Technical Report ME2009-01
January 2009
Mathematics and Mathematics Education
National Institute of Education Singapore

MATHEMATICAL INVESTIGATION: TASK, PROCESS AND ACTIVITY

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Abstract

Many writers believe that mathematical investigation is open and it involves both problem posing and problem solving. However, some teachers feel that there is a sense of doing some sort of investigation when solving problems with a closed goal and answer but they are unable to identify the characteristics of this type of investigation. Such confusion will affect how teachers teach their students and how researchers conduct their research on investigation. Therefore, this article seeks to clarify the relationship between investigation and problem solving by providing an alternative characterisation of mathematical investigation as a process involving specialisation, conjecturing, justification and generalisation. It also distinguishes between mathematical investigation as a process and as an activity: investigation, as a process, can occur when solving problems with a closed goal and answer, while investigation, as an activity involving open investigative tasks, includes both problem posing and problem solving. Implicit support for this alternative characterisation of mathematical investigation is gathered from some existing literature as these writers did not state this perspective explicitly. The article concludes with some implications of this alternative view on teaching and research.

1. Introduction

Many researchers and educators believe that mathematical investigation must be open. For example, Bastow, Hughes, Kissane and Mortlock (1991) defined mathematical investigation as the "systematic exploration of open situations that have mathematical features" (p. 1) and Lee and Miller (1997) wrote that "investigations, by their very nature, demand an open-minded, multifaceted approach" (p. 6). Delaney (1996) believed in the more open spirit of the process-dominated investigation while Bailey's (2007) idea of a mathematical investigation is "an open-ended problem or statement that lends itself to the possibility of multiple mathematical pathways being explored, leading to a variety of mathematical ideas and/or solutions" (p. 103). Ernest (1991) described investigation as the exploration of an unknown land where "the journey, not the destination, is the goal" (Pirie, 1987, p. 2). In other words, many writers believe that mathematical investigation is open with respect to its goal, processes and answer.

Many of them also use the term 'investigation' differently. For example, Orton and Frobisher (1996) used the terms 'problems' and 'investigations' to refer to the tasks in their comparison between problems and investigations; while Evans (1987) used the terms 'problem solving' and 'investigation' to refer to the process when he contrasted problem solving with investigation. Thus the term 'investigation' can refer to the task or the process or both. In mathematics education, there has been a fairly widespread adoption of the term 'investigation' as the task itself (Ernest, 1991). This is what Jakobsen (1956, cited in Ernest, 1991) called a metonymic shift in meaning which replaces the whole activity by one of its components. Therefore, many educators do not distinguish between the investigative task and the process of investigation.

Although many writers have observed that there are overlaps between problem solving and investigation, their opinions are mixed. For example, some of them (e.g., Pirie, 1987) claimed that "no fruitful service will be performed by indulging in the 'investigation' versus 'problem-solving' debate" (p. 2) while Frobisher (1994) believed that this is a crucial issue that will affect how and what teachers teach their students. But many of these educators ended up separating investigation from problem solving by restricting investigation to 'open investigative tasks' and problem solving to 'closed problems'. For example, Orton and Frobisher (1996) wrote that "the distinction between a problem and an investigation is the existence of a clear goal specified in the statement of the [problem]" (pp. 31-32) and they claimed that "very few mathematics educators would classify [investigations] of this kind as problems" (p. 27) because investigation entails an open goal and thus it involves problem posing as well (Cai & Cifarelli, 2005). Therefore, many educators (e.g., Evans, 1987; HMI, 1985) believe that problem solving is a convergent activity with a well-defined goal and answer, while investigation is a divergent activity with an open goal and answer.

On the other hand, some teachers feel that there is a sense of doing some sort of investigation when solving 'closed problems' but they are unable to pinpoint the similarities and thus there are few writings on this. Such confusion about the characteristics of investigation will affect how educators teach (Frobisher, 1994) and how researchers conduct their research. Therefore, it is the purpose of this article to clarify these issues. It begins by providing an alternative characterisation of mathematical investigation as a process and its implication of this on the relationship between investigation and problem solving. Implicit support for this alternative view is gathered from some existing literature as these writers did not state this perspective explicitly. The article clarifies the relationship between investigation and problem posing by looking at investigation from the viewpoint of a mathematical activity. The distinction between open investigative tasks, the process of investigation, and investigation as an activity, is crucial in our understanding of the relationship between investigation and problem solving. The article concludes with some implications of this alternative characterisation of mathematical investigation on teaching and research.

2. What constitutes a mathematical investigation?

There is a big difference between a task and the process of doing the task. Thus, equating mathematical investigation with open investigative tasks has added to the current confusion about mathematical investigation. Therefore, in this article, the task will be called 'an open investigative task' to emphasise the openness of such a task (see Task 1 below) and the term 'mathematical investigation' will be used to refer to the process of investigating. In fact, a further distinction between investigation as a process and investigation as an activity will be dealt with later in this article. An example of an open investigative task is:

> **Task 1: Powers of 3**
> Powers of 3 are 3^1, 3^2, 3^3, 3^4, 3^5, … Investigate.

In this task, the goal is ill-defined: investigate, but investigate what? There are two approaches: students may set a specific goal by posing a *specific* problem to investigate (Cai & Cifarelli, 2005) or they can set a general goal by searching for any pattern (Height, 1989). The latter can be called the posing of the *general* problem "Is there any pattern?" so that both approaches can be called problem posing. Because students have the freedom to choose any goal, the goal is considered open (Orton & Frobisher, 1996). As there are also many correct answers, the task is said to have an open answer (Becker & Shimada, 1997). Therefore, this type of open investigative tasks is open in its goal and answer, and it involves the process of mathematical investigation. But what is investigation?

It may be helpful to look at how the term ‘investigation’ is used in everyday life. Cambridge Dictionaries Online (Cambridge University Press, 2008) defines the word ‘investigate’ as ‘examine a crime, problem, statement, etc. carefully, especially to discover the truth’. In crime investigation, the police examine *specific* pieces of evidence in order to come to certain conclusions, such as, “Who is the culprit? How does he or she commit the crime? What is his or her motive? Is there enough evidence to prosecute him or her?” During the investigation, the police will formulate some conjectures and test them. This type of conjectures cannot be proven beyond doubt but the deciding factor is whether there is enough evidence (e.g., concrete evidence, circumstantial evidence and reliable eye-witnesses) to support these conjectures. Thus the processes involved in crime investigation are the examination of specific pieces of evidence, conjecturing, testing and proving conjectures, and coming to some conclusions.

Another common usage of the term ‘investigation’ is in educational research. Many of these research studies involve some kind of investigation. There used to be a journal called *Investigations in Mathematics Education* (21 volumes; discontinued in 1988) which contained annotated bibliographies of current research studies in mathematics education. Many researchers (e.g., Kaur, 1995; Lampert & Ball, 1998) also described their studies as investigations. In these studies, a group of *specific* subjects or issues were examined in order to formulate and test some conjectures. Similar to conjectures in crime investigation, conjectures in educational research cannot be proven beyond doubt but the deciding factor is whether there are sufficient valid and reliable data to support these conjectures (Kelly & Lesh, 2000). Therefore, educational research involves the examination of specific subjects or issues, formulating and testing conjectures, and coming to some conclusions.

Let us turn our attention to mathematical investigation. In Task 1, students may start by evaluating some powers of 3 and then comparing them to find out if there is any pattern. This involves examining *specific* examples (specialisation) in order to generalise. But the pattern observed may not be true. So there is a need to test the conjecture. When a conjecture is proven or justified, the mathematical truth discovered can be viewed as the underlying general structure and is called a generalisation (Mason, Burton & Stacey, 1985) or mathematisation (Wheeler, 1982). Thus mathematical investigation, as a process, involves four core thinking processes: specialisation, conjecturing, justification and generalisation. This alternative view of mathematical investigation is in line with the everyday usage of the term ‘investigation’ described earlier.

There is a need to clarify three issues. The first issue is the use of the term ‘process’. From one angle, an investigation is one whole process (Frobisher, 1994), just like problem solving is one whole process (Shufelt, 1983), but from another perspective, there are many processes involved in an investigation (Frobisher, 1994). Thus there are two different viewpoints of ‘process’ in mathematical investigation.

The second issue is the use of the term ‘specialisation’ to describe the examination of specific examples in order to generalise. The specific examples chosen for examination in a mathematical investigation may not be special cases and there is no need to treat the specific pieces of evidence left behind in a crime scene as special. Furthermore, the antonym for the word ‘general’ is ‘specific’, not ‘special’ (Synonyms Thesaurus with Antonyms & Definitions, 2008). However, Mason et al. (1985) defined specialising as “choosing examples randomly, to get a feel for the question; systematically, to prepare the ground for generalising; artfully, to test a generalisation” (p. 24) but these specific examples may not be special cases. Schoenfeld (1985) also included specific examples which are not special cases when discussing the heuristic of examining special cases. For example, if there is an integer parameter n in the problem statement, the special cases are when $n = 1, 2, 3, 4$, etc. but what is so special about these cases? Nevertheless, this article will follow Mason et al. (1985) and Schoenfeld (1985) in defining the process of specialisation more broadly to include the examination of both special cases and specific examples. Therefore, mathematical investigation involves the process of specialisation in order to generalise.

The third issue is the use of the term 'justification' as a core process in mathematical investigation. Justification only occurs when a conjecture is proven. But students will not know beforehand whether a conjecture is true or false, and so they will engage in the testing of conjectures during an investigation. If the conjecture is found to be false, then it is refuted; if it is found to be true, then it is proven or justified. Thus, in mathematical investigation, a conjecture is to be tested. But from the viewpoint of mathematical thinking, mathematisation or generalisation occurs only when a conjecture is justified. That is why Mason et al. (1985) wrote that justification, and not the testing of conjectures, is one of the four main mathematical thinking processes while Hatch (2002) believed that "the idea of justification … is at the heart of a true mathematical ethos" (p. 138). Therefore, mathematical investigation involves the core thinking process of justification.

Further support for this alternative view of mathematical investigation as a process can be found among certain writers although they did not openly define investigation in this manner. For example, Jaworski (1994), in her book *Investigating Mathematics Teaching*, agreed with the many writers whom she cited that investigation must be open, but in the latter part of her book, she described the teaching of the three teachers whom she observed as "investigative in spirit, embodying questioning and inquiry" (p. 96) and "making and justifying conjectures was common to all three classrooms, as was seeking generality through exploration of special cases and simplified situations" (p. 171). So you can find traces of evidence that she seemed to view an investigative approach to mathematics teaching as involving specialisation, conjecturing, justification and generalisation although she did not define it explicitly.

Some writers who believed that mathematical investigation must be open sometimes also ended up using closed investigative tasks. For example, consider the following task which is supposed to have an open goal.

> **Task 2: Number Trick**
> Jill has a trick she does with numbers. Here it is. How do you think it works?
>
> $$\begin{array}{r} 854 \\ \underline{-\ 458} \\ 369 \\ \underline{+\ 693} \\ \underline{1089} \end{array}$$
>
> Jill says that every time she does her trick, the answer is always 1089. Investigate Jill's trick. (Orton & Frobisher, 1996, p. 39)

The first goal is clearly defined: how do you think the trick works? The second goal is implied: whether the trick will always work, since Jill claims it does and the students are supposed to investigate her trick. Although there is more than one goal, the goals are specified or implied, and so the students cannot select any other goal to investigate. But a task has an open goal if and only if the task does not specify any goal and students can choose any goal to pursue (Orton & Frobisher, 1996). Thus this task has a closed goal but it is still considered an investigation by Orton and Frobisher who believed that investigation must have an open goal. Therefore, the characterisation of mathematical investigation does not lie in the open goal of the investigative task itself, but in what it entails, i.e. the processes. Hence, an alternative characterisation of mathematical investigation is a process that involves the four core thinking processes of specialisation, conjecturing, justification and generalisation. Interestingly, Mason et al. (1985) used all these four processes for solving problems with a closed goal and answer. So how is investigation similar to problem solving?

3. How is investigation related to problem solving?

Whether a task or a situation is a problem or not depends on the individual (Henderson & Pingry, 1953; Reys, Lindquist, Lambdin, Smith & Suydam, 2004). So this article will follow certain writers (see, e.g., NCTM, 1991, p. 25; Schoenfeld, 1985, p. 74) in using the phrase 'mathematical task', rather than the term 'problem', to describe a task. However, the term 'problem' may still be used in this article when referring to a task, but when such a term is used, it implies that the task is a problem to the particular student. The phrase 'solving a problem' will also be used instead of 'solving a task' because if the task is not a problem to the student, then there is really no need for him or her to solve it.

There are many types of mathematical tasks. In this article, procedural tasks refer to the usual textbook questions whose main purpose is for students to practise procedural skills after they have learnt the procedures in class (Lester, 1980; Moschkovich, 2002). This is in contrast to problem-solving tasks (see Task 3 below) which require the use of some problem-solving strategies to solve. Although the phrase 'problem-solving tasks' can be misleading because the term 'problem-solving' may suggest that the task is a problem to the student when it may not be so, the advantage of this terminology is that it immediately brings to attention that this type of tasks involve the use of some problem-solving heuristics to solve, unlike procedural tasks (Yeo, 2007). An example of a problem-solving task is:

> **Task 3: Handshakes (Problem-Solving Task)**
> At a workshop, each of the 100 participants shakes hand once with each of the other participants. Find the total number of handshakes.

This task requires certain problem-solving strategies to solve, for example, drawing a diagram for a smaller number of participants to see if there is any pattern, although this may not be a problem to some students, especially for those who have solved it before. Although a problem-solving task is closed in its goal, many educators (e.g. Frobisher, 1994) believe that you can always open up the task by rephrasing it. For example, Task 3 can be rephrased as the following open investigative task which has an open goal:

> **Task 4: Handshakes (Open Investigative Task)**
> At a workshop, each of the 100 participants shakes hand once with each of the other participants. Investigate.

Such a rephrasing is not trivial because students can now pose different problems in Task 4 to solve (e.g., how many handshakes are there if there are n participants? or if they shake hand with each other m times?), as compared to only one question in Task 3. Although students can be taught to extend Task 3 by posing more problems to solve, they cannot be penalised if they do not do so because the task statement in Task 3 does not explicitly specify that the students must extend the problem. But students who solve only one problem for Task 4 will not do as well as students who solve more than that one problem.

Now, let us look at the type of processes that students engage in when they attempt both tasks. Since Task 3 is a problem-solving task, then the students are engaged in problem solving (Evans, 1987) when they try to find the total number of handshakes. Since Task 4 is an open investigative task, then the students are engaged in mathematical investigation (Evans). Suppose the first problem that the students want to solve for Task 4 is to find the total number of handshakes, which is the same as the question in Task 3. If the students use the *same* method to find the total number of handshakes for both the problem-solving task and the open investigative task, then the *same* process is called problem solving for the former task with a closed goal but it is called investigation for the latter task with an open goal. So this suggests that mathematical investigation can be equivalent to problem solving in some instances, that investigation can occur in closed problem-solving tasks and not just in open investigative tasks, and that investigation does not depend on whether the task has a closed or open goal.

Rather, the characterisation of mathematical investigation lies in the processes that it entails. For example, if the students try to solve the problem in Task 3 by starting with a smaller number of participants in order to find a pattern for the total number of handshakes, then they have engaged in specialisation, conjecturing, justification and generalisation, which is mathematical investigation (see previous section), but some educators (e.g., Holding, 1991) call this ‘induction’ which is defined as “drawing a general conclusion from clues gathered (from specific to general)” (Ministry of Education of Singapore, 2000). However, if the students argue that the total number of handshakes must be 99 + 98 + 97 + … + 1 because the first participant must shake hand with the other 99 participants, the second participant must shake hand with the remaining 98 participants and so forth, then the students have used deductive reasoning and this is not mathematical investigation. Some high-ability students may also use a formal proof directly and this is also not investigation. Therefore, there are essentially two approaches to solve a problem: investigation (or induction) and ‘other means’ (e.g., deduction or formal proof). Figure 1 illustrates the relationship between the process of problem solving and mathematical investigation as a process.

There is a need to clarify two issues. The first issue is viewing induction as investigation because investigation includes justification but some educators (e.g., Pólya, 1957) believed that induction is not proof. The problem lies in the different meanings of the terms: ‘induction’, ‘inductive observation’ and ‘inductive reasoning’. If students observe a pattern during an investigation, this is only a conjecture and some educators called it ‘inductive observation’ (e.g., Lampert, 1990, p. 49) which is not a proof. Some educators (e.g., Tall, 1991) believed in the use of formal proofs but Stylianides (2008) considered the use of any non-proof argument as sufficient. Mason et al. (1985) also advocated the use of the underlying mathematical structure to argue that the observed pattern will always continue. The latter involves rather rigorous reasoning and so it can be called ‘inductive reasoning’ which is good enough to justify a conjecture although it is not a formal proof. Thus there is a difference between inductive observation and inductive reasoning, but unfortunately some writers (e.g., Holding, 1991) used the two terms interchangeably. Similarly, the word ‘induction’ can mean inductive observation, inductive reasoning or both. In this article, the term ‘induction’ is used to include both inductive observation and inductive reasoning, and so induction involves justification.

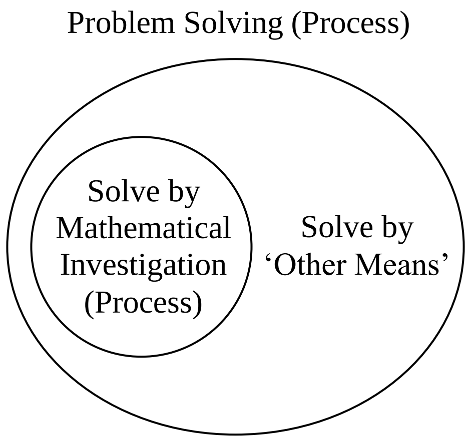

Figure 1. Relationship between Problem Solving and Investigation as Processes

The second issue is how investigation is related to heuristics. Generally speaking, problem-solving heuristics can be divided into two broad categories. The first category involves any form of specialisation. For example, if students draw a diagram or use systematic listing to examine specific examples, then this is mathematical investigation. The second category does not involve any specialisation. For example, if students use deductive reasoning directly, then this is not an investigation. But what about establishing a subgoal as a heuristic to solve a problem? By itself, this is not an investigation. The question is what happens after establishing a subgoal. If the students use deductive reasoning to achieve the subgoal, then this is not an investigation. However, if the students use some form of specialisation in order to attain the subgoal, then this is mathematical investigation. Therefore, using heuristics to solve problems are similar to solving problems by investigation or by ‘other means’.

Although this alternative view of the relationship between investigation and problem solving seldom appears in literature, there is evidence that some writers have almost the same idea but phrased in a different way. For example, Pólya (1957) advocated using what he called heuristic reasoning when solving problem, as opposed to using rigorous proof. “Heuristic reasoning is often based on induction, or on analogy” (p. 113). “Induction is the process of discovering general laws by the observation and combination of particular instances” (p. 114) and so induction involves specialising in order to generalise. “Analogy is a sort of similarity. Similar objects agree with each other in some respect, analogous objects *agree in certain relations* of their respective parts.” (p. 37) Thus analogy involves examining the relations of some parts of a *specific* object in order to discover the underlying mathematical structure of another analogous object. Therefore, heuristic reasoning involves examining specific examples in order to generalise or to infer about some mathematical fact by analogy. But isn’t this what mathematical investigation is all about? On the other hand, ‘other means’ of solving problem-solving tasks include the use of rigorous proof or deductive reasoning.

Lakatos (1976) also contrasted two approaches to problem solving: the deductivist approach and the heuristic approach. The deductivist approach uses formal proofs and Lakatos believed that very few people would ever devise this type of rigorous proofs out of nowhere. He wrote that the "deductivist style hides the struggle, hides the adventure" (p. 142) of discovering a solution to a problem or a proof for a theorem and "the zig-zag of discovery cannot be discerned in the end-product" (p. 42) of the deductivist approach. Lakatos then advocated the heuristic approach which involves exploring the problem by examining specific examples in order to formulate some conjectures which will have to be proven or refuted. Isn't this mathematical investigation? So Lakatos' approaches are very similar to the two approaches to problem solving discussed in this section: solving by investigation or by 'other means'.

Mason (1978) described seven energy states in problem solving (namely, getting started, getting involved, mulling, keep going, insight, checking and looking back) which he applied to mathematical investigation, thus suggesting that certain aspects of problem solving are similar to investigation. In the same manner, when Mason et al. (1985) explained the four key processes that underlie mathematical thinking (i.e. specialising, conjecturing, justifying and generalising), the writers were referring to mathematical thinking in solving problem-solving tasks, but these are also the key processes in mathematical investigation. The *Curriculum and Evaluation Standards for School Mathematics* stated that "our ideas about problem situations and learning are reflected in the verbs we use to describe student actions (e.g. to investigate, to formulate, to find, to verify) throughout the Standards" (NCTM, 1989, p. 10). Therefore, the Standards also recognise investigation as a means of dealing with problem situations. Bloom, Engelhart, Furst, Hill and Krathwohl (1956) classified the "ability to integrate the results of an investigation into an effective plan or solution to solve a problem" in the *synthesis* class in Bloom's taxonomy of educational objectives in the cognitive domain, thus implying that investigation is a method to solve mathematical problems. Hence, these writers have almost the same view that problem solving entails solving by investigation.

However, there is a conflict. In this alternative view of the relationship between mathematical investigation and problem solving, the process of investigation is a subset of the process of problem solving. But this contradicts the belief of many writers (e.g., Cai & Cifarelli, 2005; Frobisher, 1994) that investigation consists of both problem posing and problem solving, i.e., problem solving is a subset of investigation. This will be the focus in the next section.

4. Is problem posing part of mathematical investigation?

There are many types of problem posing. Problem "posing can occur before, during, or after the solution of a problem" (Silver, 1994, p. 19). Problem posing can also occur when a task asks the students to pose an original, interesting and complex 'word problem' that satisfy some given conditions (Getzels & Jackson, 1962; Yeap, 2002). However, in this section, we are interested in problem posing whose main purpose is to generate new problems to solve, e.g., at the start of an open investigative task, or during extension of a problem-solving task. The main issue to be addressed here is, "Does investigation involve problem posing?"

We will use two analogies. The first analogy is a real-life example of cooking (Berinderjeet Kaur, 2007, personal communication). Before you can cook, you need to plan the menu, buy the ingredients and prepare them (unless you have a kitchen hand to do that for you). After cooking, you need to scoop the food onto a dish and perhaps decorate the food to make it more presentable. Thus, to cook a dish, you need to do more than just the mere process of cooking. The second analogy is Pólya's (1957) four phases of problem solving for problem-solving tasks: understand the problem, devise a plan, carry out the plan and look back. The first phase is what to do before problem solving, the second and third phases are the *actual* problem-solving process, and the fourth phase is what to do after problem solving. But *all* the four phases are considered part of Pólya's problem-solving model. So there is a need to differentiate between the actual process and the whole activity.

We will begin by distinguishing between a task and an activity although these two terms are often treated as synonyms (Mason & Johnston-Wilder, 2006). A task refers to what the teacher sets while the activity refers to what the student does in response to the task (Christiansen & Walther, 1986). "The purpose of a task is to initiate mathematically fruitful activity [by] learners" (Mason & Johnston-Wilder, 2006, p. 25). Therefore, when students solve problem-solving tasks, they are engaged in a mathematical activity which we will call a problem-solving activity. Then a problem-solving *activity* involves Pólya's (1957) four phases of problem solving: what students should do before, during and after problem solving; but only the second and third phases entails the actual *process* of problem solving.

In the same way, there is a need to distinguish between mathematical investigation as a process and as an activity. When students attempt open investigative tasks, they are engaged in a mathematical activity which we will call an open investigative activity. Similar to Pólya's (1957) problem-solving model, the main phases in an open investigative activity are: understand the task, pose a problem, solve the problem, look back, and pose more problems to solve or extend the given investigative task. Now, which of these phases involve the actual process of mathematical investigation? The *key question* to ask is, "When students pose a problem in an open investigative activity, have they started investigating?" We believe the answer is negative. Hence, mathematical investigation, as a *process*, does not involve problem posing, but mathematical investigation, as an *activity* for open investigative tasks, involves problem posing, problem solving as a *process*, and other phases such as understanding the task and looking back. Figure 2 illustrates the relationship between mathematical investigation as an activity, mathematical investigation as a process and problem solving as a process.

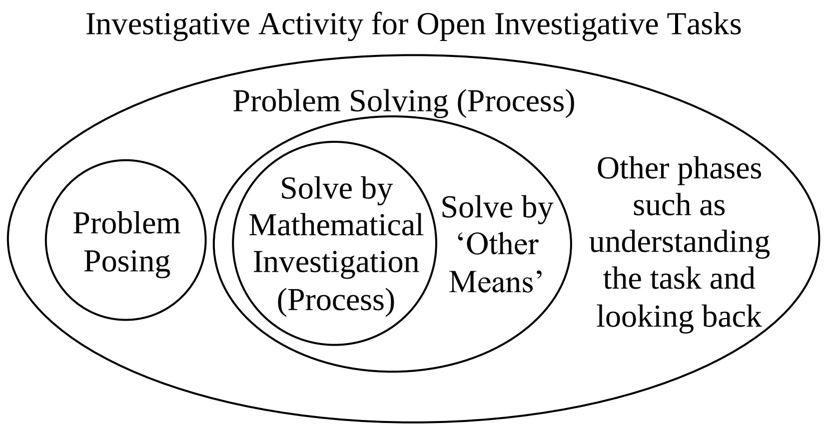

Figure 2. Relationship between Investigation as an Activity and as a Process

From Figure 2, you can observe that mathematical investigation, as a process, is a subset of the process of problem solving; but problem solving, as a process, is a subset of mathematical investigation as an activity. Therefore, distinguishing between investigation as a process and an activity can help to clarify the relationship between mathematical investigation, problem solving and problem posing.

5. Conclusion and Implications

A lot of confusion can arise because we often use the term ‘mathematical investigation’ to mean different things. This article recommends distinguishing between open investigative tasks, investigation as a process involving specialisation, conjecturing, justification and generalisation, and investigation as an activity involving open investigative tasks. With such a characterisation, the relationship between problem solving and mathematical investigation becomes apparent: the process of problem solving involves investigation as a process and solving by ‘other means’ while an open investigative activity includes both problem posing and problem solving as a process. Therefore, mathematical investigation, as a process, can occur in both open investigative tasks and problem-solving tasks.

The first implication is that the clarification of the relationship between problem solving and investigation may help to inform teachers on how and what they teach their students. This agrees with Frobisher’s (1994) belief that if teachers are unclear about the distinction between problem solving and investigation, then they will not be able to teach their students effectively. The second implication is that teachers no longer need to restrict their students to open investigative tasks but they can also use closed problem-solving tasks to expose their students to mathematical investigation. What is not so clear is the difference in the type of learning that students may gain from doing investigation involving problem-solving tasks as compared to investigation using open investigative tasks. This calls for further research. The third implication is that the characterisation of investigation as comprising of the four core thinking processes may help researchers to study how students think when they investigate. Many studies on mathematical investigation only reported its general benefits, such as the students becoming more interested (Davies, 1980) or more open to working mathematically (Tanner, 1989). Boaler (1998) went one step further to study the effectiveness of process-based teaching using open-ended activities, which is similar to open investigative activities described in this article, by looking at how the students fared in a new form of GCSE examination that rewarded problem solving (this exam was discontinued in 1994). But there are very few studies that examine the thinking processes when students engage in investigation, partly because it is hard to define clearly the processes that constitute mathematical investigation. Therefore, with this new perspective of investigation as involving specialisation, conjecturing, justification and generalisation, it may help researchers to study these processes more effectively.

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