FIFTEEN:

Combining Magic Squares and Tic-Tac-Toe

The game Fifteen can motivate students to develop problem-solving skills and mathematical thinking.

Joseph B. W. Yeo

The game Fifteen immediately engages students. Let's jump right in.

> Nine disks marked with the digits 1 through 9 are placed faceup on the table. Two players take turns picking one disk at a time. The winner is the first player to obtain the sum of exactly 15 from among any three of his or her disks. (Adapted from Mason, Burton, and Stacey 1985)

I start by picking 8. Which number will you choose? If you did not select 5, I will definitely win. For example, you pick 4. Then I choose 6. You have to pick 1 to prevent me from winning with $8 + 6 + 1 = 15$. Then I choose 2. Now I can win by picking 5 or 7 because $8 + 2 + 5 = 15$ and $6 + 2 + 7 = 15$. You cannot win in the next step (because you only have 4 and 1), so I will win regardless of whether you choose 5 or 7. To know which numbers to pick to win, see **figure 1**.

Playing Fifteen is like playing tic-tac-toe on a 3×3 magic square in which the three numbers in each row, column, and diagonal add up to 15. The game can be used in the classroom to motivate and help students learn mathematics and develop heuristic skills such as examining all possible scenarios and systematic listing (Pólya 1957), spatial visualization (Presmeg 2008), and thinking skills, including predicting, conjecturing, generalizing, and checking (Ainley 1990).

A REVIEW OF MAGIC SQUARES

When my students finally realized that playing Fifteen was like playing tic-tac-toe on a magic square, they usually felt that this was enough to give them command of the game. Students may need to be guided to explore other possibilities, however.

For example, if a player can win by choosing three numbers as shown by the Xs in **figure 2**, then playing Fifteen will be quite different from playing tic-tac-toe on a magic square. Thus, students need to prove that the only combinations of three numbers that sum to 15 are those that lie in each row, column, and diagonal of the magic square.

One method that some students have found to determine all possible sums is to start with 1 and divide the sum of the other two numbers (i.e., 14) by 2 to give 7. Because $7 + 7$ is impossible, the

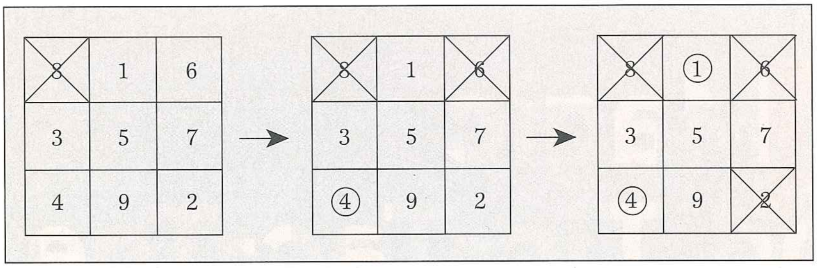

Fig. 1 Choosing 8 as a first move will enable the first player to win a game of Fifteen.

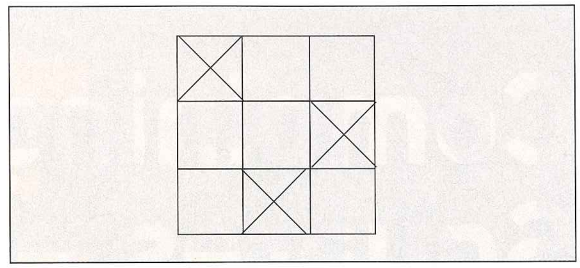

Fig. 2 Can a player get a sum of 15 with three markers that are not in the same row, column, or diagonal?

odd	even	odd
even	odd	even
odd	even	odd

Fig. 3 Placing odd numbers in the corners of a magic square cannot result in a sum of 15.

next logical choice is 6 + 8 (decrease the first number by 1 and increase the second number by 1), followed by 5 + 9 (repeat the same process), and the impossible result, 4 + 10. At this point, we are sure that there are only two possibilities if we start with 1.

Using the same method with 2, 3, and 4 as the starting number, we can sieve out all the eight combinations, as follows:

$$\begin{aligned} 15 &= 1 + 6 + 8 \\ &= 1 + 5 + 9 \\ &= 2 + 6 + 7 \\ &= 2 + 5 + 8 \\ &= 2 + 4 + 9 \\ &= 3 + 5 + 7 \\ &= 3 + 4 + 8 \\ &= 4 + 5 + 6 \end{aligned}$$

The next starting number is 5. Because 5 + 5 + 5 is not possible, if we use the same process as earlier, one of the last two numbers will be less than 5—for example, 5 + 4 + 6—but we would have found this sum using 4 as the starting number. So there are no more new combinations if the starting number is 5. Many students were tempted to stop here, but they need to reason that if the starting number is larger than 5, then all the sums obtained will be repetitions.

The next step is to construct a magic square. We will discuss a method in which students can reason through the problem instead of memorizing procedures.

Students usually remember that 5 is in the center square, probably because 5 is the average, or "middle," of the nine numbers. The important question is whether all the corners of the magic square should be occupied by odd or even numbers. Students can reason this way: If the corners are filled by the remaining odd numbers (see **fig. 3**), then the first row (or column) will show that the arrangement is impossible because odd + even + odd is even but the sum 15 is odd. Therefore, all even numbers must be in the corners.

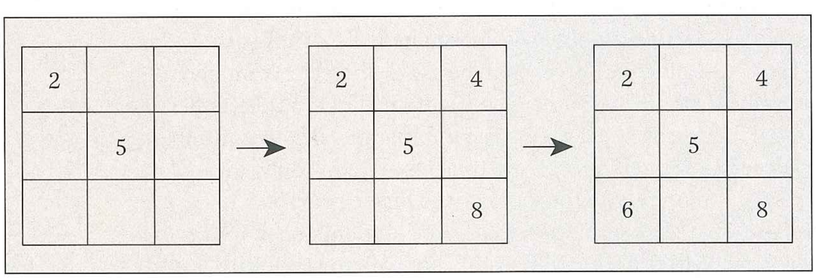

Fig. 4 This magic square is oriented differently (it is rotated through 180°) from the one shown in **figure 1**. In fact, all eight 3 × 3 magic squares can be obtained from one another by rotation or reflection.

Without loss of generality, we can put 2 in any corner (see **fig. 4**). The opposite corner must be 8 because 2 + 5 + 8 = 15. Without loss of generality, we can put 4 in either remaining corner. The last corner must be 6 because 4 + 5 + 6 = 15. The remaining squares can then be filled with the remaining odd numbers.

We next examine how students can win at tic-tac-toe, a skill that will help them strategize when playing Fifteen.

WINNING STRATEGIES FOR TIC-TAC-TOE

Many students do not understand the concept of a winning strategy. When Civil (2002) asked students to find a winning strategy for the game Nim, the strategies that students found were often dependent on some particular moves by the opponent; if the opponent made a different move, the strategy might not work.

Moreover, the term *winning strategy* has different meanings. In Nim, a winning strategy that ensures victory is possible (the only criterion being whether the player should start first). This is a sure-win strategy. But in tic-tac-toe, there are no sure-win strategies; a winning strategy is simply one that will maximize the chance of a player winning. This will be my definition of a *winning strat-*

egy. Some winning strategies for tic-tac-toe follow.

If player 1 starts with the center square and each player makes optimal moves, the game will always end in a draw.

If player 1 places an X in a corner, player 2 must place an O in the center; otherwise, player 2 will lose. For example, if player 2 puts the first O in the corner opposite the first X (see **fig. 5**), player 1 should place an X in any remaining corner, thus forcing player 2 to put an O in between the two Xs to prevent player 1 from winning. Player 1 can then win by placing an X in the last corner. Readers can figure out how to win the game if player 2 has chosen to put the first O somewhere other than the center square.

Therefore, if player 1 starts at a corner, player 2 will lose unless he or she puts an O in the center. If player 2 does not know this strategy and is equally likely (which may not be true) to put an O in any one of the remaining eight squares, then the probability of player 1 winning is 7/8 = .875.

But this probability is actually higher because player 1 can still win if player 2 puts the first O in the center (see **fig. 6**). Player 1 should place the second X at the corner opposite the first X and hope that player 2 puts the second O at a remaining corner. Then player 1 will win by placing the third X in the last corner. Readers can show that the game will be a draw if player 2 puts the second O elsewhere.

Therefore, if player 2 puts the first O in the center and is equally likely to place the second O in any one of the remaining six squares, then the probability of player 1 winning is 2/6 = 1/3. Hence, if player 1 begins by placing X at a corner, then the probability of him or her winning is

$$\frac{7}{8}+\frac{1}{8}\cdot\frac{1}{3}=\frac{11}{12}\approx .917.$$

However, player 2 is more likely to put the first O in the center if player 1 starts at a corner, so the assumption that player 2 is equally likely to put an O in any square is invalid. Thus, if player 1 starts at a corner, the probability of winning is actually less than .917, but it is still better than starting at the center, which will usually lead to a draw.

If player 1 starts with an edge-square, the game is wide open. Either player can win, although the game will most often end in a draw. Therefore, a winning strategy for player 1 is to first place an X at a corner.

Player 2 needs to know how to maximize his or her chances of winning or what must be done to prevent a loss. If player 1 starts at a corner, player 2 (as discussed previously) can prevent him or her from winning. But if player 1 starts with an edge-square, there are many cases. We will discuss only two.

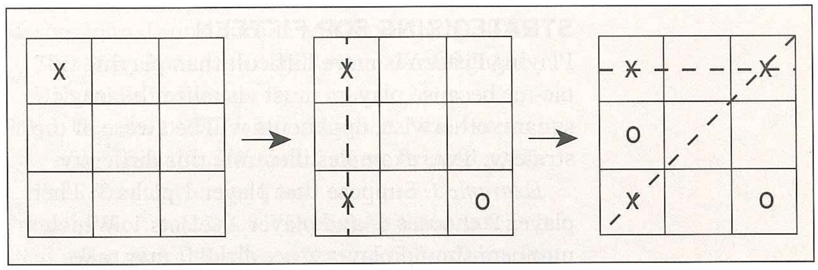

Fig. 5 If player 2's first move is not in the center, player 1 will always win.

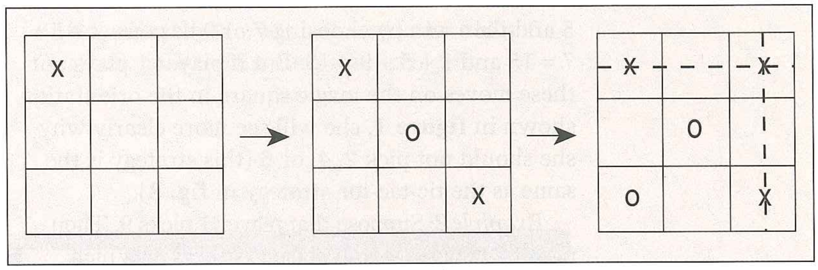

Fig. 6 If player 2's first move is to place O in the center square, player 1 can still win.

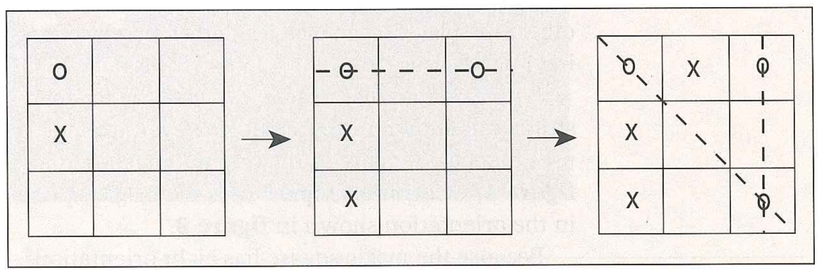

Fig. 7 If player 1 does not place his first X at a corner or at the center, it is possible (but not likely) that player 2 will win.

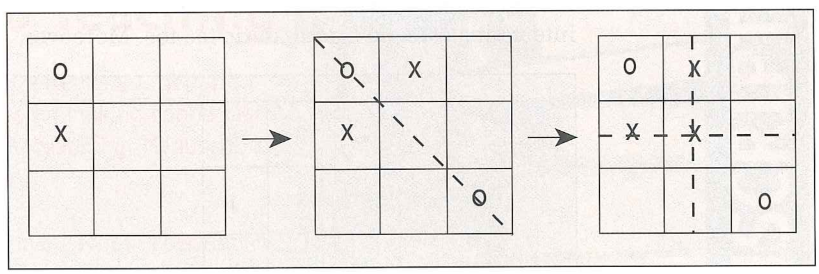

Fig. 8 In this scenario, player 2 loses.

Figure 7 shows player 1 starting with an X on an edge-square and player 2 then placing an O on a corner next to the X. If player 1 puts the second X in the other corner next to the first X, then player 2 can win as shown.

Figure 8 shows the same start as **figure 7**. But if player 1 chooses to put the second X as shown in **figure 8**, then player 2 must not place the second O in any remaining corner; otherwise, he or she will lose when player 1 puts the third X in the center. If player 2 is equally likely to place a second O in any one of the six remaining squares, then the probability that he or she loses at this stage is 3/6 = .5.

STRATEGIZING FOR FIFTEEN

Playing Fifteen is more difficult than playing tic-tac-toe because players must visualize the magic square; otherwise, opponents will be aware of the strategy. Two examples illustrate this difficulty.

Example 1: Suppose that player 1 picks 3. Then player 2 chooses 8, and player 1 selects 1. Which numbers should player 2 *not* pick? It may take player 2 some time to figure out that she should not choose 2, 4, or 6; otherwise, player 1 can pick 5 and then win by choosing 7 or 9 because $3 + 5 + 7 = 15$ and $1 + 5 + 9 = 15$. But if player 1 plays out these moves on the magic square in the orientation shown in **figure 1**, she will see more clearly why she should not pick 2, 4, or 6 (this strategy is the same as the tic-tac-toe strategy in **fig. 8**).

Example 2: Suppose that player 1 picks 9. Then player 2 chooses 4, and player 1 selects 3. Which number should player 2 *not* pick? Again, it takes some time to reason that she should not choose 2, 6 or 8; otherwise, player 1 can pick 5 and then win by choosing 1 or 7 because $9 + 5 + 1 = 15$ and $3 + 5 + 7 = 15$.

Both examples use exactly the same tic-tac-toe strategy as shown in **figure 8**. However, example 1 uses the magic square in the orientation shown in **figure 1**, whereas example 2 uses the magic square in the orientation shown in **figure 9**.

Because the magic square has eight orientations, the same tic-tac-toe strategy illustrated in **figure 8** will translate to eight different scenarios in Fifteen. For this reason, Fifteen is a more difficult but more interesting game to play than tic-tac-toe. Moreover, there are many possibilities if player 1 starts with an odd number other than 5.

4	3	8
9	5	1
2	7	6

Fig. 9 If player 1 chooses 9 and then 3 after player 2 chooses 4, where should player 2 *not* play?

Figure 10 shows the work of a student who was investigating the same scenario as shown in **figure 8** but with the magic square in yet another orientation. This student was not able to find a strategy because she kept playing the game to the end without realizing that after a critical move, player 2 will always lose or draw. The critical move in this scenario is what player 2 will choose at the second move.

Many students are unable to move away from the details to gain an overview of the game. For some, it may be easier to investigate a winning strategy for tic-tac-toe first (without all the numbers confusing them) and then apply it to Fifteen than to investigate a winning strategy for Fifteen by directly playing tic-tac-toe on a magic square.

Some scenarios that illustrate the main winning strategies for playing Fifteen follow. Try to investigate them without looking at a magic square first.

1. If you play first, should you choose an even or an odd number?
2. If your opponent plays first and picks an even number, what number must you choose to avoid a loss?
3. Suppose that you play first and pick 8. Your opponent then chooses 3. What are the three numbers that you can choose to ensure a win?
4. Suppose that your opponent plays first and picks 6. Then you choose 5, and your opponent picks 4. Which two numbers should you *not* pick?
5. Suppose that your opponent plays first and picks 7. Then you choose 2, and your opponent picks 9. Which three numbers should you *not* pick?

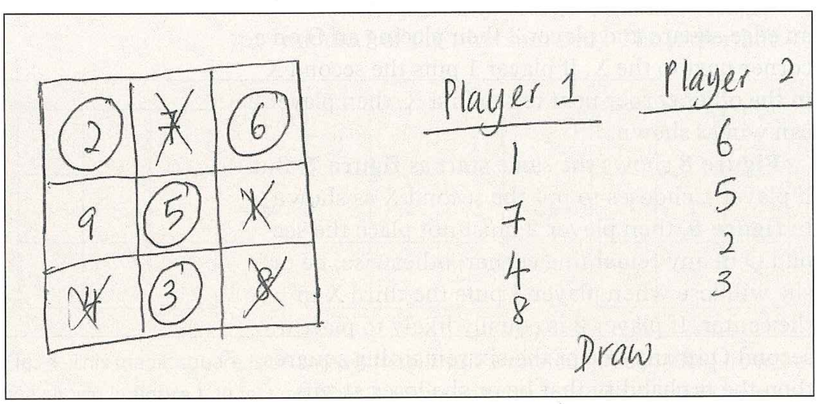

Fig. 10 After player 2 has chosen the number 5, the game will always end in a draw.

AN EXTENSION

One way to extend this game to ensure a sure-win strategy follows:

> Nine disks marked with the digits 1 through 9 are placed faceup on the table. Two players take turns picking one disk at a time. The winner is the first player to obtain the sum of exactly 15 from among any *two or more* of his or her disks.

Because it is possible to win by using two disks, there are two more winning combinations: $6 + 9 = 15$ and $7 + 8 = 15$. So player 1 will definitely win if he chooses 6 or 8 because player 2 has to pick 9 or 7, respectively, to prevent player 1 from winning. But, as we have discussed, player 2 must choose 5 on her first move to avoid losing if player 1 starts with an even number. Therefore, this extension has a sure-win strategy for player 1.

CONCLUSION

Most students love to play games. Ernest (1986) believed that games could be used to teach mathematics effectively in four areas: motivation, concept development, reinforcement of skills, and practice of problem-solving strategies. Fifteen is an interesting and thought-provoking game that helps students learn mathematics at the same time.

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JOSEPH B. W. YEO, josephbw.yeo@nie.edu.sg, teaches student teachers at the National Institute of Education, Nanyang Technological University, in Singapore. He is interested in mathematical investigation and problem solving and the use of games, puzzles, videos, math songs, and real-life examples and applications to engage students.

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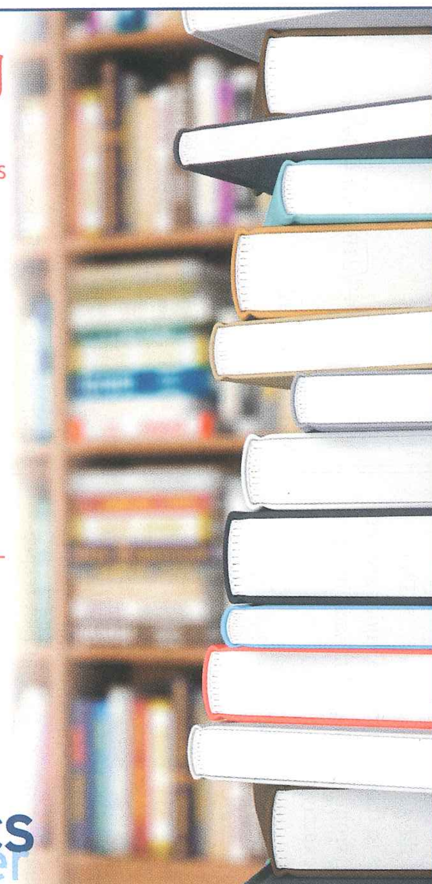

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