disjoint.
This contradicts the assumption that X is connected. Hence f(X) must be connected.

8.2.2 Corollary

Let X and Y be connected spaces. For any a in X, $\{a\} \times Y$ is a connected subset of $X\times Y$.
Similarly, $X \times\{b\}$ is connected for any b in Y.

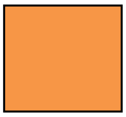
8.2.3 Theorem If X and Y are connected, then the product space $X \times Y$ is connected.

<u>Sketch of the proof</u>:
(i) For any points $A=(x, y)$, $B= (x', y')$, if they have a common component, then there is a connected subset
C(A, B) of the product space containing them

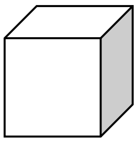
(ii) Fixed a point A =(a, b) . For any B=(z, w),
let D=(b, z). Then C(A, D) and C(D, B) are disjoint connected sets. Then C(A, D) ∪ C(D, B) (denote it by F(A, B)) is a connected set containing A and B.
(iii) Now X× Y = ∪ { F(A, B): B is an arbitrary point in X× Y, is connected by Lemma 2.1

By Induction, we can show the product of any finite number of connected spaces is connected.

8.2.4 Corollary
(1) $\mathbf{R}^2$ and any $\mathbf{R}^n$ are connected spaces.
(2) The square $[0, 1] \times [0, 1]=\{ (x, y): 0 \leq x, y \leq 1 \}$ is connected .



(3) The cub $[0, 1]^3$ is connected.

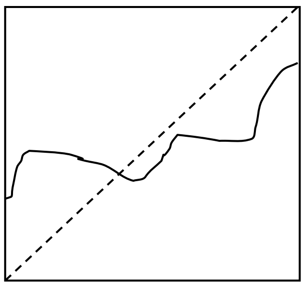


8.3 Some applications

8. 3. 1 Theorem (Intermediate Value Theorem)
If $f: [0, 1] \to \mathbf{R}$ is a continuous function, and m is a number between $f(0)$ and $f(1)$, then there is a $c \in [0, 1]$ such that $f(c) = m$.

8.3.2 Theorem (Fixed point Theorem)

If f: [0, 1] $\rightarrow$ [0, 1] is a continuous function, then f has a fixed point, that is there is x_0 in [0, 1] such that $f(x_0) = x_0$.



There are hundreds different fixed points theorems. The one we proved just now for [0, 1] is called the Brouwer fixed point theorem, named after Luitzen Brouwer. There are many other proofs for this theorem.

The fixed point theorem also true for any closed convex set of $R^{n.}$

See http://en.wikipedia.org/wiki/Brouwer_fixed_point_theorem for more about this.

Appendix:
The real line **R** is connected.

Proof:
Assume that **R** is not connected. Then **R**=A ⋃ B where A and B are non-empty and disjoint closed sets of **R**. Choose a in A and b in B. Then a and b are different points, assume that $a< b$. Let A*'* = A∩[a, b] , B*'* =B∩[a, b]. Then A*'* has an upper bound (e.g. B), so A*'* has a supremum, say b*'*. It can be shown that b*'* is in the closure of A*'*, so b*'* must be in A*'* (as A*'* is closed). Then $b'<b$ (otherwise b=b' is in both A and B). Also (b', b] must be contained in B (otherwise there is a d in (b', b] which is in A*'*, contradicting the assumption of b'). But then cl((b', b])=[b', b] is contained in B*'*, implying b' is in B'. Then A ∩ B is non-empty, a contradiction.

Here we use the property of real numbers:
Every upper (lower) bounded subset has a supremum (infimum).

Summary

- A topological space X is connected if ____________________________
- If f: X→Y is a continuous mapping and X is connected then __________________
- The product of __________________ spaces is _______________________
- A subset A of the real line R is connected if and only if ____________________

Exercise 9

1. Let X be a discrete space. Show that X is connected iff X contains just one element.
   [Hint: If X has more than one point , then X= {a} ∪ (X- {a}).]

2. Show that if A and B are connected subsets of a topological space X such that A ∩ B≠∅ then A∪ B is connected.

3. (The **Intermediate Value Theorem** for connected spaces)
   Let X be connected and f : X → **R** be a continuous real valued function. Assume that *a, b* are points in X such that f(*a*) < m < f(*b*). Show that there is a point c in X such that f(c) = m.
   [Hint: Every connected subset of R is an interval .]

4. Prove that a subspace X of the real line **R** is connected if and only if it is an interval (finite or infinite).

5. Show that if A is a connected subset of X, then cl(A) is also connected. If cl(A) is connected, can we deduce that A must be connected?