Lim-inf convergence in partially ordered sets

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Abstract. The lim-inf convergence in a complete lattice was introduced by Scott to characterize continuous lattices. Here we introduce and study the lim-inf convergence in a partially ordered set. The main result is that for a poset $P$ the lim-inf convergence is topological if and only if $P$ is a continuous poset. A weaker form of lim-inf convergence in posets is also discussed.

1. Introduction

Let $P$ be a partially ordered set (or poset, for short). The Birkhoff-Frink-McShane definition of order-convergence in $P$ is defined as follows (see [2],[4],[10]): A net $(x_i)_{i \in I}$ in $P$ is said to o-converge to $y \in P$ if there exist subsets $M$ and $N$ of $P$ such that

1. $M$ is up-directed and $N$ is down-directed,
2. $y = \sup M = \inf N$, and
3. for each $a \in M$ and $b \in N$, there exists $k \in I$ such that $a \leq x_i \leq b$ hold for all $i \geq k$.

In general, (o)-convergence is not topological; i.e., the poset $P$ may not be topologized so that nets o-converge if and only if they converge with respect to the topology. One basic problem here is: for what posets is the o–convergence topological? Although it has long been known that in every completely distributive lattice the o–convergence is topological, one still hasn't been able to find a satisfactory necessary and sufficient condition for o–convergence to be topological.

In [5] the lim-inf–convergence in complete lattices is introduced. A net $(x_i)_{i \in I}$ in a complete lattice lim-inf-converges to $x$ if $x \leq \lim_{i \in I} x_i = \inf \{ \sup_{i \geq k} \{x_i\} : k \in I \}$. It was proved that for a complete lattice $L$, the lim-inf-convergence is topological if and only if $L$ is a continuous lattice. The notion of continuous lattice was introduced by Dana Scott as a generalization of algebraic lattices and has found its applications in many fields such as computer science, topology and logic. Later on, continuous direct complete posets (or continuous dcpos) was introduced as an appropriate generalization of continuous lattices (see [3],[6],[8],[9]). In this note we consider the lim-inf–convergence in an arbitrary partially ordered set. We prove that the lim-inf–convergence in a poset is topological if and only if the poset is a continuous poset. The definition of continuous poset is similar to that of continuous dcpo.

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We shall also consider another type of lim-inf-convergence, the counterpart of the \(\omega_2\)-convergence studied in [11] and [14], and prove a similar characterization of the poset for which this convergence is topological.

2. \(\text{lim-inf}-\text{convergence and continuous partially ordered sets}\)

A net \((x_i)_{i \in I}\) in a complete lattice is said to \(\text{lim-inf}\)-converge to an element \(x\) if 
\[ x = \lim_{i \in I} x_i = \sup \{ \inf \{ x_i : i \geq k \} : k \in I \} \] (see [3]). Since in a poset the infimum of a subset need not exist, thus we have to define the \(\text{lim-inf}\)-convergence in an arbitrary poset in a different way.

**Definition 1.** A net \((x_i)_{i \in I}\) in a poset \(P\) is said to \(\text{lim} - \text{inf}\)-converge to an element \(y \in P\) if there exists an up-directed subset \(M\) of \(P\) such that

(A1) \(\sup M = \lor M\) exists with \(\lor M \geq y\), and

(A2) for any \(m \in M\), \(x_i \geq m\) holds eventually (that is, there exists \(k \in I\) such that \(x_i \geq m\) for all \(i \geq k\)).

In this case we write \(x \equiv \text{lim} - \text{inf} x_i\).

**Remark 1.** (1) It is easy to find an example to show that the inequality for \(\lor M \geq y\) in (A1) can not be replaced by equality =.

(2) Let \((x_i)_{i \in I}\) be a net in \(P\) such that \(x = \inf \{ x_i : i \in I \}\) exists. The singleton \(A = \{ x \}\) is an up-directed set, \(\sup A = x\) and \(x_i \geq x\) holds for all \(i\), so \((x_i)_{i \in I}\) \(\text{lim-inf}\)-converges to \(x\).

(3) If \((x_i)_{i \in I}\) \(\text{lim-inf}\)-converges to \(x\), then it \(\text{lim-inf}\)-converges to every \(y\) with \(y \leq x\). Thus the \(\text{lim-inf}\)-limits of a net is generally not unique.

For a poset set \(P\), the way-below relation \(\ll\) on \(P\) can be defined in a same way as for dcpos (see [3]). We write \(x \ll y\) if \(D\) is any up-directed set of \(P\) with \(\lor D\) exists and \(y \leq \lor D\), then there is \(d \in D\) such that \(x \leq d\).

From the definition we see easily that if \(x \leq y \ll z \leq w\) then \(x \ll w\), and if \(x \ll y\) then \(x \leq y\).

**Lemma 1.** If \(x\) and \(y\) are two elements of a poset \(P\), then \(x \ll y\) if and only if for any net \((x_i)_{i \in I}\) which \(\text{lim-inf}\)-converges to \(y\), \(x_i \geq x\) holds eventually.

**Proof.** Suppose \(x \ll y\) and \((x_i)_{i \in I}\) \(\text{lim-inf}\)-converges to \(y\). Then there exists an up-directed set \(A\) such that \(y \leq \lor A\) and for each \(a \in A\), \(x_i \geq a\) holds eventually. Since \(x \ll y\), there is \(a \in A\) with \(x \leq a\). Hence \(x_i \geq a \geq x\) holds eventually.

Conversely, suppose the condition is satisfied. If \(D\) is an up-directed subset with \(\lor D \geq y\), then the net \((x_d)_{d \in D}\) \(\text{lim-inf}\)-converges to \(y\), where \(x_d = d\) for each \(d \in D\). By the assumption, there is \(x_d \in D\) such that \(x_d \geq x\). Thus \(x \ll y\).

**Definition 2.** A poset \(P\) is called a continuous poset if for each \(a \in P\), the set \(\{ x \in P : x \ll a \}\) is an up-directed set and \(a = \lor \{ x \in P : x \ll a \}\).

It is easily observed that \(P\) is continuous if and only for each \(a \in P\) there is an up-directed subset \(D\) of \(\{ x \in P : x \ll a \}\) such that \(\lor D = a\). The way-below relation \(\ll\) on a continuous poset is interpolating, i.e. if \(x \ll y\) then there is \(z\) with \(x \ll z \ll y\) (see [5] for the proof of the interpolating property of continuous dcpos).
EXAMPLE 1. For any set $X$, let $\mathcal{P}_0(X)$ be the set of all finite subsets of $X$. Then $(\mathcal{P}_0(X), \subseteq)$ is a continuous poset. This follows from the observation that for each $A \in \mathcal{P}_0(X)$, $A \ll A$. However, $\mathcal{P}_0(X)$ is not direct complete unless $X$ is a finite set.

The lemma below follows from Lemma 1.

**Lemma 2.** If $P$ is a continuous poset, then a net $(x_i)_{i \in I}$ in $P$ lim-inf-converges to $y$ if and only if for each $x \ll y$, $x_i \geq x$ holds eventually.

Let $\mathcal{L}$ be the class consisting of all the pairs $((x_i)_{i \in I}, x)$ of a net $(x_i)_{i \in I}$ and an element $x$ in a poset $P$ with $x \equiv \lim\inf_i x_i$. The class $\mathcal{L}$ is called topological if there is a topology $\tau$ on $P$ such that $((x_i)_{i \in I}, x) \in \mathcal{L}$ if and only if the net $(x_i)_{i \in I}$ converges to $x$ with respect to the topology $\tau$. By Kelley[4], $\mathcal{L}$ is topological if and only if it satisfies the following four conditions:

1. **(CONSTANTS)** If $(x_i)_{i \in I}$ is a constant net with $x_i = x, \forall i \in I$, then $((x_i)_{i \in I}, x) \in \mathcal{L}$.
2. **(SUBNETS)** If $((x_i)_{i \in I}, x) \in \mathcal{L}$ and $(y_j)_{j \in J}$ is a subnet of $(x_i)_{i \in I}$ then $((y_j)_{j \in J}, x) \in \mathcal{L}$.
3. **(DIVERGENCE)** If $((x_i)_{i \in I}, x)$ is not in $\mathcal{L}$, then there exists a subnet $(y_j)_{j \in J}$ of $((x_i)_{i \in I}, x)$ which has no subnet $(z_k)_{k \in K}$ so that $((z_k)_{k \in K}, x)$ belongs to $\mathcal{L}$.
4. **(ITERATED LIMITS)** If $((x_i)_{i \in I}, x) \in \mathcal{L}$, and if $(x_{i,j})_{j \in J(i)}, x_i \in \mathcal{L}$ for all $i \in I$, then $((x_{i,f(i)})_{i,f} \in I \times M, x) \in \mathcal{L}$, where $M = \prod_{i \in I} J(i)$.

**Lemma 3.**

1. For every poset the class $\mathcal{L}$ satisfies the axioms (CONSTANTS) and (SUBNETS).
2. If $P$ is a continuous poset then $\mathcal{L}$ also satisfies the axioms (DIVERGENCE) and (ITERATED LIMITS).

**Proof.**

1. The axiom (CONSTANTS) is clearly satisfied.

2. Suppose now that $((x_i)_{i \in I}, x) \in \mathcal{L}$ and $D$ is up-directed such that $x \ll \forall D$ and for each $a \in D$, $x_i \geq a$ holds eventually. Thus for any subnet $(y_j)_{j \in J}$ of $(x_i)_{i \in I}$ and every $a \in D$, $y_j \geq a$ also holds eventually. Thus $((y_j)_{j \in J}, x) \in \mathcal{L}$.

3. Now assume that $P$ is continuous.

4. Suppose that $((x_i)_{i \in I}, x)$ is not in $\mathcal{L}$. Since the set $D = \{z \in P : z \ll x\}$ is a directed set whose supremum equals $x$, there exists $z \in D$ such that for any $i \in I$ there is a $j \in I$ with $j \geq i$ and $x_j \not\geq z$. Let $J$ be the subset of $I$ consisting of all $k \in I$ such that $x_k \not\geq z$. Then $J$ is co-final in $I$ and $(x_j)_{j \in J}$ is a subnet of $(x_i)_{i \in I}$. In addition, from Lemma 1 it follows that there is no subnet $(z_k)_{k \in K}$ of $(x_j)_{j \in J}$ such that $((z_k)_{k \in K}, x) \in \mathcal{L}$. Hence axiom (DIVERGENCE) is satisfied.

5. Assume $((x_i)_{i \in I}, x) \in \mathcal{L}$ and $((x_{i,j})_{j \in J(i)}, x_i) \in \mathcal{L}$ for every $i \in I$. We show that $((x_{i,f(i)})_{i,f} \in I \times M, x) \in \mathcal{L}$, where $M = \prod_{i \in I} J(i)$. By Lemma 1, it is enough to show that for each $z \ll x$, $x_{i,f(i)} \geq z$ holds eventually. Choose $w$ such that $z \ll w \ll x$. There exists $i_0$ such that $x_i \geq w$ for all $i \geq i_0$. Thus $z \ll x_i$ for all $i \geq i_0$. Again as $(x_{i,j})_{j \in J(i)}$ lim-inf-converges to $x_i$, for each $i \geq i_0$ there exists $g(i) \in J(i)$ such that if $j \in J(i)$ and $j \geq g(i)$ then $x_{i,j} \geq z$. Define $h \in \prod_{i \in I} J(i)$ such that $h(i) = g(i)$ if $i \geq i_0$ and $h(i)$ is any element in $J(i)$.
otherwise. Now if \((i, f) \in I \times M\) and \((i, f) \geq (i_0, h)\), then \(x_{i,f(i)} \geq z\). The proof is complete.

\[\square\]

**Lemma 4.** If, in a poset \(P\), the class \(\mathcal{L}\) satisfies the conditions (**ITERATED LIMITS**) then \(P\) is continuous.

**Proof.** Let \(a \in P\) and let \(D_a = \{\{x_{i,j}\}_{j \in J(i)} : i \in I\}\) be the family of all up-directed subsets of \(P\) whose supremum exist and is above \(a\). For each \(i \in I\), let \(x_i = \vee \{x_{i,j} : j \in J(i)\}\). Then for each \(i \in I\), we have \(x_i \geq a\). Moreover, since \(\{a\}\) is a member of \(D_a\), we have \(\inf \{x_i : i \in I\} = a\). Let the set \(I\) be equipped with the pseudo order, that is \(i \leq k\) holds for any two \(i, k \in I\). Then \((x_i)_{i \in I}\) is a net and, by remark 1(2) it \(\text{lim-inf}–\text{converges to} a\). For each \(\{x_{i,j} : j \in J(i)\} \in D_a\), define an order on \(J(i)\) by \(j_1 \leq j_2\) if \(x_{i,j_1} \leq x_{i,j_2}\). Then \(J(i)\) is a directed set and \((x_{i,j})_{j \in J(i)}\) is a net in \(P\), which obviously \(\text{lim-inf}–\text{converges to} x_i\). Now since the condition (**ITERATED LIMITS**) is satisfied, the net \((x_{i,f(i)})_{(i,f) \in I \times M}\) \(\text{lim-inf}–\text{converges to} a\), where \(M = \prod_{i \in I} J(i)\). By the definition of \(\text{lim-inf}\) limit, there exists an up-directed subset \(D \subseteq P\) such that \(\vee D \geq a\) and for each \(d \in D\), \(x_{i,f(i)} \geq d\) holds eventually. We now show that \(D \subseteq \{x \in P : x << a\}\). Let \(d \in D\). For any directed set \(A \subseteq P\) with \(\forall A \geq a\), \(A = \{x_{m,j} : j \in J(m)\}\) for some \(m \in I\). There exists \((i_d, f_d)\) such that \((i, f) \geq (i_d, f_d)\) implies \(x_{i,f(i)} \geq d\). Now \(m \geq i_d\) (recall the order on \(I\) is the pseudo order), hence \(x_{m,f_d(m)} \geq d\). Note that \(x_{m,f_d(m)} \in A\), thus \(d << a\). Thus \(D\) is an up-directed subset of \(\{x \in P : x << a\}\) and \(\vee D = a\). Hence \(P\) is continuous.

\[\square\]

The combination of Lemma 2 and Lemma 3 deduces the following theorem.

**Theorem 1.** For any poset \(P\) the \(\text{lim-inf}–\text{convergence is topological if and only if} P\) is a continuous poset.

### 3. \(\text{lim- inf}_{f_2}\)–convergence

In [11], the \(o_{2}\)–convergence was considered (In [14] this convergence is called \(2\)–convergence). This convergence is defined by replacing the directed subsets with an arbitrary subset in the definition of order–convergence. We now consider the \(\text{lim- inf}_{f_2}\)–convergence, the counterpart of \(o_{2}\)–convergence for \(\text{lim-inf}–\text{convergence}\). We shall establish a characterization for this convergence to be topological.

**Definition 3.** A net \((x_i)_{i \in I}\) in a poset \(P\) is said to \(\text{lim- inf}_{f_2}\)–converge to \(x \in P\) if there exists a subset \(M \subseteq P\), such that
\[(B1) \ \forall M \text{ exists and } x \leq \vee M, \text{ and}
(B2) \text{ for each } m \in M, x_i \geq m \text{ holds eventually.}\]

Obviously if \((x_i)_{i \in I}\) \(\text{lim-inf}–\text{converges to} x\) then it \(\text{lim- inf}_{f_2}\)–converges to \(x\).

A complete lattice is completely distributive if it satisfies the most general distributivity(see [5]). In [12] Raney established a characterization of completely distributive lattices using the long-below relation “\(\triangleleft\)”: A complete lattice \(L\) is completely distributive if and only if for each \(a \in A\), 
\(a = \vee \{x \in A : x \triangleleft a\}\), where \(x \triangleleft y\) if for any subset \(A\) with \(\forall A \geq y\), there exists \(z \in A\) such that \(x \leq z\).

For any two elements \(x\) and \(y\) in a poset \(P\), we define \(x \triangleleft y\), if for any subset \(A \subseteq P\) with \(\forall A\) exists and \(y \leq \forall A\), there exists \(z \in A\) with \(x \leq z\). A poset \(P\) is called supercontinuous if for each \(a \in P\), \(a = \vee \{x \in P : x \triangleleft a\}\).
Example 2. (1) Every chain \((P, \leq)\) is supercontinuous. In this case, for every \(a \in P\), if \(x < a\) then \(x \triangleleft a\). If \(\{x : x < a\} < a\) then \(a \triangleleft a\). Hence it follows that 
\[ a = \bigvee \{x \in P : x \triangleleft a\}\]
holds for every \(a \in P\).

(2) Given a set \(X\). Let \(\mathcal{P}_0(X)\) be the set of all finite subsets of \(X\). Then 
\((\mathcal{P}_0(X), \subseteq)\) is supercontinuous. Again, \(\mathcal{P}_0(X)\) is generally not a complete lattice. In general, if \(m\) is a cardinal, then \(\mathcal{P}_m(X) = \{A \subseteq X : |A| \leq m\}\) is supercontinuous with respect to \(\subseteq\). This follows from the observation that \(\{x \triangleleft A\}\) holds for every \(x \in A, A \in \mathcal{P}_m(X)\).

(3) If \(L\) is a supercontinuous poset and \(A\) is a down-closed subset of \(L\), such that if \(D \subseteq A\) and \(\forall D\) exists in \(A\) then \(\forall D\) is the supremum of \(D\) in \(L\). Then \(A\) is also a supercontinuous poset.

Although in every poset, \(x \triangleleft y\) implies \(x \ll y\), but a supercontinuous poset need not be a continuous poset.

Example 3. Let \(\mathcal{E}(N) = \{A \subseteq N : |A| \leq 1\ or\ |A| = \infty\}\), where \(N\) is the set of all natural numbers. Then as a subposet of \(\mathcal{P}(N), \mathcal{E}(N)\) is a supercontinuous poset. In fact, for each \(m \in N\) one can easily see that \(\{x\} \triangleleft \{x\}\) and \(A = \bigvee\{x : x \in A\}\) holds for every \(A \in \mathcal{E}(N)\). On the other hand, \(A \ll N\) if and only if \(A\) is a singleton set. But the set of all singleton set is not a direced set, that is the set 
\[\{A \in \mathcal{E}(N) : A \ll N\}\]
is not a directed set. So \((\mathcal{E}(N), \subseteq)\) is not a continuous poset. Notice that this poset is a depo.

Definition 4. Let \(P\) be a poset.

(1) Let \(x, y \in P\). Define \(x \ll_N y\) if for any net \((x_i)_{i \in I}\) which \(\lim_{-}\inf\) converges to \(y\), \(x_i \geq x\) holds eventually.

(2) A poset \(P\) is called \(N\)-continuous if \(a = \bigvee\{x \in P : x \ll_N a\}\) holds for every \(a \in P\).

Remark 2. (1) Obviously every supercontinuous poset is \(N\)-continuous. The converse is not true. Actually it is a easy checking that every lattice is \(N\)-supercontinuous. But a finite lattice is supercontinuous if and only if it is distributive.

One poset is constructed at the end of the paper which is continuous but not \(N\)-continuous.

(2) If \(P\) is \(N\)-continuous, then for each \(a \in P\), 
\[a = \bigvee\{x \in P : \exists z \in P, x \ll_N z \ll_N a\}\]  
This is because \(a = \bigvee\{y \in P : y \ll_N a\}\) and for each \(y \ll_N a\), 
\[y = \bigvee\{x \in P : x \ll_N y\}\].

Lemma 5. If \(P\) is a complete lattice, then \(x \ll y\) if and only if \(x \ll_N y\).

Proof. Suppose \(x \ll y\) and \((x_i)_{i \in I}\) is a net that \(\lim_{-}\inf\) converges to \(y\). It then follows that 
\[\sup\{\inf\{x_i : i \geq k\} : k \in I\} \geq y\].
Since \(\{\inf\{x_i : i \geq k\} : k \in I\}\) is a directed set and \(x \ll y\), there exists \(k_0 \in I\) such that \(\inf\{x_i : i \geq k_0\} \geq x\).
So \(x_i \geq x\) holds for all \(i \geq k_0\). Thus \(x \ll_N y\).

The converse implication is true for every poset.

\[\square\]

Notice that if \(L\) is a complete lattice, then \(\{x \in L : x \ll a\}\) is a directed set for every \(a \in L\). Thus it follows that a complete lattice is continuous if and only if it is \(N\)-continuous.
Now let $S$ be the class consisting of all pairs $((x_i)_{i \in I}, x)$, where $(x_i)_{i \in I}$ is a net that $\lim - \inf_{f_2}$ converges to $x$. Again one can show that for any poset $P$, the class $S$ satisfies the axioms (CONSTANTS) and (SUBNETS).

**Proposition 1.** If $P$ is $N$-continuous, then the class $S$ satisfies the axioms (DIVERGENCE) and (ITERATED LIMITS).

**Proof.** (DIVERGENCE) Suppose that $((x_i)_{i \in I}, x)$ is not in $S$. Since $\forall \{y \in P : y <<_N x\} = x$, there is $y <<_N x$ such that $x_i \geq y$ does not hold eventually. Put $J = \{i \in I : x_i \not\geq y\}$. Then $(x_j)_{j \in J}$ is a subnet of $(x_i)_{i \in I}$ which has no subnet $(z_k)_{k \in K}$ $\lim - \inf_{f_2}$-converging to $x$.

(ITERATED LIMITS) Suppose $(x_i)_{i \in I}$ $\lim - \inf_{f_2}$-converges to $x$, and for each $i \in I$, $(x_{i,j})_{j \in J(i)}$ $\lim - \inf_{f_2}$ converges to $x_i$. By Remark 2(2), $x = \bigvee \{y \in P : \exists z \in P, y <<_N z <<_N x\}$. Thus in order to show that the net $(x_{i,f(i)})_{i \in I}$ $\lim - \inf_{f_2}$-converges to $x$, it is enough to verify that if $y <<_N z <<_N x$, then $x_{i,f(i)} \geq y$ holds eventually. But this is similar to the proof of the case for $\liminf$-convergence, so we omit it. 

**Lemma 6.** If $P$ is a poset such that the class $S$ satisfies the axiom (ITERATED LIMITS) then $P$ is $N$-continuous.

**Proof.** The proof is quite similar to that of Lemma 4. For any $a \in P$, consider the collection $\{(x_{i,j})_{j \in J(i)} : i \in I\}$ of nets $(x_{i,j})_{j \in J(i)}$ that $\lim - \inf_{f_2}$-converges to $a$. Let $(x_i)_{i \in I}$ be the constant net in which $x_i = a$, $\forall i \in I$. So for each $i \in I$, $(x_{i,j})_{j \in J(i)}$ $\lim - \inf_{f_2}$ converges to $x_i$. Thus by the assumption, the net $(x_{i,f(i)})_{i \in I} \times M$ will $\lim - \inf_{f_2}$-converge to $a$, where $M = \Pi_{i \in I} J(i)$ and $I$ has the pseudo order. Thus there is a subset $A$ of $P$ such that $x_{i,f(i)} \geq y$ holds eventually for any $y \in A$. Then one can check that $\forall A = a$ and $A \subseteq \{x \in P : x <<_N a\}$. Thus $P$ is $N$-continuous.

**Theorem 2.** For a poset $P$, the class $S$ is topological if and only if $P$ is $N$-continuous.

**Remark 3.** Suppose $P$ is a lattice and $(x_i)_{i \in I}$ $\lim - \inf_{f_2}$-converges to $x$. Then there is a subset $M$ of $P$ with $\forall M \geq x$ and for each $m \in M$, $x_i \geq m$ holds eventually. Put $K = \{\forall D : D$ is a finite subset of $M\}$. Then $K$ is up-directed and for each $k \in K$, $x_i \geq k$ holds eventually. Hence $(x_i)_{i \in I}$ $\liminf$-converges to $x$. Hence in a lattice the two convergences are equivalent.

The following is an example of a poset in which the two convergences are not equivalent.

**Example 4.** The following example is a moderation of one in [8]. Let $P = \{T\} \cup \{a_1, a_2, \cdots\} \cup \{b_1, b_2, \cdots\}$. The order $\leq$ on $P$ is defined as follows:
1. $a_i < T, b_i < T$ for all $i = 1, 2, 3, \cdots$;
2. if $k \geq i$ then $a_k \geq b_i$.

By definition, if $i \not= j$ then $a_i$ and $a_j$ are incomparable and $b_i$ and $b_j$ are incomparable, and $T$ is the top element. Let $B = \{b_1, b_2, \ldots\}$. Then clearly $\forall B = T$. Since for each $b_i \in B$, $a_n \geq b_i$ whenever $n \geq i$, thus the net $(a_i)_{i \in \mathbb{N}}$ $\lim - \inf_{f_2}$-converges to $T$. However $(a_i)_{i \in \mathbb{N}}$ does not $\liminf$-converge to $T$ because there exists no up-directed set $D$ with $\forall D = T$ and for each $d \in D$, $a_i \geq d$ holds eventually.

One can check easily that $T << T$ and $a_i << a_i, b_i << b_i$ for all $i$. Thus $P$ is a continuous poset (actually a continuous dcpo).
On the other hand this \( P \) serves also as an example of poset which is continuous but not \( N \)-continuous. In fact, consider the element \( a_1 \) of \( P \). Since the net \( (a_i)_{i \in \mathbb{N}} \) \( \lim - \inf\) converges to \( T \) so \( \lim - \inf \) converges to \( a_1 \) as well. But \( a_i \geq a_1 \) does not hold eventually, thus \( a_1 \prec \prec_N a_1 \) does not hold. The only element \( x \) satisfying \( x \prec \prec_N a_1 \) is \( b_1 \). So \( \forall \{x \in P : x \prec \prec_N a_1 \} = b_1 \neq a_1 \), hence \( P \) is not \( N \)-continuous.

References

3. Erne,M., Scott convergence and Scott topologies on partially ordered sets II, in [1], pp.61-96.
6. Hoffmann,R.E., Continuous posets-prime spectra of completely distributive lattices, and Hausdorff compactification, in [1], pp.159-208.