Dcpo-completion of Posets
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Abstract. We introduce a new type of dcpo-completion of posets, called D-completion. For any poset \( P \), the D-completion exists, and \( P \) and its D-completion have the isomorphic Scott closed set lattices. This completion is idempotent. A poset \( P \) is continuous (algebraic) if and only if its D-completion is continuous(algebraic). Using the D-completion, we construct the local dcpo-completion of posets, that revises the one given by Mislove. In the last section, we define and study bounded sober spaces.

The dcpo-completion of a poset \( P \) is usually taken to be the ideal completion \( \text{Id}(P) \) consisting of all ideals of \( P \). As has been pointed out by several authors [13], [5], the ideal completion is not an idempotent completion. Furthermore, the assignment of \( \text{Id}(P) \) to poset \( P \) cannot be extended to a functor from the category \( \text{POS}_d \) of posets and Scott continuous mappings to the full subcategory \( \text{DCPO} \) of directed complete posets. On the other hand, given a poset \( P \), the set \( \Gamma(P) \) of all Scott closed sets forms a complete lattice. Different classes of posets yield different Scott closed set lattices. One of the most eminent classical results in domain theory is that a dcpo \( P \) is continuous if and only if \( \Gamma(P) \) is a completely distributive lattice(see [6],[11] ). Banaschewski characterized the Scott open set lattices of continuous lattices as the stably supercontinuous lattices [2]. In [7], [3], characterizations of the Scott closed set lattices of complete lattices as well as bounded complete posets are also obtained. However, very little is known about the properties of Scott closed set lattices for more general posets. One of the basic problems on Scott closed set lattices is: Do posets and dcpos define the same class of Scott closed set lattices?

In this paper we introduce a new type of dcpo-completion of posets, called the D-completion. This completion has the following properties: (i) it is idempotent and can be extended to a functor from \( \text{POS}_d \) to \( \text{DCPO} \) which is left adjoint to the forgetful functor; (ii) every poset \( P \) and its D-completion have isomorphic Scott closed set lattices, which gives a positive answer to the above problem; (iii) a poset \( P \) is continuous if and only if its D-completion is continuous; (iv) if \( P \) is a continuous noncomplete lattice then its D-completion is a complete lattice. In [13], Mislove defined a local dcpo \( BSpec(P) \) for each poset \( P \), and claimed that

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BSpec$(P)$ is the reflection of the poset $P$ in the subcategory $LDCPO$ of local dcpos. We shall point out that, although the poset $BSpec(P)$ is indeed a local dcpo, it is generally not the reflection of $P$ in $LDCPO$. We shall define a local dcpo $B(P)$, using the D-completion, for each poset $P$ and show that it is the actual reflection of $P$ in $LDCPO$. We will also introduce and study a generalized sobriety of topological spaces.

1. Preliminaries

In this section we recall the basic definitions and results on posets that will be used later.

A poset $P$ is a directed complete poset, or dcpo for short, if every directed subset of $P$ has a supremum in $P$.

If $X$ is a subset of a poset $P$, then $\downarrow X = \{ y \in P : y \leq x \text{ for some } x \in X \}$ and dually, $\uparrow X = \{ y \in P : y \geq x \text{ for some } x \in X \}$. A subset $X$ is called a lower set if $X = \downarrow X$.

A subset $X$ of a poset $P$ is Scott closed if i) $X = \downarrow X$ and ii) for any directed subset $D$, $D \subseteq X$ implies $\bigvee D \in X$ whenever $\bigvee D$ exists. The complements of Scott closed sets are called Scott open sets, such subsets of a poset $P$ form a topology on $P$, denoted by $\sigma(P)$. The set of all Scott closed subsets of $P$ will be denoted by $\Gamma(P)$. A mapping $f : P \to Q$ between posets is Scott continuous if for any directed subset $D \subseteq P$ with $\bigvee D$ existing in $P$, $f(\bigvee D) = \bigvee f(D)$ holds. It is well known that if $P$ and $Q$ are dcpos, then $f$ is Scott continuous if and only if it is a continuous mapping between the spaces $(P, \sigma(P))$ and $(Q, \sigma(Q))$. The above equivalence is also true for mappings between arbitrary posets. See [6] for more related results on posets, Scott topology and Scott continuous mappings.

Let $POS_d$ denote the category of all posets and Scott continuous mappings. Since every Scott continuous mapping is monotone, $POS_d$ is a subcategory of the category $POS$ of posets and monotone mappings. The full subcategory of $POS_d$ consisting of dcpos is denoted by $DCPO$.

An element $a$ of a poset $P$ is called a join-prime element if $x \lor y \geq a$ implies either $x \geq a$ or $y \geq a$ whenever $x \lor y$ exists. We use $Spec(P)$ to denote the set of all join-prime elements of $P$ excluding the bottom element of $P$ (if it exists).

The following properties of $Spec(P)$ can be verified straightforwardly.

**Lemma 1.** (i) For any poset $P$, $Spec(P)$ is closed under existing joins of directed sets, that is, for any directed subset $D$ of $Spec(P)$, $\bigvee D \in Spec(P)$ whenever it exists in $P$.

(ii) For any complete lattice $L$, $Spec(L)$ is a dcpo under the inherited order from $L$.

In the following, for any subset $X$ of a poset, we shall use $cl(X)$ to denote the closure of $X$ with respect to the Scott topology.

The following lemma is a special case of Exercise O-5.15 (vi) of [6].

**Lemma 2.** For any directed subset $D$ of a poset $P$,

$$cl(D) \in Spec(\Gamma(P)).$$
A monotone mapping \( f : P \to Q \) between posets is a lower adjoint of a monotone mapping \( g : Q \to P \) if for any \( x \in P \) and \( y \in Q \), \( f(x) \leq y \) if and only if \( x \leq g(y) \) (the mapping \( g \) is then called the upper adjoint of \( f \)). Every lower adjoint preserves existing joins and every upper adjoint preserves existing meets (see [6]). If \( f : L \to M \) is a mapping between complete lattices that preserves arbitrary meets, then \( f \) is a lower adjoint [6].

From the properties of upper adjoints we can deduce the following easily.

**Lemma 3.** If \( f : P \to Q \) has an upper adjoint that preserves existing finite joins, then \( f(\text{Spec}(P)) \subseteq \text{Spec}(Q) \).

## 2. D-completion of posets

In this section we define a new type of dcpo completion of posets. This completion is idempotent and the Scott closed set lattice of the completion of \( P \) is isomorphic to the Scott closed set lattice of \( P \).

A subset \( X \) of a dcpo \( P \) is called a subdcpo of \( P \) if for any directed subset \( D \subseteq X \), \( \bigvee D \in X \). The empty set and \( P \) itself are subdcpos. Every Scott closed set of \( P \) is a subdcpo of \( P \). The union of two subdcpos is a subdcpo and the intersection of any collection of subdcpos is a subdcpo.

A subset \( A \) of a poset \( P \) is called D-closed if for any directed subset \( D \subseteq A \), if \( \bigvee D \) exists then \( \bigvee D \in A \). The set of all D-closed sets of \( P \) forms the set of all closed sets of a topology on \( P \), which will be called the D-topology.

If \( X \) is a subset of \( P \), we denote by \( \text{cl}_d(X) \) the closure of \( X \) with respect to the D-topology.

The following lemmas can be verified straightforwardly:

**Lemma 4.** Let \( P \) be a poset. Then

1. every Scott closed set of \( P \) is D-closed;
2. every lower set of \( P \) is D-open;
3. a non-empty subset of a dcpo is D-closed if and only if it is a subdcpo;
4. for any directed set \( D \) of \( P \), if \( \bigvee D \) exists, then the directed set \( D \) converges (as a net) to \( \bigvee D \) with respect to the D-topology.

**Lemma 5.** Let \( f : P \to Q \) be a function between posets. Then the following are equivalent:

1. \( f \) is Scott continuous;
2. \( f \) is continuous with respect to the Scott topology;
3. \( f \) is monotone and continuous with respect to the D-topology.

**Corollary 1.** If \( f : P \to Q \) is a Scott continuous function between posets, then for any \( X \subseteq P \), \( f(\text{cl}_d(X)) \subseteq \text{cl}_d(f(X)) \).

By Lemma 1, for any poset \( P \), \( \text{Spec}(P) \) is D-closed.

**Remark 1.** Note that if in a topological space \( Y \), \( y \in \text{cl}(A) \) for some \( y \in Y \) and \( A \subseteq Y \), then for any open set \( U \) containing \( y \), one has \( y \in \text{cl}(U \cap A) \). Now let \( X \) be a subset of a poset \( P \) and \( p \in \text{cl}_d(X) \). Since \( \downarrow p \) is D-open and contains \( p \), it follows that \( p \in \text{cl}_d(\downarrow p \cap X) \).
LEMMA 6. If $X$ is a subset of a poset $P$ and $f, g : \text{cl}_d(X) \to Q$ are Scott continuous mappings into a poset $Q$ such that $f|_X = g|_X$, then $f = g$.

**Proof.** It is enough to note that if $f, g$ are Scott continuous, then $\{x \in \text{cl}_d(X) : f(x) = g(x)\}$ is $D$-closed.

**Definition 1.** A D-completion of a poset $P$ is a dcpo $A$ together with a Scott continuous mapping $\eta : P \to A$, such that for any Scott continuous mapping $f : P \to B$ into a dcpo $B$ there exists a unique Scott continuous mapping $\hat{f} : A \to B$ satisfying $f = \hat{f} \circ \eta$.

By a standard argument, it can be shown that if the $D$-completion of $P$ exists, it is unique up to isomorphism. We shall use $E(P)$ to denote the $D$-completion of $P$ if it exists.

**Theorem 1.** Let $P$ be a poset. Then $\text{cl}_d(\Psi(P))$, which is the smallest subdcpo of $\Gamma(P)$ containing $\Psi(P) = \{\downarrow x : x \in P\}$, is a $D$-completion of $P$.

**Proof.** Define $\eta : P \to \text{cl}_d(\Psi(P))$ by $\eta(x) = \downarrow x$ for all $x \in P$. Then $\eta$ is Scott continuous. Suppose that $f : P \to B$ is any Scott continuous mapping into a dcpo $B$. The mapping $f^{-1} : \Gamma(B) \to \Gamma(P)$, sending $X \in \Gamma(B)$ to $f^{-1}(B)$, preserves arbitrary meets and finite unions. Let $f^* : \Gamma(P) \to \Gamma(B)$ be the mapping that maps $X \in \Gamma(P)$ to $\text{cl}(f(X))$. Then $f^*$ is the lower adjoint of $f^{-1}$. Thus $f^*$ preserves arbitrary joins, and so it is Scott continuous. By Lemma 2, $f^*(\text{cl}_d(\Psi(P))) \subseteq \text{cl}_d(f^*(\Psi(P)))$. For each $x \in P$, $f^*(\downarrow x) = \downarrow f(x) \in \Psi(B)$. Since $B$ is a dcpo, $\text{cl}_d(\Psi(B)) = \Psi(B)$. Also the mapping $k : \Psi(B) \to B$ defined by $k(\downarrow y) = y$ is an isomorphism. Let $\hat{f} = k \circ f^*$. Then $\hat{f}$ restricts to a Scott continuous mapping from $\text{cl}_d(\Psi(P))$ to $B$ satisfying $f = \hat{f} \circ \eta$. If $g : \text{cl}_d(\Psi(P)) \to B$ is any Scott continuous mapping satisfying $f = g \circ \eta$, then for each $x \in P$, $g(\downarrow x) = g(\eta(x)) = f(x)$, thus for each $C \in \Psi(P)$, $g(C) = f(C)$. By Lemma 4, $g = \hat{f}$.

**Corollary 2.** The subcategory DCPO of POS$_d$ consisting of all dcpos is reflexive in POS$_d$.

By the definition of D-completion, if $P$ is a dcpo then $P \cong E(P)$. Hence the D-completion of posets is idempotent.

**Example 1.** (1) Let $P = \mathbb{Q}$ be the set of all rational numbers with the ordinary order of real numbers. Then $\Gamma(P) \setminus \{\emptyset\}$ is isomorphic to the poset $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ of all real numbers with the top element $\infty$ adjoined. Furthermore, $E(P) = \Gamma(P) \cong \mathbb{R}^*$, which is different from the set $\text{Id}(P)$ of all ideals of $P$.

In general, let $P$ be a chain, then $E(P) \cong \mathcal{D}_0(P)$, where $A \in \mathcal{D}_0(P)$ if and only if it is a lower set and either $\bigvee A$ does not exist in $P$, or $A = \downarrow a$ when $\bigvee A = a$ exists in $P$.

(2) Let $X$ be a nonempty set and $\mathcal{P}_0(X)$ be the set of all finite subsets of $X$. Then $\mathcal{P}_0(X)$ is a poset under the order of inclusion, which is not a dcpo except when $X$ is a finite set. Note that a directed subset of $\mathcal{P}_0(X)$ has a supremum
only when it is a finite set. Thus $\Gamma(P_0(X)) = D(P_0(X))$, the set of all lower sets of $P_0(X)$. It is easy to see that $E(P_0(X)) = Id(P_0(X))$, the set of all ideals of $P_0(X)$.

(3) Let $P = P_N(X)$ be the set of all countable subsets of $X$. Then $P$ is not a dcpo under the order of inclusion unless $X$ is a countable set. If $X$ is not countable, then $\Gamma(P) \neq Id(P)$.

3. The Scott closed set lattice of a D-completion

One of the most important theorems in domain theorem, originally proved independently by Lawson [11] and Hoffmann [8], is that a dcpo $P$ is a continuous dcpo if and only if the lattice $\Gamma(P)$ is a completely distributive lattice. Later, Banaschewski [2] proved that $P$ is a continuous lattice if and only if $\sigma(P)$ is a stably supercontinuous lattice. In [7], [3], the Scott closed set lattices of complete lattices as well as bounded complete posets are also obtained. It follows that the subcategories CLAT of complete lattices, $BCPOS$ of bounded complete posets, $ConCLAT$ of continuous lattices, and ConPOS of continuous posets yield different classes of complete lattices under the mapping $P \mapsto \Gamma(P)$. Now when we extend the mapping $P \mapsto \Gamma(P)$ from dcpos to arbitrary posets $P$, a natural question arising is: would we obtain a larger class of Scott closed set lattices?

Recall that if $f : P \to Q$ is the lower adjoint of a mapping $g : Q \to P$, then $f$ is injective(surjective) if and only if $g$ is surjective (resp. injective) (see [6, Proposition O-3.7]).

**Lemma 7.** Suppose that $A$ is a subset of a poset $P$ such that $P = cl_d(A)$. Then for any $F \in \Gamma(P)$, $F = cl(F \cap A)$.

**Proof.** Since each $F \in \Gamma(P)$ is a lower set, by Lemma 3, it is D-open. As $A$ is dense in $P$ with respect to the D-topology, $A$ is dense in every D-open set, thus $cl_d(F \cap A) = F$. But $cl_d(F \cap A) \subseteq cl(F \cap A) \subseteq F$, thus $F = cl(F \cap A)$.

**Theorem 2.** If $E(P)$ is a D-completion of poset $P$, then $\Gamma(P)$ is isomorphic to $\Gamma(E(P))$.

**Proof.** Let $\eta : P \to E(P)$ be the Scott continuous embedding of $P$ into its D-completion. The mapping $\eta^* : \Gamma(P) \to \Gamma(E(P))$ which sends $X \in \Gamma(P)$ to $cl(\eta(X))$ is a Scott continuous mapping and it is the lower adjoint of the mapping $\eta^{-1} : \Gamma(E(P)) \to \Gamma(P)$. We show that $\eta^*$ is an isomorphism. By the proof of Theorem 1, $cl_d(\eta(P)) = E(P)$. Thus by Lemma 7, for each $F \in \Gamma(E(P))$, $F = cl(F \cap \eta(P))$. Let $G = \eta^{-1}(F)$. Then, as $\eta$ is Scott continuous, $G \in \Gamma(P)$ and $\eta^*(G) = cl(\eta(G))$. Note that $\eta(G) = \eta(\eta^{-1}(F)) = F \cap \eta(P)$, so $cl(\eta(G)) = cl(F \cap \eta(P)) = F$. Thus $\eta^*$ is surjective. For each $A \in \Gamma(P)$, the function $j : P \to 2$ into the two elements chain with $j^{-1}(\{0\}) = A$ is Scott continuous, so there is a Scott continuous mapping $h : E(P) \to 2$ such that $h \circ \eta = j$. Then $A = j^{-1}(\{0\}) = \eta^{-1}(h^{-1}(\{0\}))$, where $h^{-1}(\{0\}) \in \Gamma(E(P))$. Thus $\eta^{-1} : \Gamma(E(P)) \to \Gamma(P)$ is surjective. Therefore, the lower adjoint $\eta^*$ of $\eta^{-1}$ is
injective. It follows that the mapping \( \eta^* \) is an isomorphism between \( \Gamma(P) \) and \( \Gamma(E(P)) \). The proof is thus completed.

**Remark 2.** We say that a subcategory \( \mathcal{C} \) of \( \text{POS}_d \) is \( \Gamma \)-faithful if for any two posets \( P \) and \( Q \) in \( \mathcal{C} \), \( \Gamma(P) \cong \Gamma(Q) \) implies \( P \cong Q \). The category \( \text{ConDCP} \) of all continuous dcpos is \( \Gamma \)-faithful. This follows from the fact that if \( P \) is a continuous dcpo, then \( P \) is isomorphic to the poset \( \text{Spec}(\Gamma(P)) \) of join-prime elements of the completely distributive lattice \( \Gamma(P) \) (see [6]). The category \( \text{BCDCPO} \) of bounded complete dcpos is \( \Gamma \)-faithful [7].

Since each poset \( P \) and its D-completion have isomorphic Scott closed set lattices, the category \( \text{POS}_d \) is not \( \Gamma \)-faithful. As far as the authors know, it is still not known whether the category \( \text{DCPO} \) of all dcpos is \( \Gamma \)-faithful.

Although \( \Gamma(P) \) and \( \Gamma(E(P)) \) are isomorphic, the two spaces \( (P,\sigma(P)) \) and \( (E(P),\sigma(E(P))) \) could be very different. Recall that a topological space \( X \) is sober if it is \( T_0 \) and for each nonempty irreducible closed set \( F \) (join-prime element of the lattice of all closed sets) there is an element \( x \) such that \( F = \text{cl}\{x\} \).

Let \( P = \{ r \in Q : 0 \leq r < 1 \} \) be the poset of all rational numbers which are greater than or equal 0 and less than 1. Then \( E(P) \) is a complete chain, which is therefore a continuous lattice. Thus \( (E(P),\sigma(E(P))) \) is sober. The set \( A = \{ x \in P : 0 \leq x < 1 \} \) is a join-prime element of \( \Gamma(P) \) but there is no \( x \in P \) satisfying \( \text{cl}(\{x\}) = A \). Thus \( (P,\sigma(P)) \) is not sober.

**4. Properties invariant under the D-completion**

The category \( \text{POS}_d \) has many different subcategories, such as \( \text{MSLAT} \) of meet-semilattices, \( \text{LAT} \) of lattices, and \( \text{ConPOS}_d \) of all continuous posets. In this section we shall examine which order properties are invariant under the D-completion.

A complete lattice \( L \) is a frame if

\[
a \land \bigvee B = \bigvee \{ a \land x : x \in B \}
\]

holds for every \( a \in L \) and \( B \subseteq L \) (see [10] for more about frames).

A semilattice (or meet-semilattice) \( P \) is a poset that has a top element, such that \( x \land y \) exists for any two elements \( x, y \) in \( P \).

A semitopological semilattice is a topological space \( X \) which is also a semilattice such that for any \( s \in X \), the mapping \( x \mapsto s \land x \) is continuous.

A semilattice \( S \) is called meet-continuous if

\[
x \land \bigvee D = \bigvee \{ x \land y : y \in D \}
\]

holds for every directed subset \( D \) whose join exists. This is equivalent to the property that \( (P,\sigma(P)) \) is a semitopological semilattice. It is easy to verify that every semilattice that is a continuous poset is meet-continuous.

By Exercise O-3.23 of [6] we have

**Lemma 8.** If \( P \) is a meet-continuous semilattice, then \( \Gamma(P) \) is a frame.
Let $X$ be a semitopological semilattice and $A$ a subsemilattice of $X$ (i.e. $A$ contains the top element and is closed under finite meets). For any $x \in A$, let $f_x : X \to X$ be defined by $f_x(y) = x \land y, y \in X$. Then $f_x$ is continuous, so $f_x(\text{cl}(A)) \subseteq \text{cl}(f_x(A)) = \text{cl}(A)$. Now for any $u \in \text{cl}(A)$, the mapping $f_u : X \to X$ is continuous and, by the above argument, $f_u(A) \subseteq \text{cl}(A)$. In addition, $f_u(\text{cl}(A)) \subseteq \text{cl}(f_u(A)) \subseteq \text{cl}(\text{cl}(A)) = \text{cl}(A)$. All these show that $\text{cl}(A)$ is a subsemilattice of $X$.

**Lemma 9.** If $P$ is a meet-continuous semilattice, then $E(P)$ is a sub meet-semilattice of $\Gamma(P)$.

**Proof.** Note that $E(P)$ is the D-closure of the subset $\Phi(P)$ of $\Gamma(P)$ and $E(P)$ is a subsemilattice of $\Gamma(P)$. By Lemma 8, $\Gamma(P)$ is a frame, it is thus a semitopological semilattice with respect to the D-topology. Now it is enough to invoke the general fact proved before Lemma 9.

**Theorem 3.** Let $P$ be a semilattice. Then the following conditions are equivalent:

1. $P$ is meet-continuous;
2. $E(P)$ is meet-continuous;
3. $\Gamma(P)$ is a frame.
4. $\Gamma(E(P))$ is a frame.

**Proof.** By Lemma 8, (1) implies (3) and (2) implies (4).

The equivalence of (3) and (4) is a result of Theorem 2.

Suppose that $\Gamma(P)$ is a frame. For any $a \in P$ and any directed set $D \subseteq P$ with $\bigvee D$ exists in $P$, it holds that

$$\downarrow a \cap \bigvee \{\downarrow d : d \in D\} = \bigvee\{\downarrow a \cap \downarrow d : d \in D\},$$

which implies that $\downarrow (a \land \bigvee D) = \downarrow a \cap \bigvee \{\downarrow d : d \in D\} = \bigvee\{\downarrow a \cap \downarrow d : d \in D\} = \bigvee\{a \land d : d \in D\}$. It thus follows that $a \land \bigvee D = \bigvee\{a \land d : d \in D\}$. So $P$ is meet-continuous. Thus (3) implies (1). Hence (1), (3) and (4) are equivalent.

If (3) is valid, then (1) is valid, thus by Lemma 9, $E(P)$ is a subsemilattice of $\Gamma(P)$. Since $E(P)$ is also closed under directed joins in $\Gamma(P)$, $E(P)$ is a meet-continuous semilattice.

Thus all the four conditions are equivalent.

Let $P$ be a poset. For two elements $a$ and $b$ in $P$, we say that $a$ is way-below $b$, denoted by $a << b$, if for any directed subset $D$ of $P$, if $\bigvee D$ exists and $b \leq \bigvee D$ then there exists $d \in D$ such that $a \leq d$.

A poset $P$ is called a continuous poset if for every element $a \in P$, the set $\{x \in P : x << a\}$ is a directed set and

$$a = \bigvee\{x \in P : x << a\}.$$

It is well known that if $P$ is a continuous poset, then the way-below relation has the interpolation property, that is, if $a << b$ then there exists $x$ such that
a \ll x \ll b$ (see [6] for more about continuous posets). By [11], a dcpo $P$ is continuous if and only if $\Gamma(P)$ is completely distributive. This result has been generalized to $Z$-continuous posets [4].

Recall that in a complete lattice $L$, for $x, y \in L$, we say that $x$ is strongly way below $y$, denoted $x \ll y$, if for any $D \subseteq L$, $y \leq \bigvee D$ implies that $x \leq d$ for some $d \in D$.

**Lemma 10.** Let $P$ be a poset.
1. If $\downarrow x \ll \downarrow y$ holds in $\Gamma(P)$, then $x \ll y$ holds in $P$.
2. If $P$ is continuous, then $x \ll y$ in $P$ implies $\downarrow x \ll \downarrow y$ in $\Gamma(P)$.

**Theorem 4.** For every poset $P$, the following statements are equivalent:
1. $P$ is continuous;
2. $\Gamma(P)$ is completely distributive under the inclusion order.
3. $E(P)$ is a continuous dcpo.

**Proof.** Since $\Gamma(P) \cong \Gamma(E(P))$, and a dcpo is continuous if and only if its Scott closed set lattice is completely distributive (see [6]), so (2) and (3) are equivalent.

The equivalence of (1) and (2) follows from Lemma 10. \qed

**Remark 3.** In [16], it was proved that if $P$ is a continuous poset, then $\text{Spec}(\Gamma(P))$ is the directed completion of $P$ (i.e. $\text{Spec}(\Gamma(P)) = E(P)$). Then by a standard result on the spectra of completely distributive lattices, $E(P) = \text{Spec}(\Gamma(P))$ is continuous.

An element $x$ of a poset $P$ is called a compact element if $x \ll x$. A poset is called an algebraic poset if for each $a \in P$, $\{x \in P : x$ is compact and $x \leq a\}$ is a directed set and $a = \bigvee\{x \in P : x$ is compact and $x \leq a\}$.

An element $x$ of a complete lattice $L$ is called a complete join-prime element if $x \leq \bigvee A$ implies $x \in \downarrow A$ for any $A \subseteq L$.

A complete lattice $L$ is called completely super continuous if every element of $L$ is the supremum of complete join-prime elements. By Raney’s characterization of completely distributive lattices [14], every completely super continuous lattice is completely distributive.

The following lemma can be proved in a similar way as for algebraic dcpos.

**Lemma 11.** A poset $P$ is algebraic if and only if $\Gamma(P)$ is completely super continuous.

Using Theorem 2 and Lemma 11 we can deduce the following

**Corollary 3.** A poset $P$ is algebraic if and only if its $D$-completion is an algebraic dcpo.

In [1], noncomplete continuous lattices are introduced and investigated. A noncomplete continuous lattice is actually a continuous poset which is also a lattice (but may not be a dcpo).

Note that the Scott topology on a continuous poset need not be sober. However we have the following
Lemma 12. If $P$ is a continuous poset and $F \in \Gamma(P)$, then $F \in \text{Spec}(\Gamma(P))$ if and only if $F = \text{cl}(D)$ for some directed subset $D$ of $P$.

Proof. The sufficiency follows from Lemma 2.

To prove the necessity, let $F \in \text{Spec}(\Gamma(P))$. Since $P$ is continuous, $\Gamma(P)$ is a completely distributive lattice. Thus it follows that $F = \bigvee \{ \downarrow x : x \in F \}$. By Theorem 1 of [18], $\{ \downarrow x : x \in F \}$ is a directed set, therefore $D = \{ x : \downarrow x \in F \}$ is directed and $\text{cl}(D) = F$. □

Proposition 1. If $P$ is a continuous poset and $P$ is a lattice then $E(P)$ is a complete lattice.

Proof. Let $P$ be a continuous poset which is a lattice. Then by Remark 3, $E(P) = \text{Spec}(\Gamma(P))$. To show that $E(P)$ is a complete lattice it is enough to show that it is a sup-semilattice because $E(P)$ is a dcpo. If $F, G \in \text{Spec}(\Gamma(P))$, then $F = \text{cl}(D), G = \text{cl}(E)$ for some directed sets $D$ and $E$. We can choose $D$ and $E$ such that $\downarrow x \in F$ for each $x \in D$ and $\downarrow x \in G$ for each $x \in E$. Let $K = \text{cl}(M)$, where $M = \{ x \vee y : x \in D, y \in E \}$. By Lemma 2, $K \in \text{Spec}(\Gamma(P))$ and $K \geq F, G$. If $H \in \text{Spec}(\Gamma(P))$ and $F, G \subseteq H$, then $H = \bigvee \{ \downarrow x : x \in H \}$ and $\{ \downarrow x : x \in H \}$ is directed. If $m = x \vee y$ where $x \in D, y \in E$, then $\downarrow x \in F \subseteq H$ implies that $x \leq u$ for some $u$ with $\downarrow u \in H$. Similarly $y \leq v$ for some $v$ with $\downarrow v \in H$. Choose $w$ such that $u, v \leq w$ and $\downarrow w \in H$. Then $m \leq w$, thus $K \subseteq H$. These show that $K = F \vee G \in \text{Spec}(\Gamma(P))$. □

5. Local dcpo-completion

In [13] Mislove defined a functor $\text{BSpec} : \text{POS}_d \rightarrow \text{LD}$ from the category $\text{POS}_d$ to the category $\text{LD}$ of local dcpo and claims that this functor is left adjoint to the forgetful functor from $\text{LD}$ to $\text{POS}_d$.

A poset $P$ is called a local dcpo ( or bounded complete dcpo ) if every upper bounded directed subset has a supremum [13]. Let $\text{LD}$ denote the full subcategory of $\text{POS}_d$ consisting of all local dcpos.

If $P$ is a local dcpo and $A \subseteq P$ is a lower subset, then $A$ is a local dcpo. The cartesian product of a collection of local dcpos is a local dcpo.

Remark 4. A poset $P$ is a local dcpo if and only if for any $a \in P$, $\{ \downarrow x : x \leq a \}$ is $D$-closed in $\Gamma(P)$. The necessity is easy to verify. For the sufficiency, let $X \subseteq P$ be a directed set with $X \subseteq \downarrow a$. By the assumption, $\text{cl}_d(X) = \downarrow y$ for some $y \leq a$. Then $y$ is an upper bound of $X$. If $b$ is any upper bound of $X$, then $X \subseteq \downarrow b$ implies $\text{cl}_d(X) \subseteq \downarrow b$ because $\downarrow b$ is $D$-closed. It follows that $y \leq b$ and hence $y = \bigvee X$.

For each poset $P$, let $\text{BSpec}(P)$ denote the set of all upper bounded join-primes of $\Gamma(P)$. The poset $\text{BSpec}(P)$ is a local dcpo [13]. Note that if $L$ is a complete lattice, then every join-prime element of $\Gamma(L)$ is upper bounded, thus $\text{BSpec}(P) = \text{Spec}(\Gamma(L))$. If the object map $P \mapsto \text{BSpec}(P)$ extends to a functor which is left adjoint to the forgetful functor $\text{LD} \rightarrow \text{POS}$, then $\text{Spec}(\Gamma(P)) = \text{BSpec}(L) = \{ \downarrow x : x \in L \}$. This may not be true if the Scott space of $L$ is not a sober space. In [9], Isbell constructed a complete lattice whose Scott topology
is not sober. Thus the map \( P \mapsto BSpec(P) \) may not define a left adjoint to the forgetful functor.

In the following we define a functor \( POS_d \to LD \) which is indeed the left adjoint of the forgetful functor.

For a poset \( P \), let \( BE(P) = \{ F \in E(P) : F \) is upper bounded\}. Then obviously \( \Psi(P) = \{ \downarrow x : x \in P \} \subseteq BE(P) \).

**Lemma 13.** For every poset \( P \), \( BE(P) \) is a local dcpo.

**Proof.** Let \( \{ A_i : i \in I \} \) be a directed subset of \( BE(P) \) that has an upper bound \( G \) in \( BE(P) \). As \( G \in BE(P) \), \( G \subseteq \downarrow a \) for some \( a \in P \). The supremum \( \bigvee \{ A_i : i \in I \} \) is still in \( E(P) \) and is bounded by \( a \), so it is in \( BE(P) \). \( \square \)

**Lemma 14.** If \( P \) is a local dcpo, then \( BE(P) = \{ \downarrow x : x \in P \} \).

**Proof.** Given a poset \( P \), let \( \eta : P \to BE(P) \) be the mapping that sends \( x \) to \( \downarrow x \). Then \( \eta \) is a Scott continuous mapping. Suppose that \( f : P \to A \) is Scott continuous with \( A \) a local dcpo. Then the mapping \( f^* : \Gamma(P) \to \Gamma(A) \) which sends \( X \) to \( cl(f(X)) \) is Scott continuous and \( f^*(BE(P)) \subseteq BE(A) \). By Lemma 14, \( A \) is isomorphic to \( BE(A) \). Then \( f^* \) restricts to a Scott continuous mapping \( \hat{f} : BE(P) \to A \) satisfying \( f = \hat{f} \circ \eta \). As in the case of \( E(P) \), such a mapping \( \hat{f} \) is unique. All these show that \( LD \) is reflexive in \( POS_d \). \( \square \)

### 6. Bounded sober spaces

In this section we introduce the bounded sober spaces and show that such spaces form a full reflexive subcategory of the category \( TOP_0 \) of \( T_0 \)-spaces. This definition is inspired by Mislove’s idea in [13].

We shall denote the set of closed sets of a topological space \( X \) by \( \Gamma(X) \). A closed set \( F \) of space \( X \) is bounded if \( F \subseteq cl(\{ x \}) \) for some element \( x \in X \). Recall that a space \( X \) is sober if it is \( T_0 \) and the closures of singleton sets are the only join-prime closed sets.

**Definition 2.** A \( T_0 \) space \( X \) is called bounded sober if for each bounded join-prime closed set \( F \) of \( X \) there is a point \( x \in X \) such that \( F = cl(\{ x \}) \).

Let \( X \) be a space and \( B(X) \) the set of all bounded join-prime closed subsets of \( X \). The collection of sets of the form \( K_F = \{ A \in B(X) : A \subseteq F \} \), where \( F \) is a closed set of \( X \), is a co-topology on \( B(X) \). We shall still use \( B(X) \) to denote the space \( B(X) \) associated with the above topology.

**Lemma 15.** For each \( T_0 \) space \( X \), \( B(X) \) is a bounded sober space.
Proof. First, \( K_F \subseteq K_{F'} \) if and only if \( F \subseteq F' \) because \( cl(\{x\}) \in B(X) \) for all \( x \in X \). Suppose that \( K_F \) is a bounded join-prime closed set of \( B(X) \). Then, \( F \) is a join-prime closed set of \( X \). Since \( K_F \) is bounded, there exists \( A \in B(X) \) such that \( K_F \subseteq cl(\{A\}) \). But \( cl(\{A\}) = \{C \in B(X) : C \subseteq A\} \). Hence \( F \subseteq A \). Since \( A \) is bounded in \( X \), \( F \) is also bounded in \( X \). Thus \( F \in B(X) \) and \( K_F = cl(\{F\}) \).

Theorem 6. The subcategory \( BSob \) of bounded sober spaces is fully reflexive in the category \( TOP_0 \) of \( T_0 \) spaces.

Proof. For any \( T_0 \) space \( X \), let \( B(X) \) be the bounded sober space defined in Lemma 15. Define \( \kappa : X \to B(X) \) be the mapping that sends \( x \in X \) to \( cl(\{x\}) \). Then \( \kappa \) is continuous. Suppose that \( f : X \to Y \) is a continuous mapping with \( Y \) a bounded sober space. For each \( F \in B(X) \), \( cl(f(F)) \) is a bounded join-prime closed set of \( Y \), so there is a unique point \( y \in Y \) such that \( cl(\{y\}) = cl(f(F)) \), let \( \hat{f}(F) = y \). Thus we have defined a mapping \( \hat{f} : B(X) \to Y \).

(i) Obviously \( f = \hat{f} \circ \kappa \).

(ii) \( \hat{f} \) is continuous. In fact, take any closed set \( G \) of \( Y \), \( F \in B(X) \) and \( \hat{f}(F) \subseteq G \) if and only if \( f(F) \subseteq G \), if and only if \( F \subseteq f^{-1}(G) \) Thus \( f^{-1}(G) = K_{f^{-1}(G)} \), which is a closed set in \( B(X) \). So \( \hat{f} \) is continuous.

Since \( \kappa(X) \) is dense in \( B(X) \), the mapping \( \hat{f} \) satisfying (i) and (ii) is unique. The proof is completed.

The following example shows that even when \( P \) is a continuous poset, the Scott space \( (P, \sigma(P)) \) need not be bounded sober.

Example 2. Let \( Q \) be the set of all rational numbers. With respect to the ordinary order, \( Q \) is a continuous poset but not a local dcpo. The subset \( F = \{x \in Q : x < \sqrt{2}\} \) is a join-prime Scott closed set and has an upper bound \( 2 \). But \( F \neq cl(\{x\}) \) for every \( x \in Q \).

Proposition 2. If \( P \) is a continuous local dcpo, then \( (P, \sigma(P)) \) is bounded sober.

Proof. By Theorem 4, \( \Gamma(P) \) is a completely distributive lattice. Let \( F \in Spec(\Gamma(P)) \) and \( F \subseteq cl(\{a\}) = a \) for some \( a \in P \). Then \( F = \bigvee\{x : x \in F\} \). By Theorem 1 of [18], \( \{x \in P : x \in F\} \) is a directed set which is bounded by \( a \). Hence \( b = \bigvee\{x : x \in F\} \) exists and thus \( F = cl(\{b\}) \). Thus the space \( (P, \sigma(P)) \) is bounded sober.

Example 3. Let \( \mathbb{R} \) be the set of all real numbers with the ordinary order of real numbers. Then \( \mathbb{R} \) is a continuous local dcpo. By Proposition 2, the Scott space of \( \mathbb{R} \) is bounded sober. However, the Scott space of \( \mathbb{R} \) is not sober, because \( \mathbb{R} \) is a join-prime but \( \mathbb{R} \) is not the closure of any singleton set.

The following is a dcpo whose Scott space is bounded sober but not sober.

Example 4. Let \( P = N \times (N \cup \{\infty\}) \) and define \( (m, n) \leq (m', n') \) if \( m = m', n \leq n' \), or \( n' = \infty \) and \( n \leq m' \). As was pointed out in [10], the Scott
space \((P, \sigma(P))\) is not a sober. Now if \(F \subset P\) is a Scott closed set which is join-prime with an upper bound, then \(F\) must have a top element \((m,n)\), thus \(F = cl(\{(m,n)\})\). Hence \((P, \sigma(P))\) is bounded sober.

**Remark 5.** Recently, [15] and [12] also proved the existence of the dcpo-completion of posets using different approaches. The D-topology was systematically investigated and used in [12].

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**References**
